APPLIED ELASTICITY

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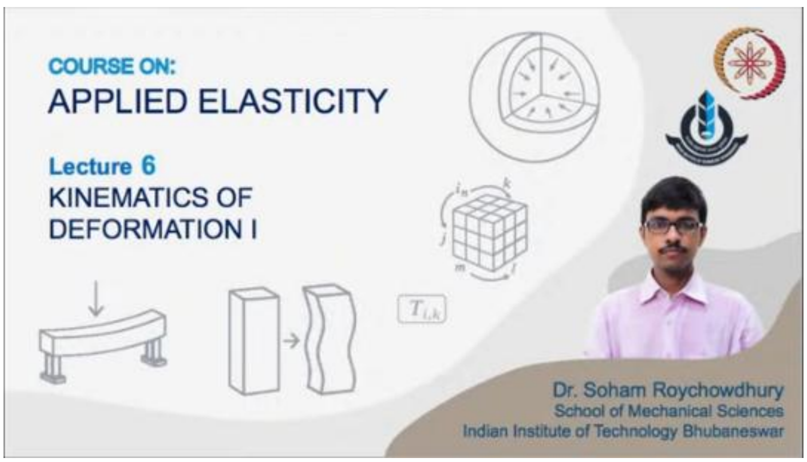
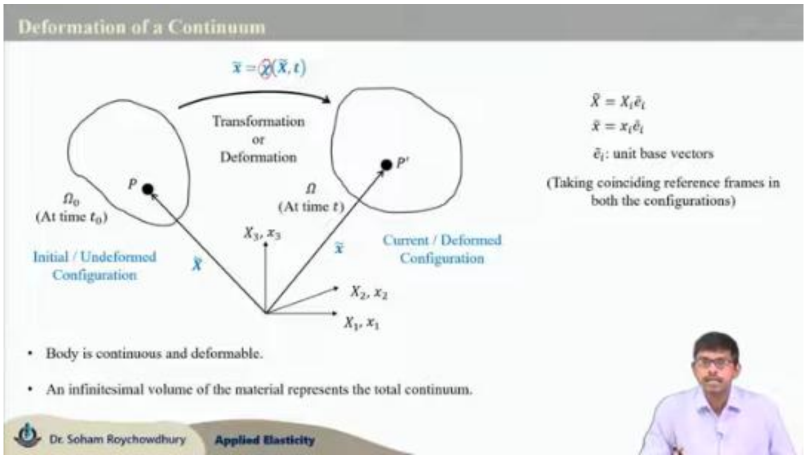
Week 2

Lecture 06: Kinematics of Deformation I



Welcome back to the course on applied elasticity. In the first week of lectures, we talked about the introduction to tensors, followed by tensor algebra and tensor calculus, which were the mathematical preliminaries required to discuss the theory of elasticity. Now, this week onward, we are going to enter into the actual subject of elasticity. We are going to start with the kinematics of deformation, which will be followed by the definition of various strain measures.



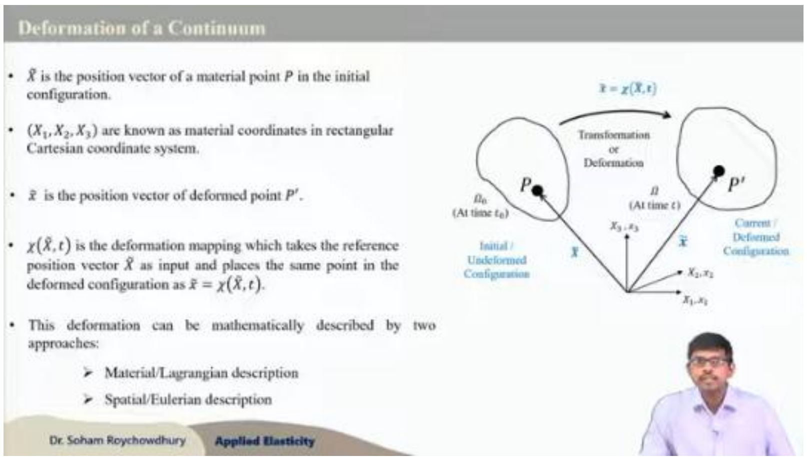
So, let us start with the kinematics of deformation. Now, we are going to consider the deformation of a continuum. So, let us consider X_1, X_2, X_3 to be the coordinates in the undeformed configuration. We are considering this particular body. P is any point on the body whose position vector is defined by $\tilde{X}$.

So, the components of the $\tilde{X}$ are X_1, X_2, X_3, all capitalized. This particular configuration is defined as initial time t_0 and defined by Ω_0. This is called the initial configuration or undeformed configuration. Now, upon loading, when this body is subjected to various external forces along with boundary conditions, this body will deform.

It will move from the initial configuration to the deformed configuration as we go from time t naught to any specific time t. So, that transformation and deformation results in the deformed body where point P has moved to another point P'. Now, P' is defined by a position vector $\tilde{x}$ with coordinates x_1, x_2, x_3. So, $\tilde{X}$ was the undeformed position vector of point P, and $\tilde{x}$ is the deformed position vector of point P'. This deformed configuration is defined at any time t, which is also named or referred to as the current configuration.

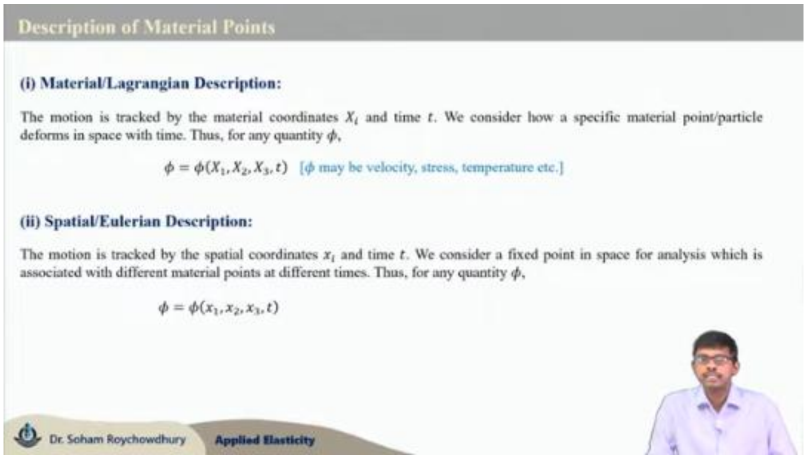
This deformation from $\tilde{X}$ to $\tilde{x}$ is defined through a transformation mapping χ. So, $\tilde{x}$ is defined as this transformation mapping χ, which is a function of the undeformed coordinate or position vector $\tilde{X}$ and time t, as shown in the figure. So, this χ is called the deformation mapping, which is a function of $\tilde{X}$, the undeformed material position vector, and time t. Now, the $\tilde{X}$ vector is $X_i\tilde{e}_i$, whereas the $\tilde{x}$ vector is $x_i\tilde{e}_i$. $\tilde{e}_i$ being the unit vectors for both the undeformed and deformed configurations.

We are choosing coinciding reference frames in both the initial and current (or deformed) configurations. Now, from this figure, we can examine the kinematics of deformation for this continuum, and two assumptions for this theory are: The body is assumed to be continuous without any discontinuity within the domain, and it is assumed to be deformable. It is not rigid; with the loading, the body is going to deform. An infinitesimal volume element (a small volume element) taken within the body represents the behavior of the total continuum.



With these assumptions we are moving forward. Now, $\tilde{X}$ is the position vector of material point P in the initial state, where X_1, X_2, X_3 are three components of the $\tilde{X}$ vector. Now, it is deformed to $\tilde{x}$ position vector through a transformation defined through χ. So, χ transformation is taking $\tilde{X}$ as input and giving $\tilde{x}$ as output through which the deformation mapping is defined.

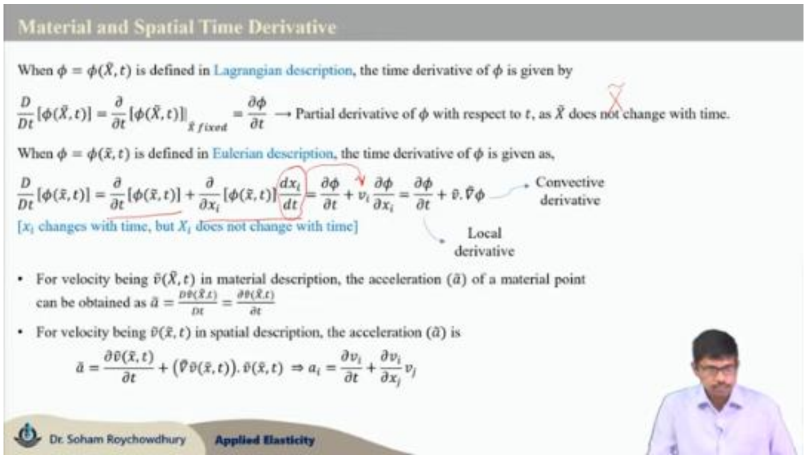
And this deformation can be mathematically described with the help of two different approaches. The first one is called material or Lagrangian approach or material or Lagrangian description, whereas the second one is called spatial or Eulerian description. Now, in case of material or Lagrangian description, we are doing all the calculations with respect to initial or undeformed state, whereas, for the case of Eulerian or spatial description all the calculations are done with respect to the deformed or the current configuration.



So, (moving on to) "description of the material points." We are choosing any material point and we are going to describe it with respect to both the approaches. The first approach is material or Lagrangian description. Now, motion of the material point within the body is traced by the undeformed material coordinates X_i and time t in the Lagrangian description approach. So, for any quantity ϕ, where ϕ is function of undeformed material coordinates X_1, X_2, X_3 as well as time.

For the case of material description, ϕ can be any quantity; it may be stress, temperature, velocity, acceleration, or anything for a particular point P within the body. So, for material description, all the quantities, properties of the body are expressed as function of undeformed coordinates X_i along with time. Now, coming to the Eulerian description of motion or spatial description of motion. Here the motion is traced with respect to spatial or the deformed coordinate x_i and time. So, for this case, we consider a fixed point in space because we are choosing x_i as our point of interest.

So, the point is fixed in space which is occupied by different material points as the time progresses. We are not tracing the motion of a particular material point in the spatial description, whereas, for the Lagrangian description as $\tilde{X}$ was chosen to be fixed for a specific material point we are observing how it is it deforming in space with time. Now, coming back to spatial description; here, any quantity ϕ is function of x_i (x_1, x_2, x_3) and time - the deformed components of the position vector Now coming to the time derivative in both material and spatial descriptions.



Considering the material or the Lagrangian description, ϕ is function of $\tilde{X}$ vector and time. So, the total time derivative of ϕ, $\frac{D\phi}{Dt}$ where D refers to total time derivative. So, the total time derivative of ϕ which is function of $\tilde{X}$ and time t equals to $\frac{\partial \phi(\tilde{X}, t)}{\partial t}$ with $\tilde{X}$ being fixed. Now, $\tilde{X}$ is defined as the position vector in undeformed or initial configuration, and this is always a fixed quantity - not changing with time. Thus, $\frac{\partial \tilde{X}}{\partial t} = 0$ and for such cases the total time derivative of ϕ is nothing but $\frac{\partial \phi}{\partial t}$.

$\frac{\partial \phi}{\partial t}$ is the total time derivative of ϕ in the Lagrangian description because ϕ is function of $\tilde{X}$ and $\tilde{X}$ doesn't change with time. So, $\frac{\partial \phi}{\partial t}$ defines the total derivative of phi with respect to t. This is called material time derivative. Now, coming to spatial description or Eulerian description, here ϕ is function of $\tilde{x}$ and time. So, here the total time derivative of ϕ is $\frac{D\phi(\tilde{x}, t)}{Dt}$.

Now, both $\tilde{x}$ as well as time are changing here. So, for the previous case, $\tilde{X}$ was not changing with time, but here the deformed position vector $\tilde{x}$ is continuously changing as the body is continuously deforming. So, here, $\frac{D\phi}{Dt}$ is equal to summation of two set of terms. First, we are taking the time derivative with respect to t, whereas, here (in the second term) we are causing the changes coming due to changes in x_i components.

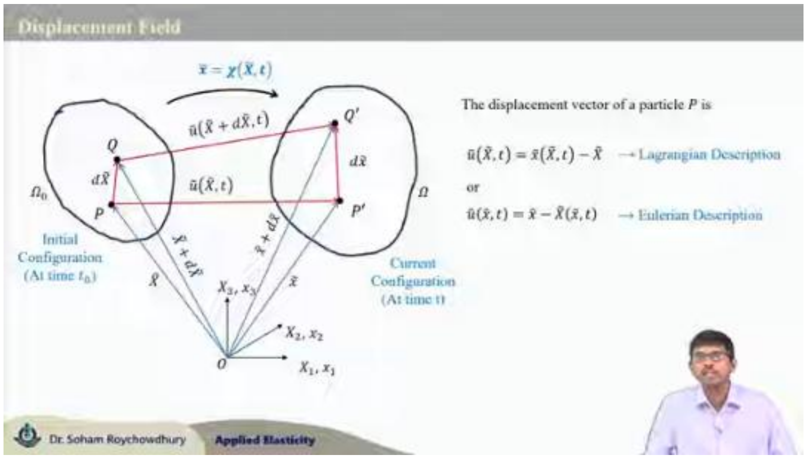
So, $\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x_i}\frac{dx_i}{dt}$. $\frac{dx_i}{dt}$ - this quantity is nothing but the velocity component v_i in the deformed configuration. So, this $(\frac{dx_i}{dt})$ is written as v_i and thus, the total time derivative becomes $\frac{\partial\phi}{\partial t} + v_i\frac{\partial\phi}{\partial x_i}$. $v_i\frac{\partial\phi}{\partial x_i}$ can also be written as $\tilde{v}.\tilde{\nabla}\phi$. So, total time derivative in the spatial or Eulerian description is having two terms:

one is $\frac{\partial\phi}{\partial t}$, which was also present for Lagrangian description, and second term is $\tilde{v}.\tilde{\nabla}\phi$. So, this first term is called local derivative with respect to time whereas, second term is called convective derivative. So, total derivative in special coordinate is summation of local derivative and the convective derivative. Now, considering ϕ to be velocity, let's say ϕ is chosen as velocity of the material and our objective is to obtain the acceleration. So, we need to take the time differentiation of ϕ which is nothing, but velocity vector.

So, in the Lagrangian description or material description, $\tilde{v}$ is function of $\tilde{X}$ and time. Thus acceleration will be $\frac{D\tilde{v}}{Dt}$ which is nothing, but $\frac{\partial\tilde{v}}{\partial t}$ - the partial derivative of $\tilde{v}$ with respect to time. Now, coming to the spatial or Eulerian description where $\tilde{v}$ is function of $\tilde{x}$ - the deformed coordinate, and time. Here, acceleration will have two terms - both local derivative as well as convective derivative. So, $\tilde{a} = \frac{\partial\tilde{v}}{\partial t} + \tilde{\nabla}\tilde{v}.\tilde{v}$.

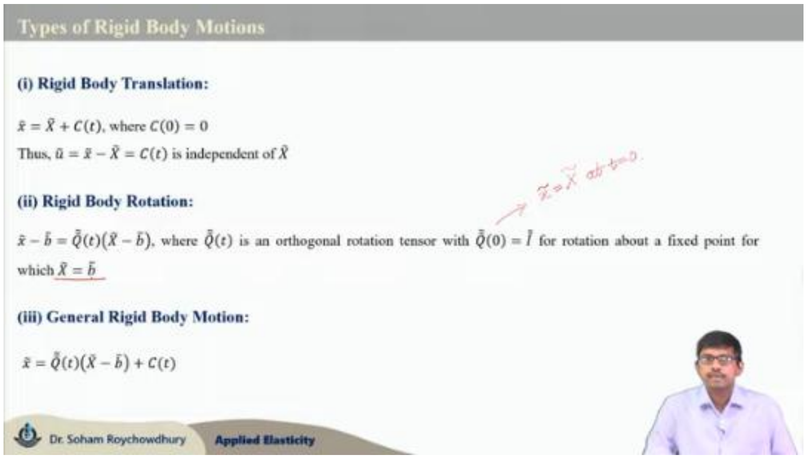
Now, we can further simplify this and write in initial format - a_i, the i th component of acceleration vector is equal to $\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}v_j$. So, this is called the spatial time derivative or acceleration in the spatial approach.

Now, moving forward to the displacement fields or definition of displacement vector. We are considering the initial configuration at time t_0, which is deformed through the transformation mapping χ to this deformed configuration where point P is moving to P'.



Now, we are choosing another point Q in the undeformed configuration, which moves to another point Q' in the deformed configuration. Now, P to Q, a small elementary length of $d\tilde{X}$, deforms to $P'Q'$, a small element in the deformed coordinate $d\tilde{x}$. Now, the displacement of point P is defined as the vector P to P', which is $\tilde{u}$. Normally, we use this symbol $\tilde{u}$ as the displacement component. In the material description, this is a function of $\tilde{X}$ and time. So, the displacement of point P is $\tilde{u}(\tilde{X}, t)$, and the displacement of point Q is $\tilde{u}(\tilde{X} + d\tilde{X}, t)$ because point Q is $d\tilde{X}$ distance away from point P.

The formal definition of the displacement vector of particle P in the Lagrangian or material description is $\tilde{u} = \tilde{x} - \tilde{X}$, where both $\tilde{u}$ and $\tilde{x}$ are functions of the undeformed material coordinate $\tilde{X}$ and time. This is the definition of the displacement vector in the material description. Now, coming to the definition in the spatial or Eulerian description, $\tilde{u}$ is once again $\tilde{x} - \tilde{X}$, but here, $\tilde{X}$ and $\tilde{u}$ are both written as functions of $\tilde{x}$. So, if we express all the quantities as functions of the deformed coordinate $\tilde{x}$, then that is the Eulerian description. So, $\tilde{u}(\tilde{x}, t) = \tilde{x} - \tilde{X}$, which is also a function of $(\tilde{x}, t)$.

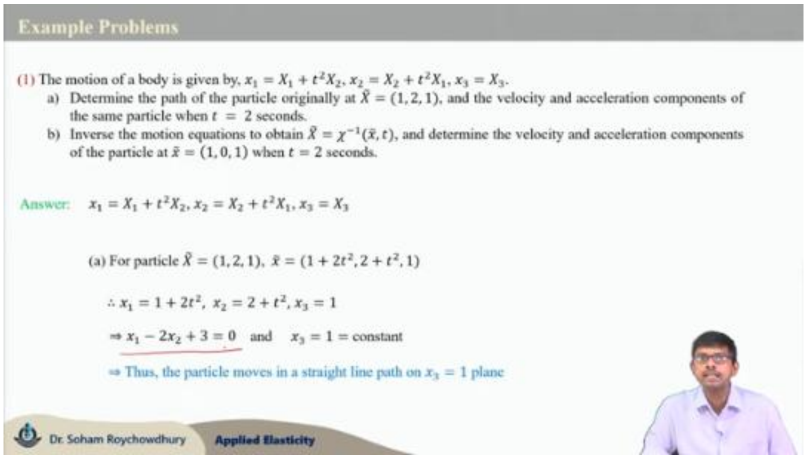


Now, coming to the types of rigid body motion, if the body is not elastic, i.e., if we consider the body to be rigid, then, with the help of the nature of the expression for $\tilde{u}$, the displacement vector, we can define various types of rigid body motion. We are going to start with rigid body translation, where $\tilde{x} = \tilde{X} + C(t)$.

This $C(t)$ is independent of $\tilde{X}$ and at time $t = 0$, $C(t)$ must be 0 which will ensure coinciding $\tilde{x}$ with $\tilde{X}$ at initial time $t = 0$. Displacement $\tilde{u} = \tilde{x} - \tilde{X} = C(t)$, independent of $\tilde{X}$, only function of time, which means the body is continuously translating along a straight line or along a curved path. This is called rigid body translation case.

Now, coming to the next type of motion - rigid body rotation. Here, the relation between $\tilde{x}$ and $\tilde{X}$ is like $\tilde{x} - \tilde{b} = \tilde{\tilde{Q}}(t)\left(\tilde{X} - \tilde{b}\right)$, where $\tilde{\tilde{Q}}$ is an orthogonal tensor referring to the rotation and at time $t = 0$, $\tilde{\tilde{Q}}(0) = \tilde{\tilde{I}}$. So, at time $t = 0$, $\tilde{\tilde{Q}}(0)$ being $\tilde{\tilde{I}}$, this should result $\tilde{x} = \tilde{X}$ at $t = 0$. So, they must be coinciding at time $t = 0$, and this refers to the rotation about a fixed point whose position vector is given as $\tilde{X} = \tilde{b}$.

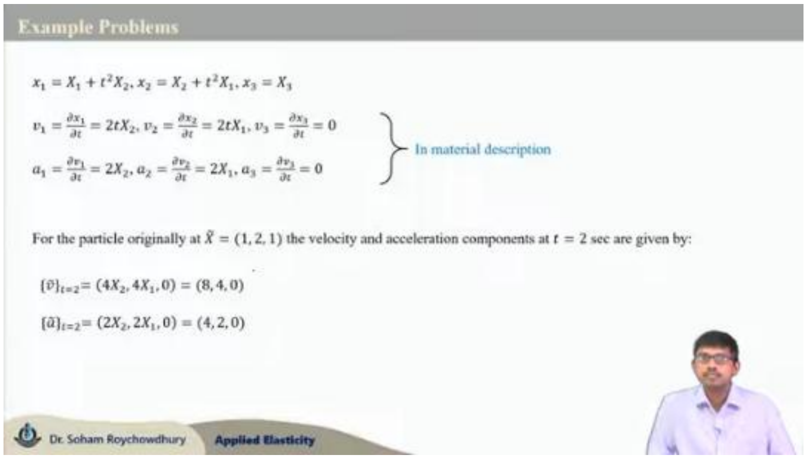
This is the case of rigid body rotation. And in general any three dimensional general rigid body motion is combination of translation and rotation. So, for such cases $\tilde{x}$ will be a combination of two previous cases, *i.e.*, $C(t) + \tilde{\tilde{Q}}(t)\left(\tilde{X} - \tilde{b}\right)$. This is the case of general rigid body motion. Now, we are going to discuss few example problems.



We are going to solve a few problems related to the kinematics of deformation. So, let us consider the motion of a body described by this given mapping: $x_1 = X_1 + t^2 X_2$; $x_2 = X_2 + t^2 X_1$; and $x_3 = X_3$. The first part is to determine the path of the particle which was originally at $\tilde{X} = (1, 2, 1)$, and the velocity and acceleration components of the same particle at time $t = 2$s.

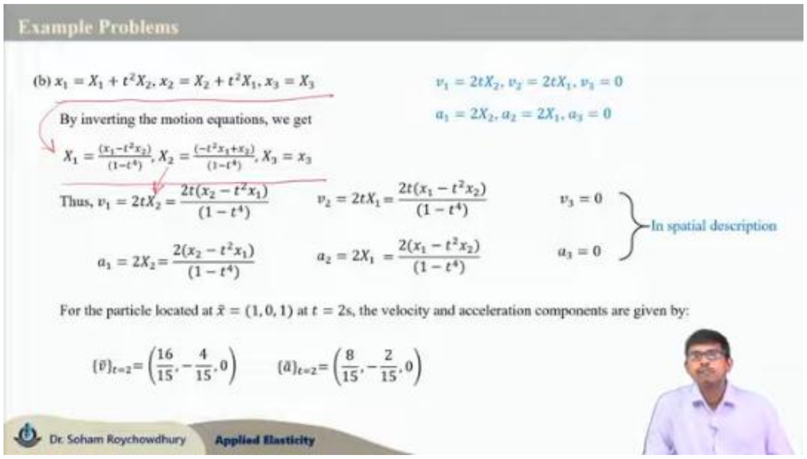
So, this is the problem of material description: $\tilde{X}$ is given, and at that point, you need to find out the path of the particle, its velocity, and acceleration. So, this is the given transformation mapping. From this, if you are writing $\tilde{X}$ and $\tilde{x}$ components, they are $(1, 2, 1)$, and in the deformed coordinates, they are $(1 + 2t^2, 2 + t^2, 1)$. Now, I have simply written x_1, x_2, and x_3 as $1 + 2t^2$, $2 + t^2$, and $x_3 = 1$. Now, we can simply combine them by the removal of time t, and that will result in two equations.

The first two equations, x_1 and x_2, can be combined with the help of the removal of t as $x_1 - 2x_2 + 3 = 0$, and $x_3 = 1$ will remain as it is. Now, this first expression is the equation of a straight line, whereas, $x_3 = 1$ refers to a plane which is parallel to the x_1 - x_2 plane at a constant value of $x_3 = 1$. So, thus the particle is going to move on a straight line defined by this equation on the plane $x_3 = 1$. This is the trace of the particle which was initially located at $\tilde{X} = (1, 2, 1)$.



Now, coming to the velocity and acceleration of that particle at a given time t. So, for finding velocity in the material description, we just need to take the partial derivative with respect to time, because all x_1, x_2, x_3 are described as functions of X_1, X_2, X_3. So, this is in the material frame or Lagrangian formulation. So, taking the time derivative of all three position components, the velocity components can be obtained as $v_1 = \frac{\partial x_1}{\partial t} = 2tX_2$. Similarly, $v_2 = 2tX_1$, $v_3 = 0$ because X_3 and x_3 were the same; it is independent of time.

Taking the derivative with respect to time once again, from the velocity components, we can obtain the acceleration components as well. Those will be $a_1 = 2X_2$, $a_2 = 2X_1$, $a_3 = 0$, and these are the velocity and acceleration in the material description. Now, putting the values of (X_1, X_2, X_3) to be $(1, 2, 1)$, and at a given time $t = 2$s, the velocity vector components can be obtained as $(8, 4, 0)$, and the acceleration vector components can be obtained as $(4, 2, 0)$.

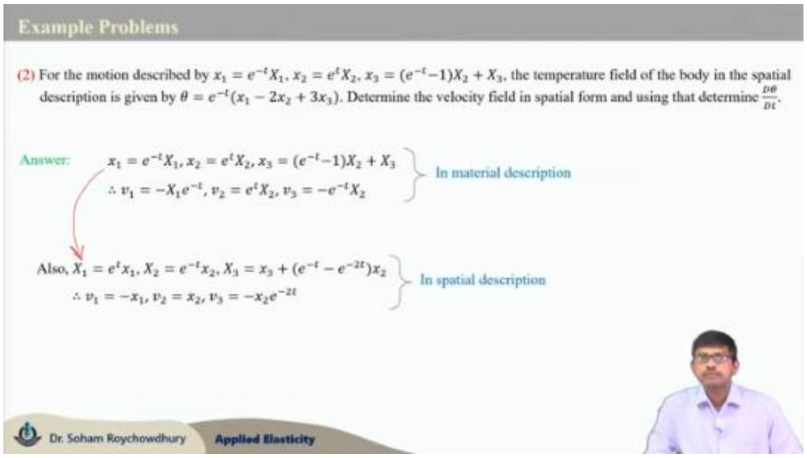


This is for the velocity and position vector solved using the material description. Now, coming to the second part of the problem, where we have to solve using the spatial description. So, x_1, x_2, x_3 were given as functions of X_1, X_2, X_3. The $\tilde{v}$ components and $\tilde{a}$ components — velocity and acceleration components — were also obtained in terms of X_i, the undeformed locations.

Now, if we want to take it back to the spatial configuration (spatial approach), then, we need to invert these equations of motion and write X_1, X_2, X_3 as functions of x_1, x_2, x_3. So, starting with these 3 equations of the material description, we can get these 3 equations of the spatial description where, $X_1 = \frac{(x_1-t^2x_2)}{(1-t^4)}$. Similarly, X_2 and X_3 can also be derived. This is by simple algebraic manipulation.

Now, v_1 which was defined as $2tX_2$. X_2 is replaced with this expression which is function of x_1 and x_2. Thus, this will become $v_1 = \frac{2t(x_2-t^2x_1)}{(1-t^4)}$. Similarly, v_2 and v_3 can be expressed as functions of x_1, x_2, x_3. Similarly, by replacing the expression of X_1, X_2, X_3, the acceleration components can also be written as function of spatial coordinates x_1, x_2, x_3.

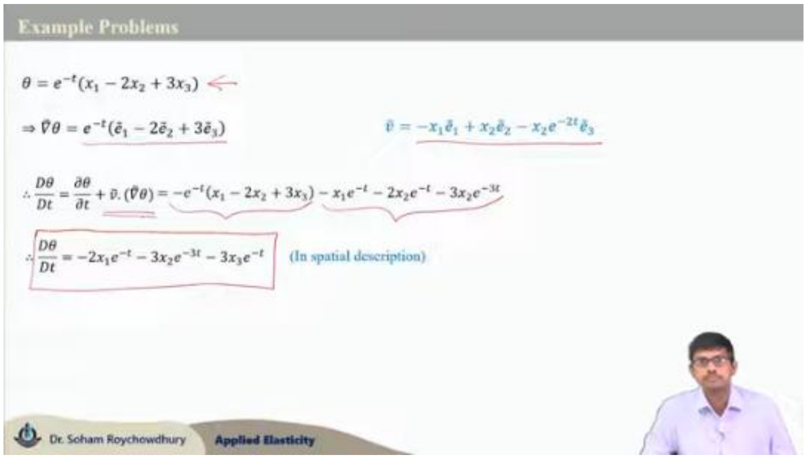
Now, for the given problem in the spatial description $\tilde{x} = (1, 0, 1)$. $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ at time $t = 2$s. So, substituting all these values, the velocity of the spatial point at time 2s can be obtained as $(16/15, -4/15, 0)$ and the acceleration components can be obtained as $(8/15, -2/15, 0)$. So, using this approach you can solve for velocity and acceleration in both material as well as spatial descriptions.



Now, coming to the second problem, where the motion is given as $x_1 = e^{-t}X_1$, $x_2 = e^{t}X_2$, $x_3 = (e^{-t} - 1)X_2 + X_3$. The temperature field of the body in the spatial description, θ, is given as $e^{-t}(x_1 - 2x_2 + 3x_3)$. You are asked to find out the velocity field in the spatial formulation and, using that, determine the time derivative of the temperature field $\frac{D\theta}{Dt}$.

So, the problem is given in the material description. The deformed coordinates (x_1, x_2, x_3) are given as functions of the material coordinates (X_1, X_2, X_3). By taking the time derivative, we can get the velocity components (v_1, v_2, v_3) in the material description as $(-X_1e^{-t}, e^{t}X_2, -e^{-t}X_2)$. These are the velocity components in the material description. Now, as we want to find out the velocity in the spatial form, we need to invert the equations of motion. So, by expressing X_i as a function of x_i, we can get this set of expressions, *i.e.*, by inverting the equation.

X_1 can be obtained as $e^{t}x_1$, X_2 is $e^{-t}x_2$, X_3 is $x_3 + (e^{-t} - e^{-2t})x_2$. Replacing these expressions of X_1, X_2, X_3 in the velocity field, we can express the velocity field in the spatial description as $v_1 = -x_1$, $v_2 = x_2$, $v_3 = -x_2e^{-2t}$. So, for any given problem, if we want to express the field in the spatial description, first, we need to invert the equation of motion: express $\tilde{x}$ components as functions of $\tilde{X}$ components and then find out the respective velocity and acceleration components as required.



Now, coming to the temperature field; the temperature field is given as $\theta = e^{-t}(x_1 - 2x_2 + 3x_3)$. We need to find out the gradient of the temperature field, which can be easily obtained as $e^{-t}(\tilde{e}_1 - 2\tilde{e}_2 + 3\tilde{e}_3)$. Why is this gradient required? This is required because we need to take the total time derivative in spatial form, which involves the local term as well as the convective term. The convective derivative term includes $\tilde{\nabla}\theta$.

Now, $\frac{D\theta}{Dt}$ in the spatial form (total derivative of the temperature field, θ) can be written as: $\frac{\partial\theta}{\partial t}$ (the local time derivative of θ) $+\tilde{v}.\tilde{\nabla}\theta$ ($\tilde{v}$ is the velocity field). Now, both $\tilde{v}$ and $\tilde{\nabla}\theta$ were obtained in the spatial configuration. So, $\tilde{v} = -x_1\tilde{e}_1 + x_2\tilde{e}_2 - x_2 e^{-2t}\tilde{e}_3$, which was obtained in the previous slide. Taking the dot product of this $\tilde{v}$ with the obtained $\tilde{\nabla}\theta$.

This $\tilde{\nabla}\theta$ and this $\tilde{v}$ — we are taking a dot product of those, which results in 3 terms — the convective time derivative terms. Whereas, if you take the partial time derivative, that would result in $\frac{\partial\theta}{\partial t}$ — the first set of terms, which are called the local time derivative terms. And if you simplify that, out of these 6 terms, a few will be cancelled, and finally, the total time derivative of the temperature field, $\frac{D\theta}{Dt} = -2x_1 e^{-t} - 3x_2 e^{-3t} - 3x_3 e^{-t}$. This is the change in the temperature field in the spatial description — the answer which was asked in the problem.

And if you were asked to obtain it in the material description, then there is no need to consider the convective term. This term, $\tilde{v}.\tilde{\nabla}\phi$, is not required to be considered in the material description. $\frac{D\theta}{Dt}$ would just become $\frac{\partial\theta}{\partial t}$ in the material description.

So, in this lecture, we introduced the basics of deformation of a continuum using both Lagrangian and Eulerian descriptions. We also talked about the material and spatial time derivatives.