APPLIED ELASTICITY

Dr. Soham Roychowdhury

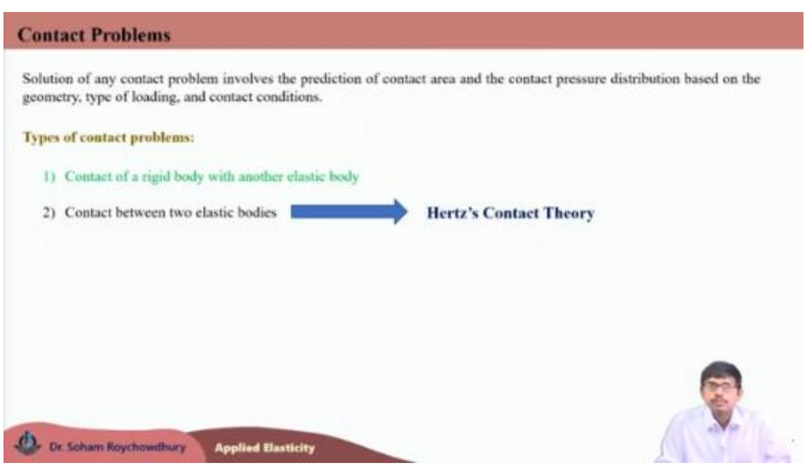
School of Mechanical Sciences

IIT Bhubaneswar

Week 12

Lecture 58: Contact Problems II

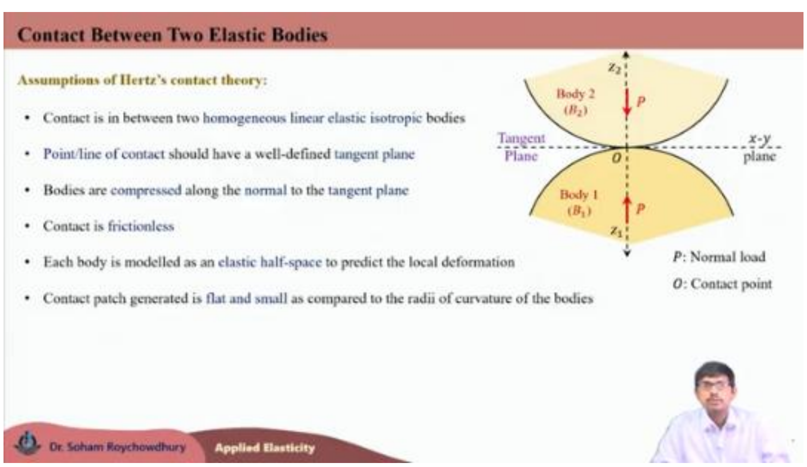
Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the contact problems that we started in the last lecture.



To have a quick recap, the contact mechanics problem basically involves the solution of the contact area and the contact pressure distribution generated within that contact patch, based on the geometry of the bodies coming in contact, the nature of friction present between the two contacting surfaces, and the type of loading. So, contact mechanics problems basically deal with the solution of the contact stresses, contact pressure distribution, and the area of contact generated when two bodies, which may be rigid or elastic (at least one should be elastic), come in contact. The solution of the contact problem depends on the geometry of the bodies coming in contact, the type of loading, and the nature of friction present between the two bodies.

We can broadly classify two types of contact problems: one is contact between a rigid body and an elastic body, and another is contact between two elastic bodies. In the previous lecture, we discussed the contact of a flat rigid indenter with an elastic half-space, which falls under the first category of contact problems.

In today's lecture, we are going to talk about the contact between two elastic bodies. Contact between two elastic bodies is mostly described by the famous contact theory proposed by Hertz. This is called Hertz's contact theory. First, we will discuss the assumptions of Hertz's contact theory, and then, using that theory, we will try to predict the solution of the contact problem, *i.e.*, the contact pressure distribution and contact patch area prediction for two elastic bodies coming in contact.



Here, let us consider these two bodies: one body is named as B_1, and body 2 is named as B_2. They are coming in contact as shown in this figure due to the application of the normal load P. P is the normal load applied between these two bodies along the z-direction. O is the point of contact. There are two contacting surfaces of body 1 and body

2 which are coming in contact at a single point O due to the application of this normal load.

z_1, z_2 are the normal directions for body 1 and body 2 respectively, and the xy plane is called the tangent plane between these two contacting bodies, which passes through point O. The xy plane is the tangent plane, and z_1, z_2 are normal axes on the xy plane, which are normal along body 1 and body 2 respectively.

Coming to the assumptions of the Hertzian contact stress theory. The first one is that both the bodies coming in contact are described as homogeneous, linear elastic, isotropic solids involving only two material constants.

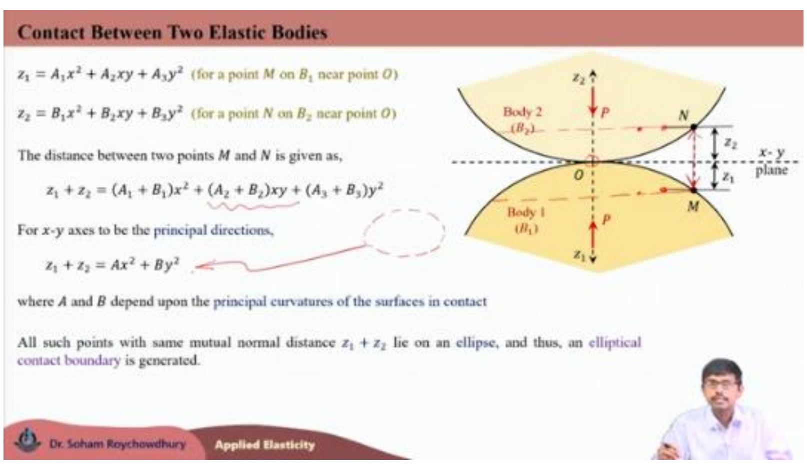
The next assumption is that there would be a well-defined tangent plane at the point or line of contact. Whether there will be a point contact or line contact depends on the initial geometry of the body. Let us say there is one spherical body which is coming in contact with another spherical body or with a flat surface. For such cases, if a sphere is coming in contact with another body, the point of contact would be there; contact will be at a single point. However, instead of a sphere, if a cylinder is coming in contact with a flat surface, in that case, a line contact would be generated. So, a sphere coming in contact with another sphere or a flat face would result in a point contact, while a cylinder coming in contact with a flat face will result in a line contact.

Now, at the point or line of contact, there should be a well-defined tangent plane. For the present figure, in the present problem of contact between body 1 and body 2, the tangent plane is the xy plane. Bodies must be compressed along the normal direction to the tangent plane. The applied loading P should be perpendicular to the tangent plane for both the bodies, and that is preserved here. z_1, z_2 are the directions along which P force compressive loads are acting and z_1, z_2 are perpendicular to the tangent xy plane for our problem.

Next, the contact is assumed to be frictionless. No friction is considered at the point of contact or within the contact span. Each body is modeled as an elastic half space to predict their local deformation around the point of contact. The bottom body is considered to be one half space subjected to load P coming from the top body B_2, and then, we will get the deformation pattern or resulting stresses on the bottom elastic half space at point O using an approach similar to the Flamant solution approach. On the other hand, top body B_2 will also be modeled as another elastic half space, which is subjected to the upward load P coming from body B_1, which would also be solved by using a similar approach, and the local deformation and local stresses on body 2 at point 2 can be obtained. So, both the bodies are being modeled as individual elastic half space for predicting their local deformation.

Also, the contact patch generated is assumed to be flat patch, and the contact patch area or radius is assumed to be small as compared to the dimensions of the body or as compared to the radius of curvatures of the body at the point of contact. Radius of curvatures of both the bodies at the point of contact is much larger as compared to the size of the contact patch and that contact patch must be flat.

These are the assumptions for the Hertzian contact stress theory, with which we will try to solve this problem of contact between two elastic bodies.



Now, we are considering one point M on body 1, which is at a distance z_1. The normal coordinate of point M is z_1, and the z_1 value must be small, which is too close to the point of contact. This state is just when two bodies are in contact, and P is not yet pressed. This is when two bodies are just coming into contact without any pressurization. Once we apply the load P, there will be a finite patch generated. Now, there is a single point contact. Once P is applied, then there will be a finite patch of contact generated.

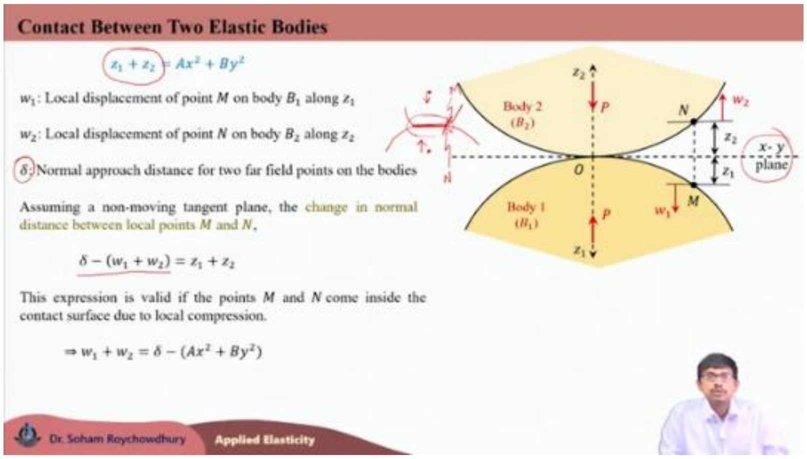
M is one point on body 1, which is initially outside the contact span, and once P is applied, point M will come inside the contact span, that would be our approach. We are choosing any point M on the body 1 surface closer to the point of contact, and the z_1 normal coordinate of that point is written as $A_1x^2 + A_2xy + A_3y^2$. This is kind of the approximate way of writing the z_1 coordinate with the help of two in-plane Cartesian variables x and y. We are considering it to be a second-order polynomial of the in-plane Cartesian variables x and y for point M on body B_1.

Similarly, we can choose another point N, which is on the same normal line. The line joining M and N must be orthogonal or perpendicular to the tangent plane xy. The normal coordinate of point N on the surface of body 2 is z_2, which is written as $B_1x^2 + B_2xy + B_3y^2$, where both these z_1 and z_2 are small. Those are the two points on body 1 and 2, which are much closer to the point of contact O.

Now, we can sum these two up: $z_1 + z_2$ will give us the total normal distance between these two points M and N, which would be $(A_1 + B_1)x^2 + (A_2 + B_2)xy + (A_3 + B_3)y^2$. x and y are the two in-plane directions. We can always choose the x, y axes along the principal directions so that this mixed term, xy term, A_2, B_2 term is set to 0. A_1, B_1, A_2, B_2, A_3, B_3 are some constants which depends on the curvature of the bodies coming in contact around point O. By choosing this x-y axis to be principal direction, it is possible to write this equation as $z_1 + z_2 = Ax^2 + By^2$, just two terms dependent with x^2 and y^2, where A and B depend on the principle curvature of the surfaces coming in contact at point O.

Now, this equation, if you carefully look, is basically equation of an ellipse. Hence, all such points M and N, with mutual normal distance $z_1 + z_2$, will form an ellipse. Thus, once P is applied, the bodies are pressed against each other and elliptical contact boundary would be generated.

One M, I have taken on body 1, and N on body 2. If you take series of such points, if we keep on taking such points, all those points on bodies 1 and 2 will generate an ellipse. If you look from top, all such M points on each of the body, and all such N points on another body will be forming an ellipse, whose equation is given like this. Once the bodies are pressed in the normal direction, M and N points will be going towards each other, mutual distance will be reducing, and thereby, an elliptical contact boundary would be generated.



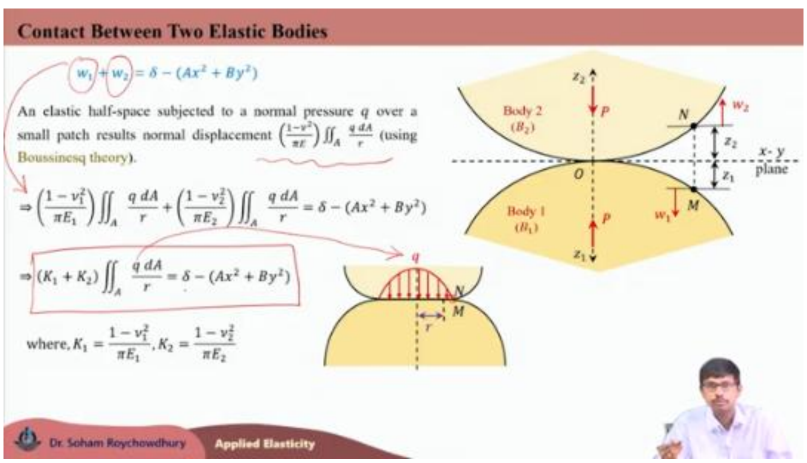
Moving forward, considering w_1 to be local displacement of point M, this is the local deformation once the load P is applied. The previous one was in the unloaded case, once just two point contacts are there. Now, once we are applying the load, there would be local deformation of points M and N, which we can obtain by using the elastic half space deformation theory. So, w_1 is considered to be local deformation of point M on body B_1 along the normal direction z_1. Similarly, w_2 is the local deformation of point N on body B_2 along the normal direction z_2. These are localized deformations which is coming due to the applied load P.

However, there will be some far field approach between these two bodies. As we are pressing two bodies with each other, there would be certain amount of contact patch generated because from this configuration, it will be going to a configuration like this, where this much amount of finite contact span is getting generated Hence, the centre of these two bodies will come closer to each other by, let us say, an amount δ.

δ is the normal approach distance for far field points of these two bodies, and w_1 and w_2 are the respective local deformations of points M and N on bodies 1 and 2 respectively. Hence, initial distance $z_1 + z_2$ for bodies 1 and 2 which was there before the generation of contact patch, that should be equal to this far field approach distance δ minus the summation of local deformation of bodies 1 and 2, *i.e.*, $w_1 + w_2$.

If we assume our tangent plane, this xy plane, to be datum, which is a non-moving one, that is remaining where it is. Top body is coming down, bottom body is going up due to application of this normal loads P along z_1 and z_2 directions. For that, we can write that the far field displacement, δ, minus the summation of local displacements, w_1 and w_2, predicted by elastic half space deformation theory should be equal to $z_1 + z_2$. This gives us the change in normal distance between these two points.

If this is true, then, these two points M and N are just coming within the contact patch due to local compression, as this load P is applied. This equation is true only when M and N points are coming closer to each other and going inside this flat contact boundary. Let us say this point is M, this point is N, now they are overlapping points, they are coming and setting on each other. From this we can write $w_1 + w_2 = \delta - (Ax^2 + By^2)$.

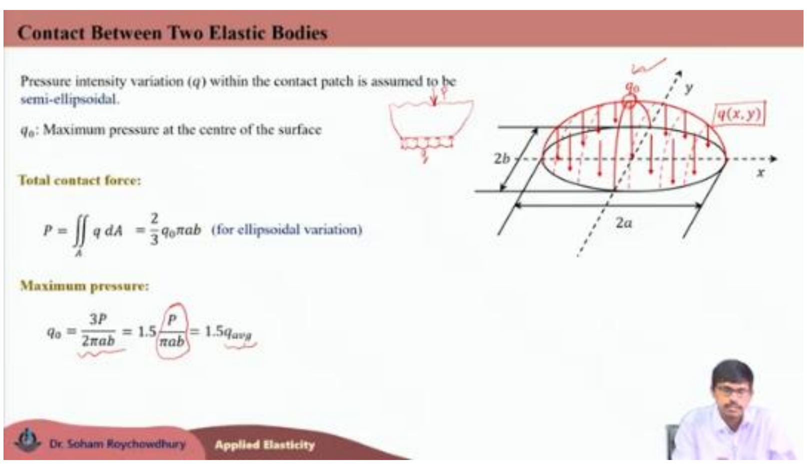


Now, for an elastic half space subjected to a normal pressure intensity, let us say q, over a small patch, for that, the normal displacement component, using the Boussinesq theory, can be obtained like this: $\left(\frac{1-\nu^2}{\pi E}\right)\iint_A \frac{qdA}{r}$. This is simply extension of the Flamant problem.

For the Flamant case, a single line load of intensity W was applied. Instead of applying the normal load at a point, If we apply the normal pressure distribution of intensity q over a small area or patch of contact, for such case, the Boussinesq theory predicts the deformation in a similar fashion as the Flamant theory in the normal direction as this: $\left(\frac{1-\nu^2}{\pi E}\right) \iint_A \frac{q dA}{r}$. We can write both w_1 and w_2 using this deformation predicted by the Boussinesq theory.

Taking E_1, E_2, ν_1, and ν_2 to be properties of bodies 1 and 2 respectively, this equation can be written in this fashion. Then, $\left(\frac{1-\nu_1^2}{\pi E_1}\right)$ and $\left(\frac{1-\nu_2^2}{\pi E_2}\right)$, these two terms can be taken as two new constants, K_1 and K_2, respectively. Hence, the equation becomes $(K_1 + K_2) \iint_A \frac{q dA}{r} = \delta - (Ax^2 + By^2)$, where q is the contact pressure profile or intensity generated within the contact patch.

Now, M and N are the two points that came into contact within the contact patch; they are coinciding points. And q is the distribution of the pressure profile within the contact patch, which can be related to δ, the approach distance, using this equation. And K_1, K_2 are two constants that we can obtain by using the material properties of the two bodies. If the two bodies are identical, K_1 and K_2 would be the same.

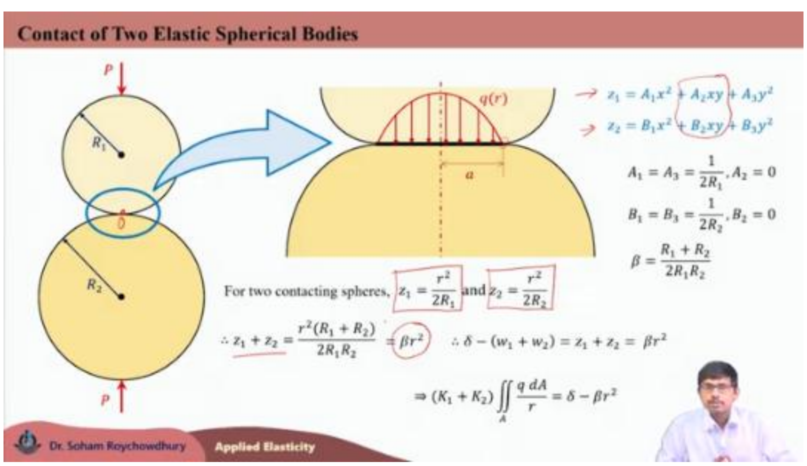


Moving forward, the pressure intensity variation q can be assumed to have a semi-ellipsoidal variation because, as we discussed earlier, the contact patch profile boundary is described by equation of an ellipse. Hence, the pressure distribution within that contact patch would be semi-ellipsoidal in nature.

Let us say, this is the contact boundary, with semi-major axis a, semi-minor axis b, and the contact pressure distribution would be something like this, which has a semi-ellipsoidal variation: $q(x, y)$ in 2D. And q_0 is the value of q at the tip, at the centre, which is the maximum pressure intensity at the centre of the contact surface. Using this semi-ellipsoidal variation, we can write the total contact force.

The total contact force applied is P, which is compressive in nature, and that results in this pressure distribution within the contact patch of the body. Hence, from the free-body diagram, if you consider one body, load P is applied downward, and here, this kind of contact pressure distribution q is generated. q integrated over this contact area should balance the net compressive load P applied on the body. So, $P = \iint_A qdA$, and for ellipsoidal variation of q, this integral can be obtained as $\frac{2}{3} q_0 \pi ab$, and this maximum pressure q_0 for this particular case can be then written in terms of P as $\frac{3P}{2\pi ab}$.

We know that the area of the ellipse is equal to πab. So, this $\frac{3P}{2\pi ab}$ can be written as $1.5P$ divided by the area of the ellipse. This, P by area of the ellipse, we can write as average contact pressure. If pressure is uniformly distributed over the elliptical area, then, that would be called as q_{avg} or average pressure. So, 1.5 times of the average contact pressure is generated at the centre of the contact surface, which we call q_0 or the maximum pressure generated at the centre of the elliptical contact patch once two elastic bodies are pressed.



Based on this theory, we are going to check the exact equation for the contact patch, the mutual approach distance, δ, and contact pressure intensity between the contact of two elastic spheres. The last one was a general one; it is valid for spherical or any other non-spherical general profile. Now, we are specifically focusing for the contact between two spherical bodies.

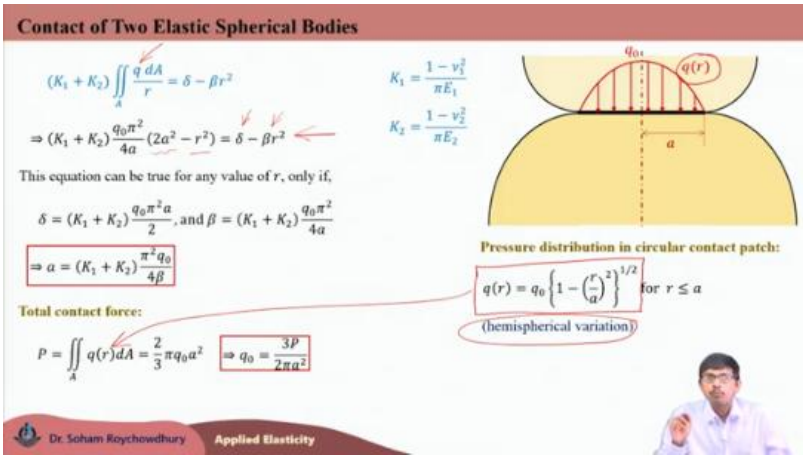
Two elastic spheres are considered with radii R_1 and R_2, which are pressed against each other in the normal direction by applying this compressive load P, where this point O is the point of contact. Let us look into the point of contact, that region, in a zoomed view, where a contact patch area is generated having radius a and $q(r)$ is the pressure profile distribution, which is varying radially over this contact patch.

Considering these equations of z_1 and z_2, which are distance of two points M and N in the normal direction lying on these two bodies B_1 and B_2, respectively. Considering the bodies to be sphere, we can get these values of A_2 and B_2 to be 0. These mixed terms are 0 because we are having the equal radius of curvature. Both the radius of curvature at any point for the spheres are equal, thus, all the directions are principle curvature directions. A_1, A_3 for sphere 1 would be $\frac{1}{2R_1}$, and for second sphere, B_1 and B_3 would be equal to $\frac{1}{2R_2}$. Hence using these equations of z_1, z_2 for the spheres, we can write, for the first sphere, $z_1 = \frac{r^2}{2R_1}$, and $z_2 = \frac{r^2}{2R_2}$.

These are the distances of those two points M and N, which are normally just above each other, and once P is applied, they will be coming within the contact patch. Hence, $z_1 + z_2$, we can write as some constant β times r^2, where β is given as $\frac{R_1+R_2}{2R_1R_2}$.

Substituting it back in the equation of that mutual approach distance, δ, which was $\delta - (w_1 + w_2) = z_1 + z_2$. This is the equation we had derived for the Hertzian contact stress theory, where w_1 and w_2 are the local deformation of two bodies around the point of contact. δ is the mutual approach distance for the far field points between two bodies that is equal to $z_1 + z_2$, and for the present problem of contact between two spherical bodies we had obtained that $z_1 + z_2 = \beta r^2$. $\delta - (w_1 + w_2)$ would be equal to βr^2, and we can write w_1 and w_2 in terms of this integral using the Boussinesq theory, which is

$(K_1 + K_2) \iint_A \frac{q dA}{r} = \delta - \beta r^2$. This is the governing equation for the present problem from which we need to find out the pressure distribution q and δ.



From this equation if we are writing the pressure distribution. If you recall the general case, the contact patch boundary was defined by an ellipse. As the two bodies coming in contact are spheres, for such cases, the contact patch boundary will be converted into a circular boundary from an elliptical boundary. Hence, the semi-ellipsoidal pressure variation will now be changed to hemispherical pressure variation within that circular boundary.

We can write the expression of the pressure distribution q, as a function of radial coordinate r using this equation, where $q(r) = q_0 \left\{1 - \left(\frac{r}{a}\right)^2\right\}^{\frac{1}{2}}$. This is valid within the contact patch, that is, for r varying between 0 to the contact patch radius a.

Substituting this hemispherical pressure variation as proposed by the Hertzian contact stress theory here in this equation, and then, evaluating this integral, we will get this particular equation, where $(K_1 + K_2) \frac{\pi^2 q_0}{4a} (2a^2 - r^2) = \delta - \beta r^2$, where K_1 and K_2 are two constants depending on the material parameters E_1, E_2, ν_1, and ν_2, *i.e.*, Young's moduli and Poisson's ratios of both the elastic bodies coming in contact.

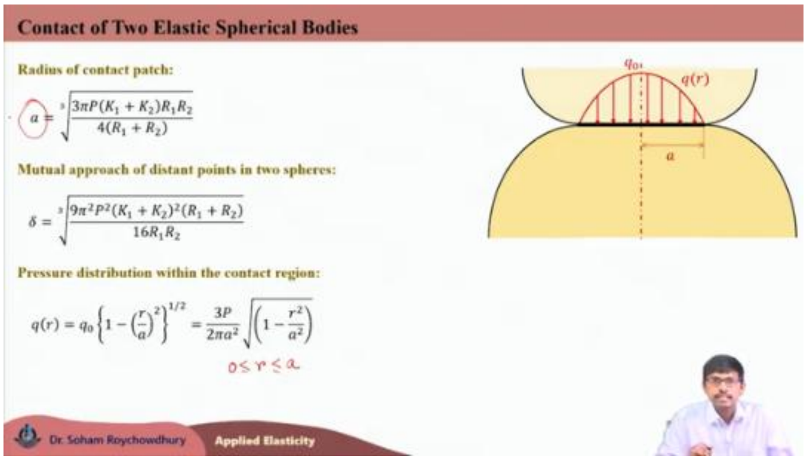
For this particular equation to be true for all values of r within the contact patch, if we compare both sides, you can see one term is constant on the right side, δ is a constant term, and βr^2 is an r-dependent term. Similarly, on the left-hand side, the first term $2a^2$ times the external constant is a constant term, and then this external term times r^2 is an r-dependent term.

So, for this equation to be valid for all values of r, both the constant term and the coefficient of r^2 must be the same for all cases. Hence, this can be true for all r only if the constant terms (independent terms) on both sides are equal, which would give $\delta = (K_1 + K_2)\frac{q_0\pi^2 a}{2}$. The coefficient of r^2 must also be the same on both sides, which would give $\beta = (K_1 + K_2)\frac{q_0\pi^2}{4a}$. Using this equation, we can relate δ with q_0 and a, and we can relate β with q_0 and a.

Moving forward, if we remove δ from this equation and relate a with β, then a would come out as $(K_1 + K_2)\frac{q_0\pi^2}{4\beta}$. Note that β was a geometric constant dependent on the radii of both the spheres, R_1 and R_2. K_1 and K_2 are related to the material constants; all these are known quantities. So, from this equation, we can relate the radius of the contact patch a with the maximum pressure intensity q_0 generated at the centre of this circular contact boundary.

The total contact force P can be related to the contact pressure distribution $q(r)$ like this: P equals the area integral of the contact pressure. This comes from the force balance of any one body in the normal direction. So, $P = \iint_A q(r)dA$, and for $q(r)$ having hemispherical pressure variation like this, if you substitute this equation here, you would get P to be $\frac{2}{3}\pi q_0 a^2$. This gives us the total contact force, using which, q_0 and P can be related as q_0, the maximum pressure intensity generated, equals 3 times applied normal load P divided by $2\pi a^2$, where a is the contact patch radius.

Note that contact patch radius is unknown, that should be our output. And P is the known; we are applying the load, which is known. We know the geometry R_1 and R_2 for both the spheres, which means β is known, and K_1, K_2, the elastic properties, are also known for both the spheres. These are the known quantities. Our objective is to find out pressure distribution $q(r)$, that is, q_0 is unknown. We also want to find out a. Using these two equations, the first equation relates a with q_0, and second equation relates q_0 with P. So, we can either remove a from this equation or remove q_0 from this equation. Using that, we can relate a, the contact patch generated, with the applied load P. We can also relate q_0 with P, which would be independent of a.

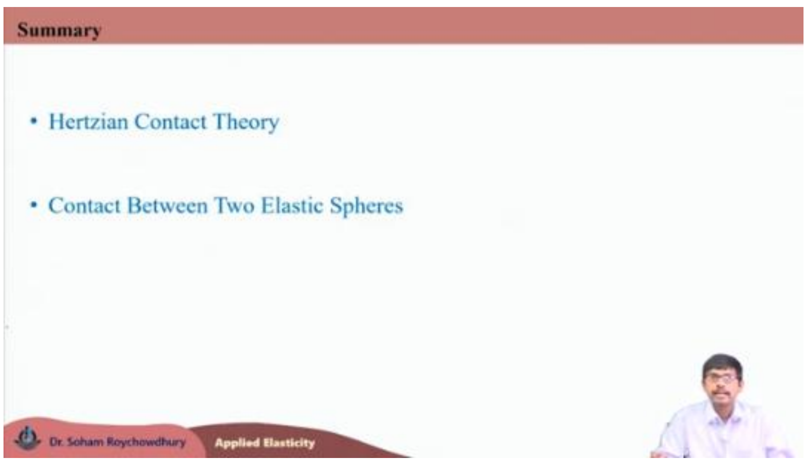


If we do so, the radius of the contact patch, a, in terms of applied load P, the geometry, *i.e.*, radii of two spheres R_1, R_2, and material properties defined by these two constants, K_1 and K_2, can be written like this: $a = \sqrt[3]{\frac{3\pi P(K_1+K_2)R_1R_2}{4(R_1+R_2)}}$. This would be the radius of contact patch generated, once the two elastic spheres are coming in contact due to the compressive load P applied along the normal direction. This is the contact patch predicted by the Hertzian contact stress theory.

Coming to δ, we can obtain the value of δ by using the equation $\delta - \beta r^2 = w_1 + w_2$. Substituting the expression of a and q_0 there, the mutual approach distance δ can be obtained like this. The radii of the two spheres coming in contact would approach each other. The centers of the two spheres and any far-field point of the two spheres would be coming towards each other by this much mutual approach distance δ.

Finally, the pressure profile distribution within the contact span $q(r)$, that is a hemispherical pressure distribution, which is $q_0\sqrt{1 - \left(\frac{r}{a}\right)^2}$. We had related q_0 with the applied total force P, which is $P = \iint_A \; q dA$. From that, q_0, the maximum intensity of the contact pressure at the centre of the contact patch, can be written as this: $\frac{3P}{2\pi a^2}$. Hence, $q(r)$, the contact pressure distribution, is written as $\frac{3P}{2\pi a^2}\sqrt{1-\left(\frac{r}{a}\right)^2}$. This is valid only within the contact span defined for the range of r from 0 to a.

This gives us the complete solution of this contact problem between two elastic spheres of radii R_1 and R_2, where we got the contact patch radius, which gives us the area of the contact patch, then, the mutual approach distance of far-field points of the two spheres, and the pressure distribution generated within the contact patch, which is $q(r)$.



In this lecture, we discussed the formulation of the Hertzian contact theory. Using that, we obtained the solution of the contact problem of two elastic spheres pressed against each other due to the application of compressive normal loading.

Thank you.