APPLIED ELASTICITY

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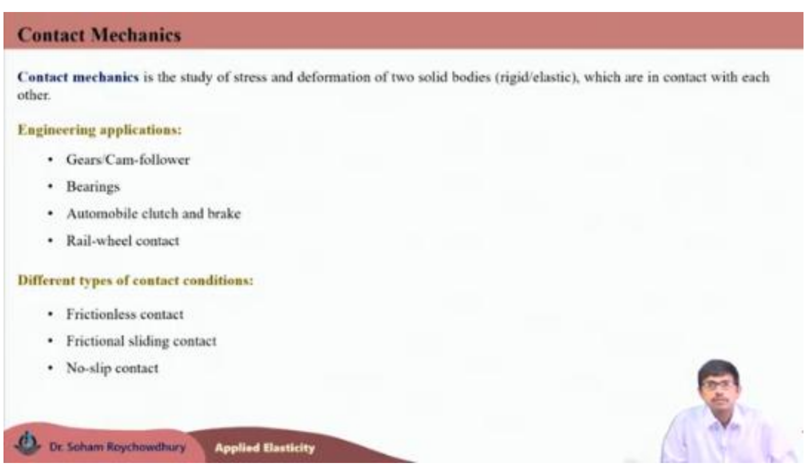
Week 12

Lecture 57: Contact Problems I

Welcome back to the course on Applied Elasticity. In today's lecture, we are going to talk about the solution of the contact problems in elasticity.



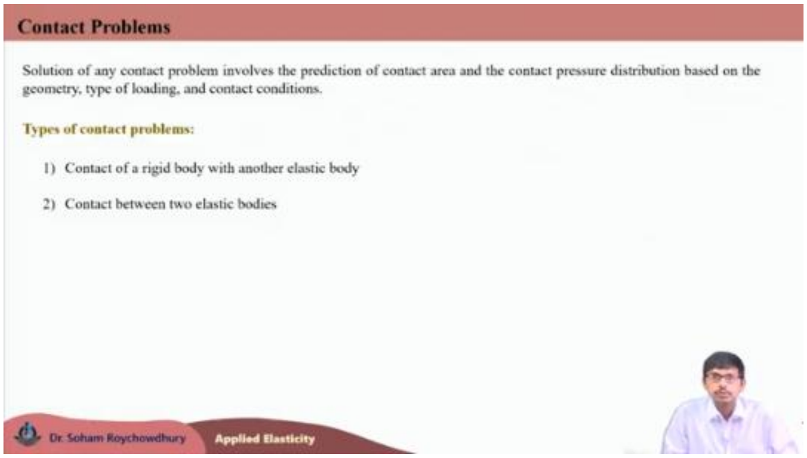
Contact mechanics is the study in which we are interested to find out the stress and deformation of two bodies which are coming in contact with each other. If there are two bodies which may either be rigid or elastic, at least one body is required to be elastic, but both may also be elastic. For such cases, if two bodies are coming into contact at a single point, along a line, or along a surface, for all those cases near the contact zone, there will be local deformation of the two bodies which are coming in contact, and hence, there will be a stress distribution generated around the contact patch. The study of these stresses and the local deformation around the contact region is called contact mechanics.

In various mechanical engineering applications, different elements are coming into contact, such as gear teeth. One gear tooth is mating with another gear tooth; there will be a line or a point contact between those two gear teeth which is used for transferring motion. To design the strength of the gear tooth, we need to analyze the contact stress developed between a pair of gear teeth once they are in contact.

Similarly, for the cam-follower mechanisms, cams are coming into contact with the follower. Also, in bearings, the bearing and the journal are coming in contact. In automobile, clutches and brakes, the braking operation between the brake disc and the rotor involves contact when we are applying the brake. For the design of the braking torque, it is important to understand the contact mechanics for the brakes or the automobile clutches.

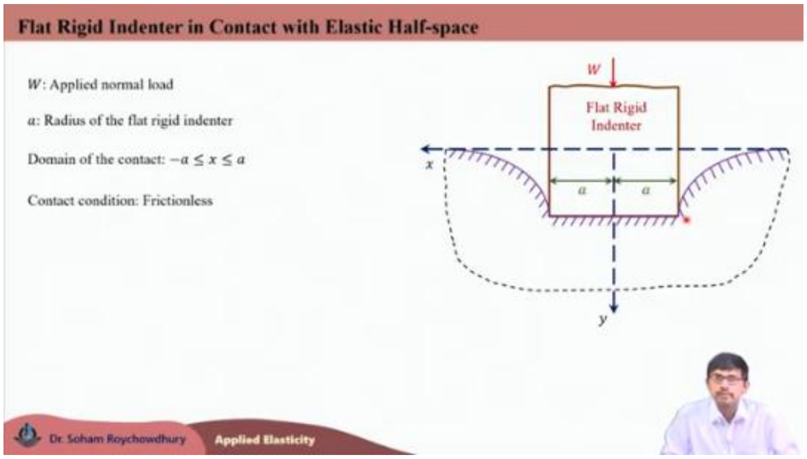
Similarly, in the rail-wheel system, we have contact between the rail and the wheel. To study the contact force, the stress generated, and the rate of wear at the contact patch, for all these purposes, the study of contact mechanics is important, which is involved in various mechanical applications as mentioned here.

There may be different types of contact conditions which we can assume between the two bodies coming in contact. First is the frictionless contact. We may not assume friction to exist at the contact surface between two bodies; that is called frictionless contact. Then we may consider friction to exist. Due to friction, there may be sliding with the presence of a non-zero friction coefficient. That is frictional sliding contact, where within the contact region, two bodies may slide, and the slipping velocity will depend on the friction coefficient between those two surfaces. There may also be no-slip contact; the friction coefficient is sufficiently high to avoid any kind of sliding in the contact zone. Based on the nature or value of the friction force, we may have different types of contact conditions between the two bodies coming in contact.



Moving forward, the solution of any contact mechanics problem involves predicting the contact area, the amount of contact patch generated near the point or line of contact, and then there will be the generation of contact pressure. What kind of contact pressure distribution is obtained once two bodies come in contact? These are predicted by solving a contact mechanics problem. Depending on the nature of the geometry, the type of bodies coming in contact, their geometry, the loading conditions, and the contact condition - whether it is frictionless, no-slip, or frictional contact - based on these parameters, the result of the contact problem - the amount of contact area and the contact pressure distribution - will be influenced.

Now, we will discuss two different contact problems. In the first case, the contact is between a rigid body, a rigid indenter, and another elastic body. Out of the two bodies, the rigid one is not allowed to deform; only the elastic body is allowed to deform. In the second case, we will consider where both bodies coming in contact are elastic. So, with the contact near the contact zone, both the bodies will be having local deformation.

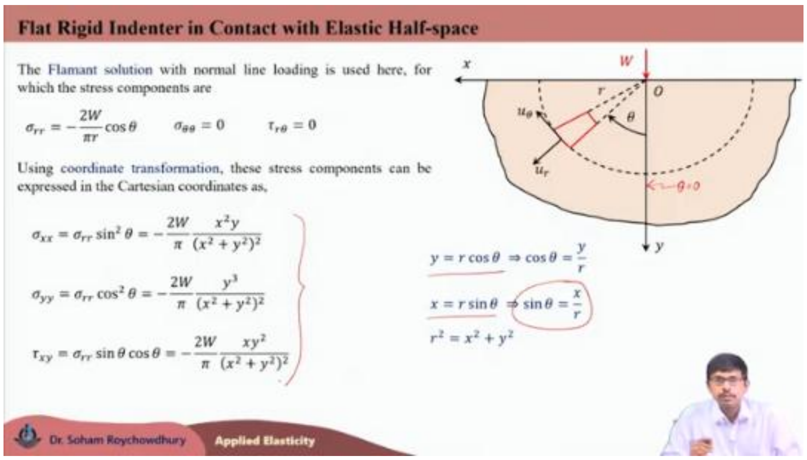


In today's lecture we are going to talk about the first case, that is, contact of a rigid body with an elastic body. Here, we are considering one flat rigid indenter. We are considering one indenter which is rigid means it cannot deform once it is loaded. This is coming in contact with an elastic half-space. The elastic half-space can deform as the indenter is being pressed against it. Let us consider this elastic half-space where x-axis defines the top free surface of the half space and y-axis is going vertically downward within the half-space.

One flat rigid indenter, flat means the face of the indenter which would be coming in contact with the half-space is flat; it is parallel to the free surface of this elastic half-space. That indenter is being vertically pressed on the elastic substrate such that there will be a finite patch of contact area generated. We are considering this flat rigid indenter which is now pressed within the elastic half-space and thus the elastic half-space is getting deformed like this. This is the deformed profile of the elastic half-space. Earlier, it was a flat elastic half-space; free surface was flat. As this rigid indenter is pressed against the elastic half-space, the local deformation around the contact region is achieved and the free surface profile looks like this.

We are considering the applied load to be W, which is applied normal to the free surface of the elastic half-space, which is basically along the positive y-axis. Then, the radius of this flat rigid indenter of circular cross section is taken to be a. Total contact patch diameter becomes $2a$. So, a circular contact area of radius a or diameter $2a$ is generated as we press this flat rigid indenter on the elastic half space with a normal load W.

We are assuming that domain of the contact by x varying between $-a$ to $+a$. This point refers to $x = -a$ whereas, this point refers to $x = +a$. As we are varying x from $-a$ to $+a$, the entire contact patch is defined in the rectangular Cartesian coordinate system, and the contact condition is assumed to be frictionless. There is no friction between the rigid indenter and the elastic half-space. With this assumption, we will try to solve this contact mechanics problem between one rigid body and an elastic half-space.



Moving forward, we will just recall about the Flamant solution which we had discussed earlier. Flamant solution is the solution of an elastic half-space problem, where the elastic half-space is subjected to normal line loading of, let us say, some intensity W. If we are having this elastic half-space, where the normal line load of intensity W in the vertically downward direction is applied, following the St. Venant's principle, the effect of W will be localized around the point of application of the force that is at point O. So, we are considering the semicircular region within which the stress distribution coming due to W would be valid for this elastic half-space, and beyond this, we are considering it to be far field, where the stresses will not get affected due to application of this load W.

If we solve this Flamant problem in the polar coordinates, we can obtain the stress components σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$. We can also obtain the displacement components u_r and u_θ that we had solved earlier. For the solution of this Flamant problem, we had obtained one purely radial stress distribution, where only σ_{rr} is non-zero and other two stresses $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0. That is a purely radial stress distribution, where σ_{rr} is obtained as $-\frac{2W\cos\theta}{\pi r}$, and rest two stress components are 0.

Using this Flamant solution, we will try to extend this for solving the present contact problem. This W would be the load coming on the elastic half-space from the flat rigid indenter, which is pressed against the elastic half-space in the normal direction. But this Flamant problem solution was considering a line loading which was acting only at a single point on the elastic half-space, that is point O. However, in the present problem, we have a finite patch of contact, where x varies between $-a$ to $+a$. For that patch of contact, we need to find out the stress generated and the deformation due to it. We will try to obtain that by extending this Flamant solution.

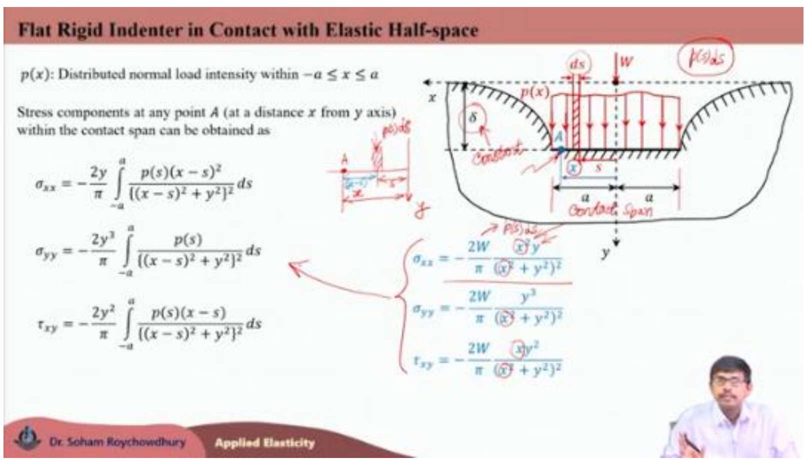
Flamant solution was obtained in polar coordinates. Here, for solving the present problem of contact between a flat rigid indenter and an elastic half-space, we will use the rectangular Cartesian coordinate system with x and y, x being oriented along the free surface of the elastic half-space and y going vertically downward within the elastic half-space, with this choice of x and y, and θ being measured from the y-axis. The y-axis is the line of $\theta = 0$.

For that, we can relate r, θ, x, and y using these two equations: $y = r\cos\theta$, $x = r\sin\theta$, for this choice of $\theta = 0$ along the y-axis. So, $\cos\theta$ is $\frac{y}{r}$, $\sin\theta$ is $\frac{x}{r}$, where r is $\sqrt{x^2 + y^2}$. Using these coordinate transformations, we can rewrite the stresses along the x and y directions.

Two normal stresses, σ_{xx} and σ_{yy}, and shear stress τ_{xy} can be obtained, where the radial component of stress σ_{rr} is known, which is $-\frac{2W\cos\theta}{\pi r}$, and the rest two, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, are 0. As $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0, those cannot have any contribution to σ_{xx}, σ_{yy}, and τ_{xy}. σ_{xx} in the stress transformation formula will have only one non-zero term, that is $\sigma_{rr}\sin^2\theta$. Replacing $\sin\theta$ as $\frac{x}{r}$, we can write σ_{xx}, the normal stress generated along x or the tangential direction. x is the tangential direction to the free surface, so, σ_{xx} is the tangential normal stress. That is coming out to be $-\frac{2W}{\pi}\frac{x^2 y}{(x^2+y^2)^2}$.

Similarly, transforming to σ_{yy}, it can be written as $\sigma_{rr}\cos^2\theta$, and replacing $\cos\theta$ with $\frac{y}{r}$, σ_{yy} would be $-\frac{2W}{\pi}\frac{y^3}{(x^2+y^2)^2}$. Lastly, τ_{xy}, the shear stress in the Cartesian coordinate can be expressed as $\sigma_{rr}\sin\theta\cos\theta$. Replacing $\sin\theta$ and $\cos\theta$ in terms of x and y, this would be $-\frac{2W}{\pi}\frac{xy^2}{(x^2+y^2)^2}$.

Here, we have obtained the stress generated in a Flamant problem in the rectangular Cartesian coordinate system, where W is the line loading acting at a single point of the free surface on an elastic half-space. Now, we will try to extend these solutions for finding the solution of the flat rigid indenter with the elastic half-space, where instead of a single point contact, a finite area contact is coming, which is defined by x varying between $-a$ to $+a$.



Once again, considering that same figure, here the deformed free surface profile is shown. Let us say as the indenter is pressed with some load W, δ is the amount of deformation for the free surface within the contact region. This portion refers to the contact region; this is the contact span.

As the flat indenter is rigid and flat, so, the face which is coming in contact with the elastic half-space of the indenter is parallel to the elastic half-space free surface. Thus, this δ would be a constant. So, in the y-direction, the deformation of the free surface is constant over the entire contact span for the present case of contact between a flat rigid indenter and the free surface of an elastic half-space.

Due to this contact, there would be a pressure profile generated on the contact span or contact patch over the elastic half-space. Let us consider $p(x)$ to be the distributed normal load intensity, which is valid between the contact span from x varying between

$-a$ to $+a$. This is the pressure distribution generated within the contact span of the elastic half-space as the rigid indenter is pressed. Note that the contact span is flat and the indenter is rigid; there is no deformation of the indenter.

Now, considering the stress components, at point A. We are choosing one point A, this point, which is the point of our interest. We are interested in finding out the stresses or displacements at any point A within the contact span. Point A is defined by its position vector $\tilde{x}$, which is located at a distance x from the y-axis. From the y-axis at a distance x, we are choosing a point A, and due to this contact pressure distribution $p(x)$, we are interested in finding out what is the deformation and what are the stress components generated at point A.

For that, first we will take a small patch of length ds. As shown here, we are considering a small patch of length ds at a distance s from the y-axis. On the small patch, $p(s)ds$ amount of line load intensity is acting, and first we are going to consider the deformation of point A due to the pressure acting on the small patch of length ds, that is net load intensity of $p(s)ds$, which is acting at a distance s from the y-axis.

The point of interest A is at a distance x from the y-axis. If we are able to find out the deflection at point A due to this small line load intensity $p(s)ds$, then we can integrate it over the entire span, and that would give us the net displacement, or net stresses generated at point A due to the entire pressure distribution for the total contact span.

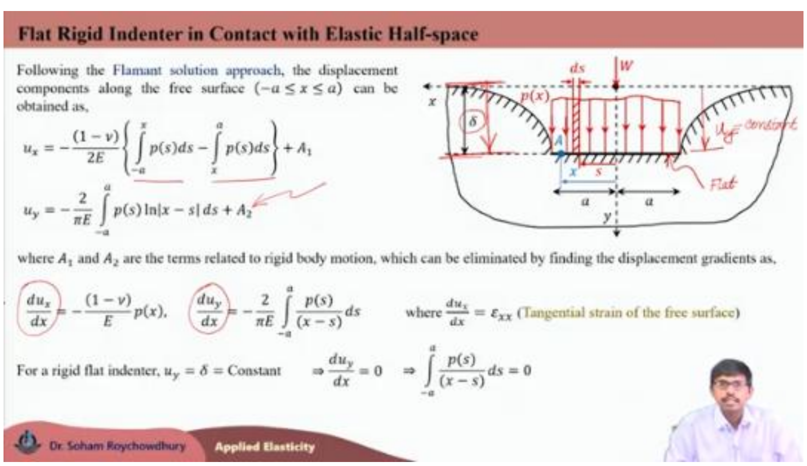
For a single line loading, we know from the Flamant solution the stress distribution in the rectangular Cartesian coordinate system is obtained like this. For our case, this $W = p(s)ds$. A small load is acting on the patch ds, which is $p(s)ds$, that is load intensity (load per unit length). This σ_{xx}, σ_{yy}, and τ_{xy} would first be obtained for the small patch of load, and then we will integrate this for the entire contact span, where ds would vary from $-a$ to $+a$.

Hence, for the total contact span, the pressure distribution σ_{xx} at any point A can be written like this. Here, in this set of equations, these x and y, y is the y coordinate. As the contact span is flat, the y-coordinate of all the points is the same. However, this x-

coordinate, x, is the distance between the point of interest and the point on which the load is applied. Here, the load is applied on a point at a distance s from the y-axis. The load is applied here, which is at a distance s from the y-axis, and the point at which we are obtaining this distribution is at a distance x from the y-axis. Hence, the mutual distance between the point of interest and the point of loading s is $x - s$. This is the quantity that you should replace here as x.

In all these x quantities, we should write this as $x - s$, which is the distance between the point of interest and the point of loading. With that, integrating it over the entire contact span, we can get σ_{xx} as: $-\frac{2y}{\pi}\int_{-a}^{+a}\frac{p(s)(x-s)^2}{[(x-s)^2+y^2]^2}ds$. Using this σ_{xx} expression, we have simply done that, y being the same over the entire contact span, that is treated as a constant and taken out, and W for the small patch is $p(s)ds$. Then we are integrating it over the entire span from $-a$ to $+a$.

In a similar fashion, σ_{yy} and τ_{xy}, that is normal stress along the y-direction and shear stress in the xy plane, can be obtained like this, using this set of equations. These give us the total stress distribution for all three components in the Cartesian coordinates for the present problem due to contact over this patch of radius a with a pressure distribution of $p(x)$.



Moving forward, following Flamant solution approach, we can also obtain the displacement after the stresses. Following a similar method, similar to the stress case as we had discussed, extending the deformation of the Flamant problem, which was for a single point loading of load intensity W, to this pressure distribution intensity $p(x)$, we can obtain the x and y components of the displacements like this. u_x, that is, the displacement along x direction, tangential displacement, is $-\frac{(1-\nu)}{2E}\left\{\int_{-a}^{s} p(s)ds - \int_{s}^{a} p(s)ds\right\} + A_1$, where A_1 is integration constant. Then, u_y, the normal displacement, would be $-\frac{2}{\pi E}\int_{-a}^{a} p(s)\ln|x-s|\, ds + A_2$.

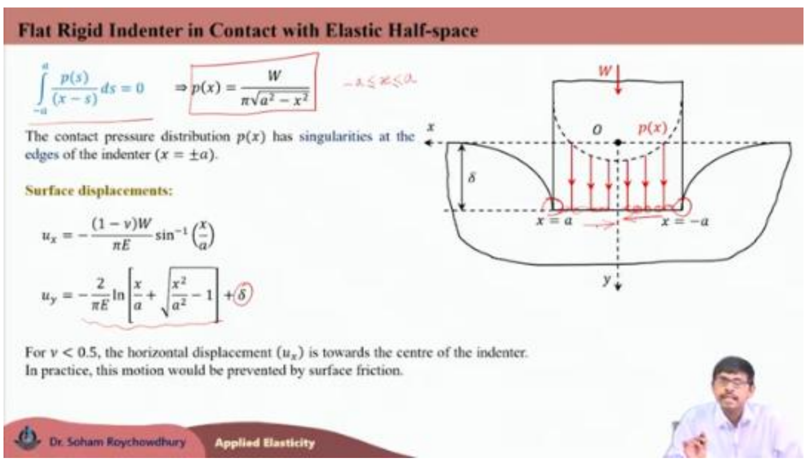
In the first term of u_x, you can see this integral is same, only limits are different and sign is different. As sign is different, we cannot write the total integral for $-a$ to $+a$, for that case a signum function is required to be used. It is written by breaking the integral into two parts: for one side it is positive for another side it is negative. This gives us the displacement components along x and y directions for any particular point within the contact patch.

Now, A_1, A_2 are two integration constants which can be determined by using the rigid body motion. For example, here the entire rigid indenter is just having a motion along y direction. Initially indenter was here at top, and along the free surface this is pressed till this point by an amount δ. δ is the rigid body motion of the indenter along the y-axis. This constant A_2 can be obtained by this information. Similarly, if the body is having any motion along x-axis, that would give us the non-zero value of A_1. In this case, that motion we are not considering. We are only pressing the indenter normal to the substrate or the elastic half space, and that will only give rise to a non-zero value of A_2.

To avoid these A_1 and A_2 in the displacement components, instead of writing u_x and u_y, we can write the displacement gradients, *i.e.*, the rate of change of u_x and u_y over the contact patch, which will be $\frac{du_x}{dx}$ and $\frac{du_y}{dx}$. Simply by taking the derivative of these two equations of u_x and u_y, we can get the displacement gradients along the x direction, $\frac{du_x}{dx}$, as $-\frac{(1-\nu)}{E}p(x)$, and along the y direction, $\frac{du_y}{dx}$, as $-\frac{2}{\pi E}\int_{-a}^{a}\frac{p(s)}{(x-s)}ds$.

Note that for the present problem, the contact patch is flat. u_y refers to the y-directional displacement for the free surface, from the undeformed to deformed configuration, the difference between their coordinates along the y-axis. This contact patch being flat in both the undeformed and deformed states, u_y must be a constant. All the points within the contact patch are moving down by the same value; thus, this quantity $\frac{du_y}{dx}$ must be 0.

And this $\frac{du_x}{dx}$ is nothing but ε_{xx}, that is, the tangential strain of the free surface. Whereas, due to rigid body displacement, δ being constant, u_y must be constant over the entire contact patch, and that would result in $\frac{du_y}{dx}$ being 0. Thus, this particular integral $\int_{-a}^{a} \frac{p(s)}{(x-s)} ds$ must be 0 for the present problem of a flat rigid indenter coming in contact with the elastic half-space.



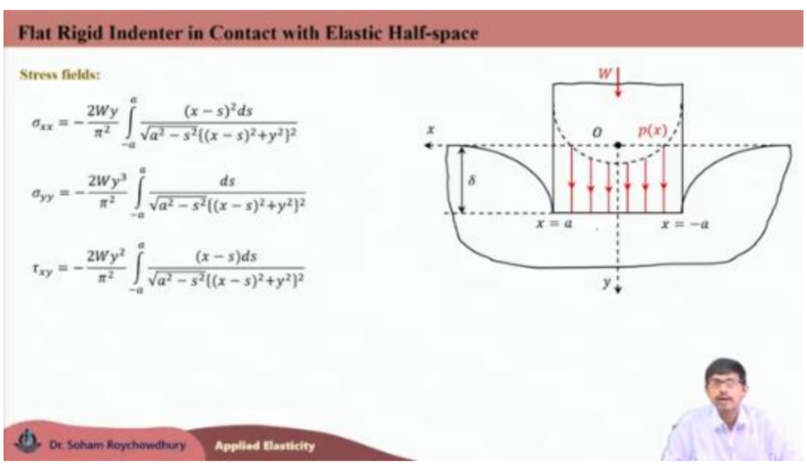
This integral equals to 0 gives us the value or the profile of the pressure distribution within the elastic half-space in the contact span. From this integral being 0, we can get $p(x)$, the pressure distribution, to be $\frac{W}{\pi\sqrt{a^2-x^2}}$. So, due to the application of this normal load W, this kind of pressure profile distribution would be generated, which is valid between $-a$ to $+a$, *i.e.*, the contact span.

If you plot this pressure profile, it would look like this. This is minimum at $x = 0$, and as you go towards the end, the value increases and finally, this has singularities, a very large value of stresses, at $x = \pm a$. These two points are the points of singularity for this particular pressure profile.

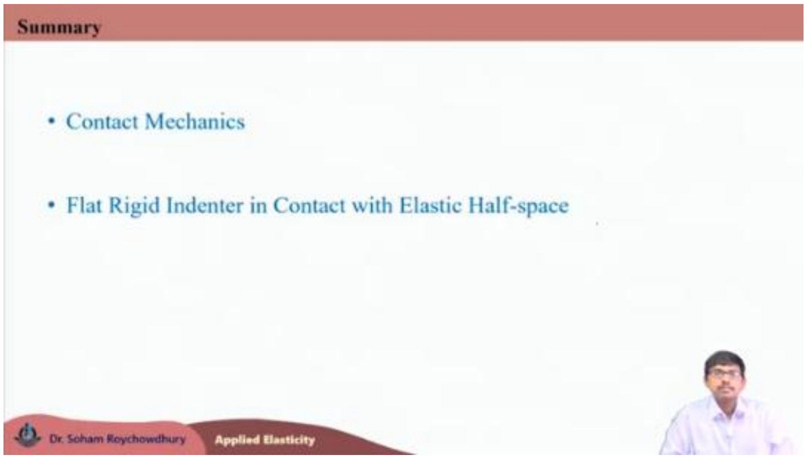
Substituting this expression of $p(x)$ as $\frac{W}{\pi\sqrt{a^2-x^2}}$ in the surface displacement components u_x and u_y, we can obtain the final equation of u_x as $-\frac{(1-\nu)W}{\pi E}\sin^{-1}\left(\frac{x}{a}\right)$, and u_y to be $-\frac{2}{\pi E}\ln\left[\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right]+\delta$. δ is the rigid body displacement, which is coming due to the movement of the rigid indenter. This additional term is the elastic displacement occurring due to the elastic nature of the half space.

Now, for $\nu > 0.5$, which is the common range or restriction on the value of the Poisson's ratio, this horizontal displacement u_x, the free surface tangential displacement, is always found to be towards the centre of the indenter. All these points will try to move towards the centre of the indenter. u_x will give us such a value that all the points on the contact will move towards the centre of the indenter.

Now, if the friction coefficient value is large, which is the case for practical problems, here we had considered it to be frictionless. If friction is present, obviously, some value of friction exists in all real-life problems, that will prevent this kind of tangential motion towards the centre for the free surface points coming in contact.



Putting this expression of $p(x)$, the contact pressure distribution, in the stress equations, we can obtain the final form of stresses σ_{xx}, σ_{yy}, and τ_{xy} like this. This is the final stress distribution generated for this contact problem, where a flat rigid indenter is coming in contact with an elastic half-space, and the contact condition is assumed to be frictionless.



In this lecture, we discussed the basic contact mechanics formulation and solved one problem of contact between a flat rigid indenter and the elastic half-space.

Thank you.