APPLIED ELASTICITY

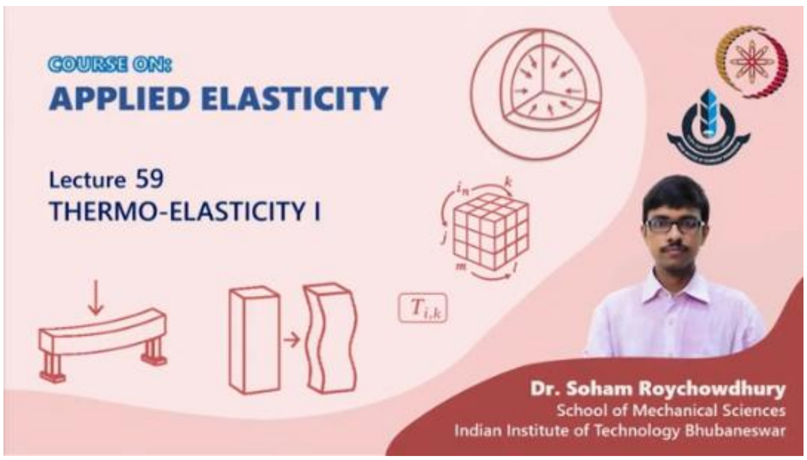
Dr. Soham Roychowdhury

School of Mechanical Sciences

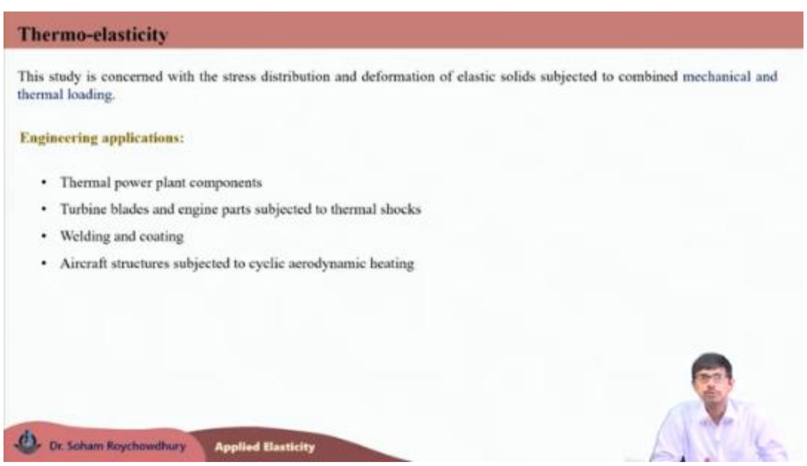
IIT Bhubaneswar

Week 12

Lecture 59: Thermo-elasticity I



Welcome back to the course on Applied Elasticity. In today's lecture, we are going to start our discussion on a new topic called thermo-elasticity.

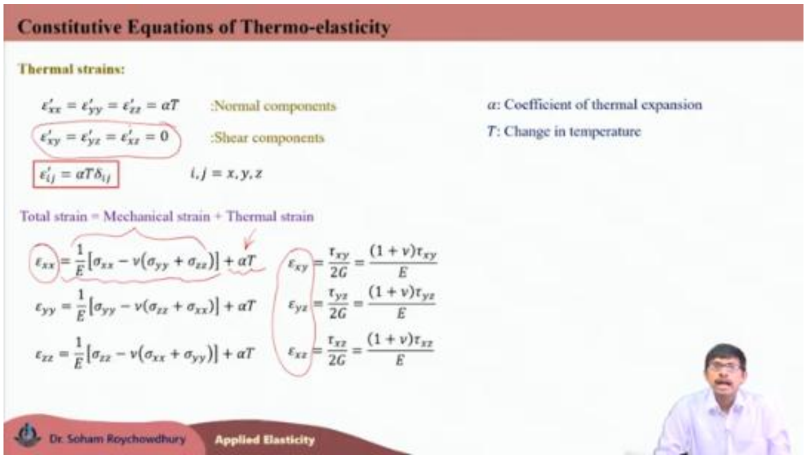


What do we mean by thermo-elasticity or thermo-elastic deformation problems? Thermo-elasticity is the study in which we predict the stress distribution and deformation of an elastic continuum, when the body or elastic continuum is subjected to combined mechanical and thermal loading. If there is a body, that is not only subjected to mechanical loading but also to combined mechanical and thermal loading, that problem falls under the category of thermo-elasticity, or the domain of thermo-elasticity. Using the thermo-elastic problem solution approach, you would obtain the stress distribution and displacement deformation patterns of the elastic continuum when it is subjected to combined mechanical and thermal loading.

Let's discuss various engineering applications where thermo-elastic problems need to be solved. For example, components in thermal power plants are subjected to very high temperatures along with mechanical loads. These loads may come from fluid pressure within pipes or other components of the thermal power plant, such as heat exchangers, reheaters, and superheaters, which are subjected to fluid pressure as well as high temperatures. Thus, it is a combination of thermal and mechanical loads.

Similarly, gas turbine blades, or different parts of an I.C. engine are also subjected to mechanical and thermal shocks. This is another example of combined mechanical and thermal loading. During welding and coating, thermo-mechanical loads are also present, as is the case with aircraft structures. They are subjected to cyclic aerodynamic heating. The temperature is going up and down in a cyclic manner, which is giving the cyclic thermal loading, and along with that, the mechanical loads are also acting on the aircraft structure due to the high pressure. Hence, all these are various real-life engineering applications, where we can see the bodies or the elastic continuums being subjected to mechanical as well as thermal loadings.

Now, the formulation which we had discussed till now, all such elastic problems were only considering the application of mechanical loads. Various mechanical loads were there; it may be a normal point force, concentrated force, distributed force, moment, twisting moment, anything, but all those were purely mechanical loads. If thermal loads are also acting on the problem, then there should be some extra amount of stress or strains generated due to the thermal loading as well. For the thermo-elasticity solution approach, we are going to consider the effect of both mechanical loading on the continuum or body and thermal loading on the elastic body and continuum.



First, moving to the constitutive equations of thermo-elasticity. For the mechanical loading, mechanical stress and generated mechanical strain were related with the help of constitutive equations. We had obtained the constitutive relations for various types of materials, and for the linear elastic isotropic solid, we have seen that the stresses and strain components can be related with the help of only two material constants. But that is for a purely mechanical loading problem; if we add thermal loading to the system, along with mechanical strains, thermal strains would be generated, and thus, that constitutive equation is required to be modified.

So, the first task for studying thermo-elasticity is to understand the relation between the state of stress and the state of strain for a body subjected to both mechanical and thermal loading. Starting with the concept of thermal strain. This is the strain produced within the body due to the applied temperature variation or due to the thermal loading. Due to thermal loading, there will be a variation of temperature change within the body which will give rise to the thermal strain.

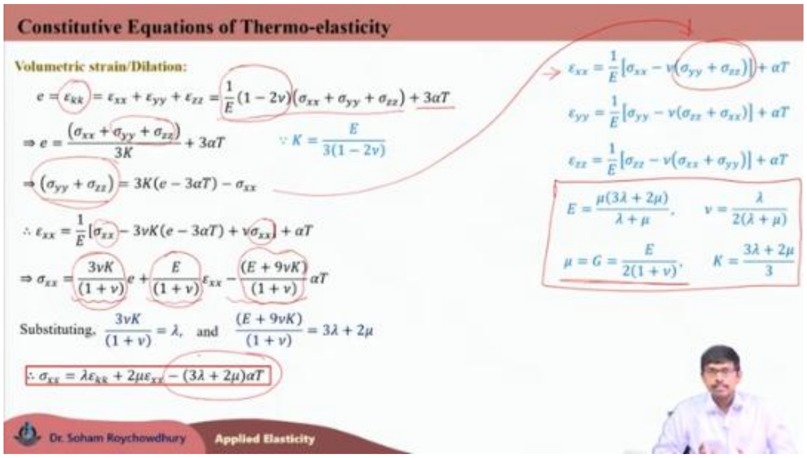
Let us write ε'_{xx}, ε'_{yy}, and ε'_{zz} as the thermal strain component along three normal directions, which is produced only due to thermal loading. We are not considering mechanical loading here, and all of them are equal to αT, whereas, the shear components of the thermal loading for all three directions, ε'_{xy}, ε'_{xz}, ε'_{yz} would be 0. α is the coefficient of thermal expansion of the material, and T is the change in temperature due to the application of the thermal loading.

With the help of this material constant α, this is another material constant. Apart from the Poisson's ratio ν and Young's modulus E and modulus of rigidity G, this coefficient of thermal expansion would be another material constant, which would be coming in the constitutive equation for the case of thermo-mechanical problems. And T is the change in temperature, which would be multiplied with α to predict the normal thermal strains, but no thermal shear strains can be generated due to increase in the temperature T.

Now, these strain equations can be combinedly written in the indicial notation as $\varepsilon'_{ij} = \alpha T \delta_{ij}$, where this δ_{ij} is the Kronecker delta. If $i = j$, δ_{ij} is 1, and if $i \neq j$, δ_{ij} is 0. So, for all shear stress cases δ_{ij} would be 0 and thus, ε'_{ij} is 0 for $i \neq j$. For $i = j$ case, δ_{ij} being 1, all three normal thermal strain components would be equal to αT. Here, i and j can take the values of x, y, and z, and this ε'_{ij} refers to the thermal strain component generated within the body due to thermal loading.

Now, the total strain within the body will be coming due to the contribution of both mechanical strain and thermal strain. So, total strain is the summation of mechanical strain generated due to mechanical loading and thermal strain generated due to thermal loading. Expressions of the mechanical strain are well known to us. ε_{xx}, written on the left-hand side, is the total normal strain generated along the x-axis, which is normal strain due to the mechanical component of the loading, which is the first term. This is the mechanical normal strain along the x-axis, and this is the thermal normal strain along the x-axis, which is nothing but αT, and in terms of the mechanical stresses, we can write ε_{xx} mechanical as $\frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right)$. Total ε_{xx} will be the summation of the first term, which is the mechanical component of the normal strain, plus αT, which is the thermal normal strain.

In the same fashion, all other normal strain components ε_{yy} and ε_{zz} can be written as shown here. Whatever mechanical strains were there for the normal part, we are just adding one αT with all the normal strains. Whereas, the shear strain components ε_{xy}, ε_{yz}, ε_{xz} remain unaltered because the thermal shear strains are 0. Total shear strain is equal to only mechanical shear strain; there is no contribution coming to the shear strain component because of the thermal loading. These are the six equations which relate stress components with the strain components and the temperature variation T, and α is another property coefficient of thermal expansion coming in this constitutive equation.



Now, we will try to write this constitutive equation in a concise form using the indicial notation. We know that the volumetric strain, e or ε_{kk}, the trace of the strain tensor $\tilde{\tilde{\varepsilon}}$, is written as $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$. All these ε_{xx}, ε_{yy}, and ε_{zz} are the three normal strain components for the thermo-mechanical problem, which we had obtained like this just in the previous slide.

Substituting these total normal strain components ε_{xx}, ε_{yy}, ε_{zz} in the expression of the volumetric strain or dilation ε_{kk}, this dilation can be written like this: $\frac{1}{E}(1-2\nu)\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) + 3\alpha T$. For only the mechanical problem, this first term was there, and due to this thermal part, each of the ε is contributing one αT term. So, in the volumetric strain, there would be a $3\alpha T$ term, which is coming due to the three different normal thermal strain components.

Rewriting e from this, we can write e as $\frac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{3K} + 3\alpha T$. What is K? K is the bulk modulus. We had discussed about all these different properties of the linear elastic homogeneous isotropic solids: E, K, ν, G, λ, and μ. λ and μ are Lame constants. They can be related with the commonly known constants: Young's modulus E, shear modulus or modulus of rigidity G, Poisson's ratio ν, and bulk modulus K. Those relations we had earlier derived.

One such equation we are using here, which is $K = \frac{E}{(1-2\nu)}$. So, in the first term, this $\frac{(1-2\nu)}{E}$ I am writing as $3K$. Thus, the volumetric strain or dilation, ε_{kk} or e, would be $\frac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{3K} + 3\alpha T$.

Now, from this, we can write the summation of σ_{yy} and σ_{zz}. Our objective is to replace σ_{yy} and σ_{zz} in the first strain expression, ε_{xx}, in terms of σ_{xx}. These parts of σ_{yy} and σ_{zz}, I want to write in terms of σ_{xx}, so that, on the right-hand side of ε_{xx}, only one normal stress corresponding to ε_{xx} is present, that is σ_{xx}. From this, we are keeping these parts $(\sigma_{yy} + \sigma_{zz})$ on one side and sending all the rest to the right-hand side. So, $(\sigma_{yy} + \sigma_{zz})$ would be $3K(e - 3\alpha T) - \sigma_{xx}$.

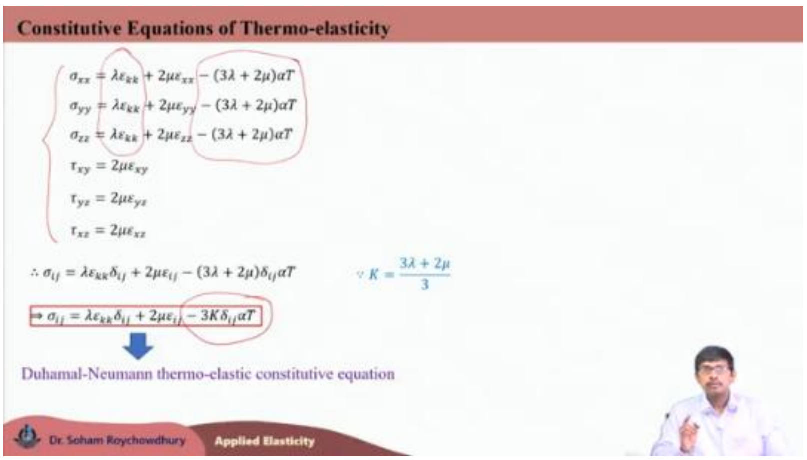
Replacing this expression of $(\sigma_{yy} + \sigma_{zz})$ here in the ε_{xx} equation's right-hand side, ε_{xx} can be written like this: $\frac{1}{E}[\sigma_{xx} - 3\nu K(e - 3\alpha T) + \nu\sigma_{xx}] + \alpha T$. If I expand this in terms of σ_{xx}, keeping this σ_{xx} on one side and sending everything on another side. The normal strain ε_{xx} and the thermal part αT, all those are sent on the right-hand side.

The σ_{xx} expression in that case would be looking like this. We are having three terms: σ_{xx} is dependent on dilation or volumetric strain, e. It is dependent on corresponding normal strain ε_{xx}. It is also dependent on the thermal strain αT. All these terms are having coefficients $\frac{3\nu K}{(1+\nu)}$, then $\frac{E}{(1+\nu)}$, and $-\frac{(E+9\nu K)}{(1+\nu)}$, respectively. All these are material constants. They are related; these constants are directly related with different material constants.

From this E, ν, K, all these material constants, we will try to write in terms of the Lame constants λ and μ. If you recall the relation between these commonly available material constants and the Lame constants, all those relations are presented here, which we had derived once while discussing the constitutive equation for isotropic solids. Using these equations of Young's modulus E, shear modulus G or μ, Poisson's ratio ν, and bulk modulus K, we can write each of them in terms of λ and μ, which are the two quantities known as the Lame constants.

With that, the first term $\frac{3\nu K}{(1+\nu)}$ can be written as λ. This last coefficient, $\frac{(E+9\nu K)}{(1+\nu)}$, can be written as $3\lambda + 2\mu$. This central term, the coefficient of the middle term, $\frac{E}{(1+\nu)}$, using this equation, is nothing but 2μ. Hence, σ_{xx} becomes $\lambda\varepsilon_{kk} + 2\mu\varepsilon_{xx} - (3\lambda + 2\mu)\alpha T$,. This is the relation between the normal stress σ_{xx} and the normal strain ε_{xx}, along the x-direction for a thermo-mechanical problem.

If you recall the corresponding equation for a purely mechanical problem, this term was not there. This is the extra term coming. For a mechanical problem, we had σ_{xx} to be $\lambda\varepsilon_{kk} + 2\mu\varepsilon_{xx}$, that is all. Here, this additional term, $-(3\lambda + 2\mu)\alpha T$, is coming due to the thermal strain.

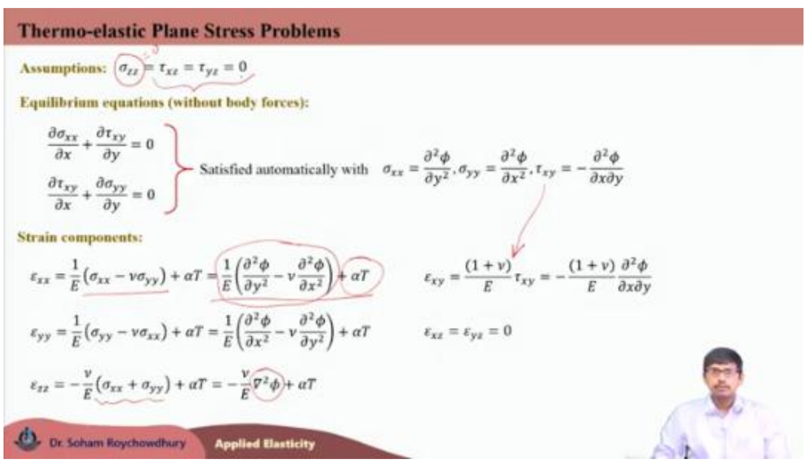


Similarly, you can write σ_{yy} and σ_{zz}, the other two normal stress components, in a similar fashion like this. And as there is no thermal shear strain generated, the constitutive equation for the shear stress and shear strain components remains the same. τ_{xy}, τ_{yz}, τ_{xz} are equal to $2\mu\varepsilon_{xy}$, $2\mu\varepsilon_{yz}$, and $2\mu\varepsilon_{xz}$, respectively.

These are the six constitutive equations for any thermo-mechanical problem, which can be combined into a single equation written in indicial notation form like this. σ_{ij} equals $\lambda\varepsilon_{kk}\delta_{ij}$, this term, due to the presence of the Kronecker delta, δ_{ij}, would be non-zero only if $i = j$. That term results in $\lambda\varepsilon_{kk}$ only for the normal stress components. For the shear stress components, $i \neq j$, so this term would vanish. The second term is present for all six cases, which is $2\mu\varepsilon_{ij}$. The last term, $-(3\lambda + 2\mu)\alpha T\delta_{ij}$, is also present only for the normal terms. That is why δ_{ij} or the Kronecker delta is added here in the last term. So, the last term is also present only for $i = j$, or for the normal stress-normal strain relations.

Rewriting K as $\frac{(3\lambda+2\mu)}{3}$, which is the relation between the bulk modulus and the Lame constants, we can rewrite this equation as $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - 3K\alpha T\delta_{ij}$. This is the constitutive equation, known as the Duhamel-Neumann thermo-elastic constitutive equation, which is required to be used for relating the total stress with the total thermo-elastic strain, containing both mechanical and thermal components.

This last term is extra, which gives us the effect of the change in temperature T, involving two material constants: the bulk modulus K and the coefficient of linear thermal expansion α.



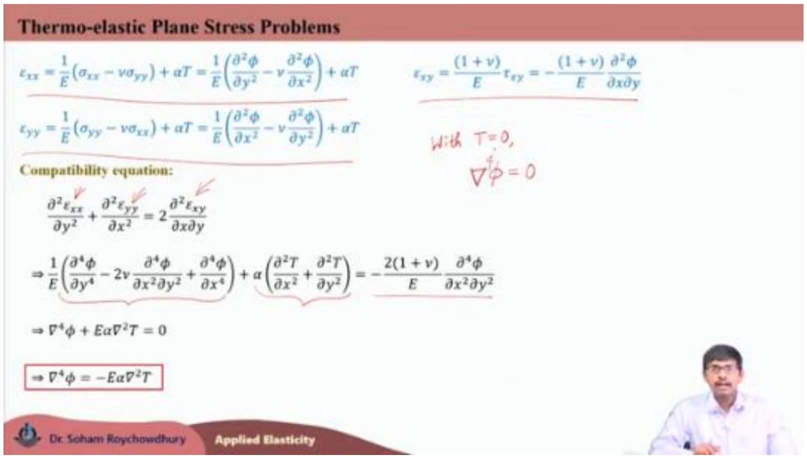
Moving forward to the formulation of 2D thermo-elasticity problem, which may be either of plane stress type or plane strain type. Coming to the assumptions of the thermo-elastic plane stress problem. First, we are considering a thermo-elastic problem with plane stress approximation for all the out of plane stresses. σ_{zz}, out of plane normal stress, τ_{xz} and τ_{yz}, the out of plane shear stresses, are neglected for this plane stress assumption, which is valid with geometry of the body being thin in the z direction.

For such cases, these three out of plane stress components being 0, the equilibrium equations after neglecting the body forces would be like this: $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0$ and $\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0$. These are the only two equilibrium equations available for the thermo-elastic plane stress problem, where these two equations can be automatically satisfied by choosing the in-plane stress components in terms of a stress function known as Airy's stress function, ϕ, as $\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$, and $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$. If we choose the stress components like this, we know that then, both the equilibrium equations would be automatically satisfied.

Now, we will try to write the strain components for this thermo-elastic plane stress problem. Strain components are having this additional thermal strain term present. Considering the normal strain in the x direction, ε_{xx}, that is equal to $\frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right)$. Note that as $\sigma_{zz} = 0$, we are not having any term $(-\nu\sigma_{zz})$ here in the ε_{xx}. σ_{zz} is already 0 because of the plane stress nature of the problem.

Substituting σ_{xx} as $\frac{\partial^2\phi}{\partial y^2}$ and σ_{yy} as $\frac{\partial^2\phi}{\partial x^2}$, ε_{xx} would be like this: $\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right) + \alpha T$. The first term, this one, is the mechanical normal strain in x direction, and αT is the thermal normal strain in x direction. Similarly, another normal strain in y direction, ε_{yy} can be written as $\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right) + \alpha T$. The third one, ε_{zz}, the axial normal strain can be written as $-\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) + \alpha T$. Replacing σ_{xx} and σ_{yy} as $\frac{\partial^2\phi}{\partial y^2}$ and $\frac{\partial^2\phi}{\partial x^2}$, respectively, these terms would give us $\nabla^2\phi$. So, ε_{zz} would be $-\frac{\nu}{E}\nabla^2\phi + \alpha T$. These are the three normal strain components for a thermo-elastic plane stress problem.

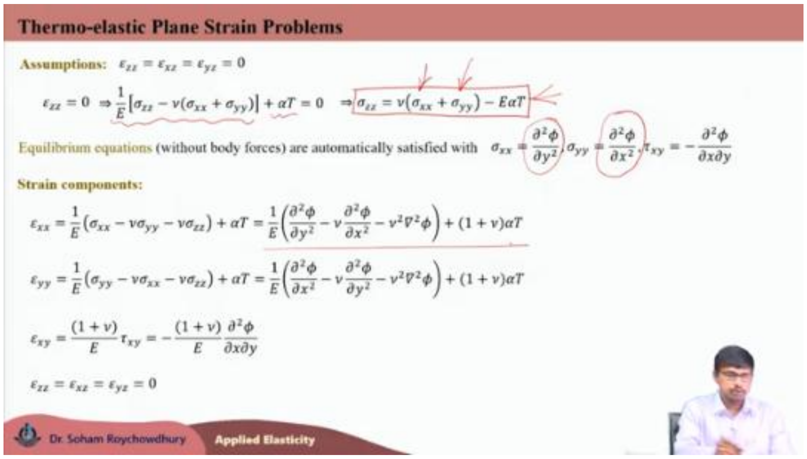
The only non-zero shear strain component ε_{xy} can be written in terms of corresponding shear stress component τ_{xy} as $\frac{(1+\nu)}{E}\tau_{xy}$. Replacing τ_{xy} as $-\frac{\partial^2\phi}{\partial x\partial y}$, we can write this as $-\frac{(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}$. And other two out of plane shear strains, ε_{xz} and ε_{yz}, would be 0, as we have τ_{xz} and τ_{yz} as 0. These are the six thermo-elastic plane stress problem strain components.



Using these non-zero in plane strains, we must satisfy the strain compatibility equation for this thermo-elastic plane stress problem, and only then, unique displacement field can be ensured. For the 2D problem, only compatibility equation which is left out to be satisfied is this: $\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{2\partial^2\varepsilon_{xy}}{\partial x\partial y}$.

Here, these ε_{xx}, ε_{yy}, and ε_{xy} are replaced in terms of these equations. Replacing these here, if I simplify this. All the terms on the left hand side, the first set of terms are the terms which are having $\frac{1}{E}$ as common factor. Then, the α dependent terms are written here, and the right hand side are the terms coming from the ε_{xy}. If I rewrite this using the Laplacian and biharmonic operator, this equation would be $\nabla^4\phi + E\alpha\nabla^2 T = 0$, or $\nabla^4\phi = -E\alpha\nabla^2 T$. This is the governing equation for choosing the stress function ϕ for a thermo-elastic plane stress problem.

If you recall for a purely mechanical 2D plane stress problem, our Airy stress function should just satisfy $\nabla^4\phi = 0$. Now, from this equation, if we want to go back to the purely mechanical problem, then we must force $T = 0$. If you say there is no change in temperature with $T = 0$, this governing equation for thermo-elastic plane stress problem would be reduced to the governing equation of purely mechanical plane stress problem where ϕ has to be just biharmonic. Here, the biharmonic operator acting over ϕ, $\nabla^4\phi$, should be equal to $-E\alpha\nabla^2 T$. So, for a given change in temperature field, T, we need to find $\nabla^2 T$, then $-E\alpha\nabla^2 T$ should be equal to $\nabla^4\phi$. Accordingly, the stress function should be chosen for the thermo-elastic plane stress problem.



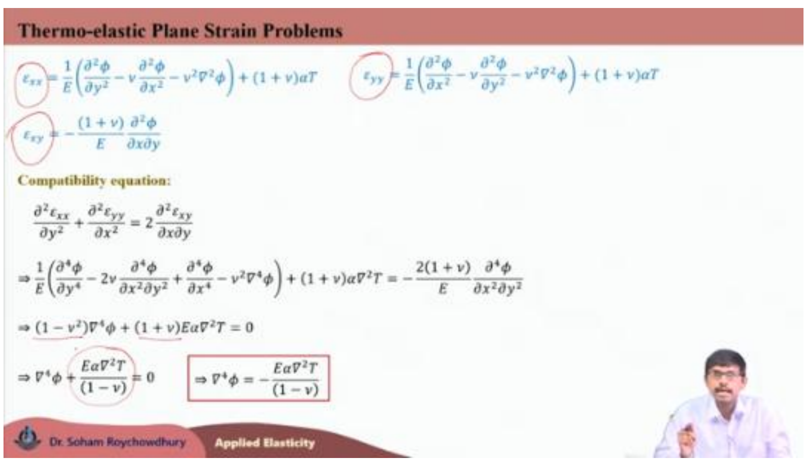
Moving forward to the thermo-elastic plane strain problem in a similar manner, where the body is supposed to be or assumed to be very large along the z direction, so that the out-of-plane strain components ε_{zz}, ε_{xz}, ε_{yz} can be neglected. ε_{zz} being 0, we can write the total mechanical plus thermal component of ε_{zz} is 0. So, $\frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \alpha T = 0$.

The first term is the normal strain in the z direction for mechanical, and αT is normal strain in the z direction for thermal loading. From this, we can write σ_{zz} as $\nu\left(\sigma_{xx} + \sigma_{yy}\right) - E\alpha T$. So, σ_{zz} is non-zero for the plane strain problem and that is related with the other normal stresses and the temperature field using this particular equation.

Moving forward to the equilibrium equation, they are the same as the case of the plane stress problem. Equilibrium equations for any 2D problem, plane stress or plane strain, are the same. Thus, we can have the same type of choice of stress components in terms of the Airy stress function ϕ as: $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$, and $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$.

In terms of these chosen stress components, we can write the strain components like this. The first one, normal strain along the x direction for a plane strain thermo-elastic problem, ε_{xx}, would be: $\frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right) + \alpha T$, where replacing σ_{xx} with $\frac{\partial^2 \phi}{\partial y^2}$, σ_{yy} with $\frac{\partial^2 \phi}{\partial x^2}$, and σ_{zz} using this equation, in which once again these σ_{xx} and σ_{yy} are required to be replaced in terms of ϕ. If I do so and simplify, the final expression of ε_{xx}, the normal strain along the x direction, would be: $\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2} - \nu^2\nabla^2\phi\right)$, this is the mechanical part, plus $(1+\nu)\alpha T$, this is the thermal part. This gives us the total expression for normal strain along the x direction for the thermo-elastic plane strain problem.

Similarly, we can get the other strain components ε_{yy}, another normal strain like this, and the in-plane shear strain ε_{xy} would be $-\frac{(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}$, which is the same for both plane stress and plane strain problems. The remaining three out-of-plane normal and shear strains ε_{zz}, ε_{xz}, ε_{yz} are 0 for the plane strain assumption. We got the strain fields written in terms of ϕ, the Airy stress function.



Moving forward to the strain compatibility equation, which is required to be satisfied for ensuring a unique displacement field. The only strain compatibility left to be satisfied for the thermo-elastic plane strain problem is: $\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{2\partial^2\varepsilon_{xy}}{\partial x\partial y}$, same as the plane stress compatibility equation for the 2D problem.

Substituting ε_{xx}, ε_{yy}, and ε_{xy} using these equations in terms of ϕ, and simplifying this further, the strain compatibility equation gives us this: $(1-\nu^2)\nabla^4\phi + (1+\nu)E\alpha\nabla^2 T = 0$. Cancelling this $(1+\nu)$ factor from this $(1-\nu^2)$, we would be left with only one term, which is $\nabla^4\phi + \frac{E\alpha\nabla^2 T}{(1-\nu)} = 0$, and if you are taking this term on the other side, in that case, $\nabla^4\phi = -\frac{E\alpha\nabla^2 T}{(1-\nu)}$. This is the governing equation for choosing the stress function ϕ for the thermo-elastic plane strain problems.

Here also, if you substitute T to be 0, it will be reduced to the corresponding mechanical plane strain problem with $\nabla^4\phi = 0$. So, ϕ should be bi-harmonic. But here, it is the biharmonic operator $\nabla^4\phi$ equal to $-\frac{E\alpha\nabla^2 T}{(1-\nu)}$ if we are considering the thermo-elastic plane strain problem. So, for both thermo-elastic plane strain and thermo-elastic plane stress, we are having one governing differential equation based on which the choice of ϕ, the stress function, is dictated.

In this lecture, we discussed the introduction to the thermo-elastic problems. We derived the thermo-elastic constitutive relation for isotropic solids, and finally, discussed the formulation for thermo-elastic plane stress and thermo-elastic plane strain problems.

Thank you.