APPLIED ELASTICITY

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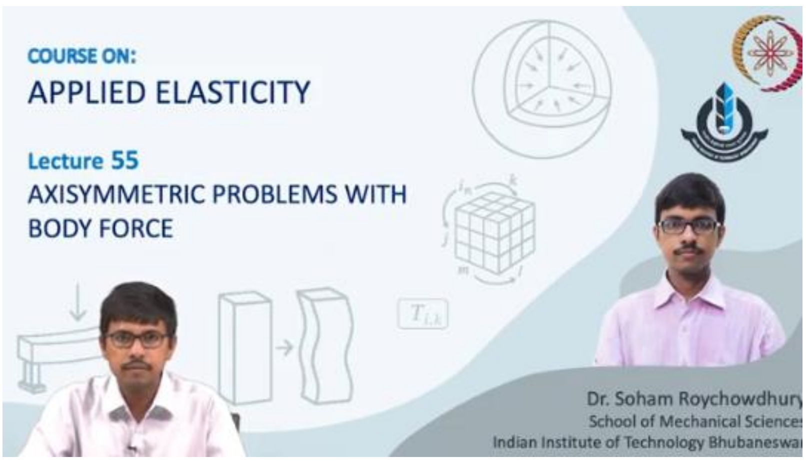
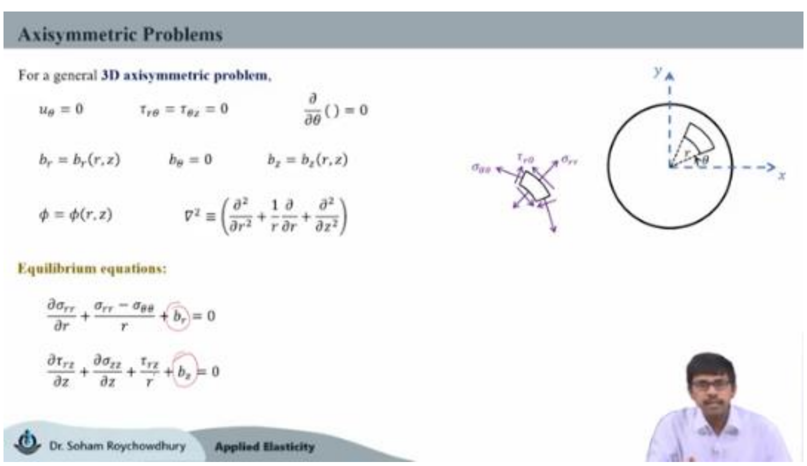
WEEK: 11
Lecture- 55



Welcome back to the course of applied elasticity. The topic of today's lecture is axisymmetric problems with body forces. So in In previous lectures, we have considered different types of axisymmetric problems. But in none of those were the effects of body forces considered.



In today's lecture, we are going to see the effect of the body forces if you are trying to solve any axisymmetric problem.

Starting with a quick recap of a general three-dimensional axisymmetric problem, if you are considering a problem to be axisymmetric, the geometry of the problem must have one axis of symmetry. So, in the figure, the z-axis is the axis of symmetry, and we are considering this circular body, which has z as its axis of symmetry.

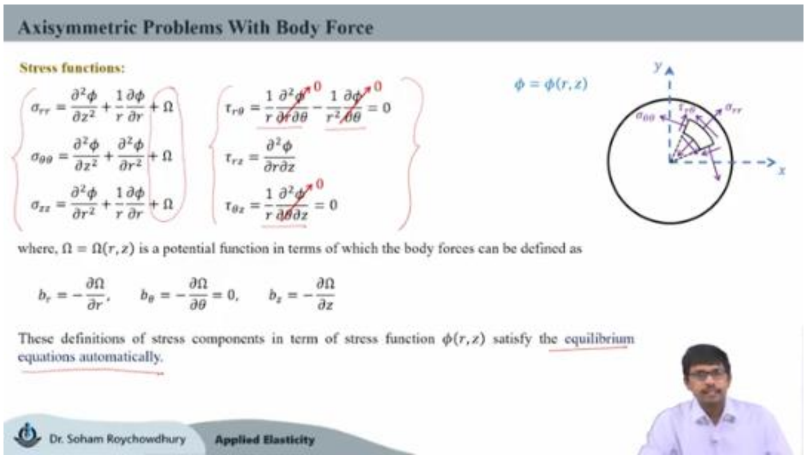
The loading, boundary conditions, and material properties all should also be symmetric about the same axis of symmetry; only then can the problem be called an axisymmetric problem. So, if you are considering one small element in the polar coordinate which is subjected to σ_{rr},$\sigma_{\theta\theta}$ (these two normal stresses), and $\tau_{r\theta}$ (the shear stress). For that, if you try to use the assumptions of axisymmetric problems, we will have $\frac{\partial}{\partial\theta}(\,) = 0$ of any quantity to be 0. That is, none of the quantities vary over θ. All the field variables are independent of the angular coordinate θ for axisymmetric problems. Also, both shear strains involving theta in the subscript, So, $\varepsilon_{r\theta}$ and $\varepsilon_{\theta z}$ both should be 0, which results in the corresponding shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ also being 0. To ensure the axis symmetry of the deformed shape, the u_θ displacement along the θ direction is also forced to be 0. Coming to the body forces, the body force in the θ direction must be 0.

Body forces along the radial (r) and axial (z) directions may be non-zero. However, b_r and b_z can only be functions of the radial coordinate r and axial coordinate z. They should be independent of θ. Then, only the body forces can be called the axisymmetric body force. For such cases, the stress function ϕ is chosen as a function of r and z only for any general axisymmetric problem, and this stress function ϕ must satisfy the bi-harmonic condition. Now, for the axisymmetric problem, the Laplacian operator is reduced to this form: $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$. This is the Laplacian operator.

So, this operator will act twice over ϕ, and the biharmonic of ϕ should be 0. So, here note that the Laplacian operator is independent of the partial derivatives with respect to θ for axisymmetric problems. Now, the equilibrium equations in general, we have three equilibrium equations, but for an axisymmetric problem, the equilibrium equation in the θ direction is automatically satisfied.

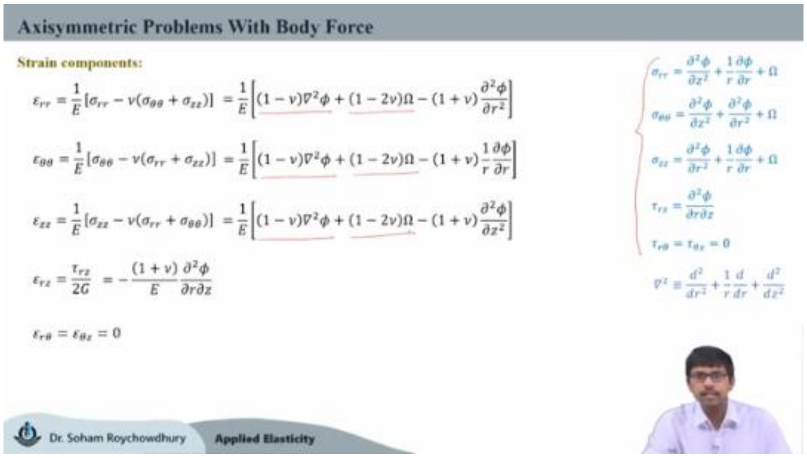
So, we are only left with the two equilibrium equations along the r and z directions, which are given here. $\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = 0$. And another one is $\frac{\partial\tau_{rz}}{\partial z} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + b_z = 0$. So, these are the two equilibrium equations for any general axisymmetric $3D$ problem. which include the body force terms b_r and b_z.

Earlier, whatever axisymmetric problems we had solved, whether it was the thick cylinder problem, the compound cylinder problem, or some other stress concentration problems, they had partial solutions which were axisymmetric in nature. So, for all such cases, the body force terms were dropped. Here, we are going to see what the effect of the body force would be on the solution of the axisymmetric problem in this particular lecture.



So, moving forward to the axisymmetric problem with body forces, the stress function ϕ is chosen to be a function of r and z in terms of which the stress components σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, these three normal stress components are written in terms of ϕ as this and the shear stress components $\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$ are written like this in terms of a stress function ϕ, so that all these, when substituted in the equilibrium equation, would satisfy the equilibrium equation directly. That was the aim; that is the objective. Now, ϕ being a function of r and z, and independent of θ, $\frac{\partial\phi}{\partial\theta}$ terms. So, whenever we have this $\frac{\partial}{\partial\theta}$ derivative acting on ϕ, all those terms will go to 0. So, hence we have two of the stress components $\tau_{r\theta}$ and $\tau_{\theta z}$ to be 0 and the remaining four stress components are present. Now, you can see in the normal stress components we have this term of capital omega, which is called a potential function. So, capital Ω is a potential function which is a function of r and z only which is related to the body forces in the following form. b_r, the body force along the radial direction per unit volume, is defined as $-\frac{\partial\Omega}{\partial r}$.

And b_z, the body force along the axial direction z per unit volume, is defined as $-\frac{\partial\Omega}{\partial z}$. These are the only two non-zero body forces for the axisymmetric problems. Whereas, b_θ, the body force along the angular direction, must be zero to maintain axisymmetry. So, if you are writing b_r and b_z in terms of capital omega like this, and then all these stress components, non-zero stress components, if we substitute in the two equilibrium equations shown in the previous slide, those two equilibrium equations would be satisfied automatically. Thus, we choose our stress components in terms of this axisymmetric stress function phi r comma z in these particular forms.



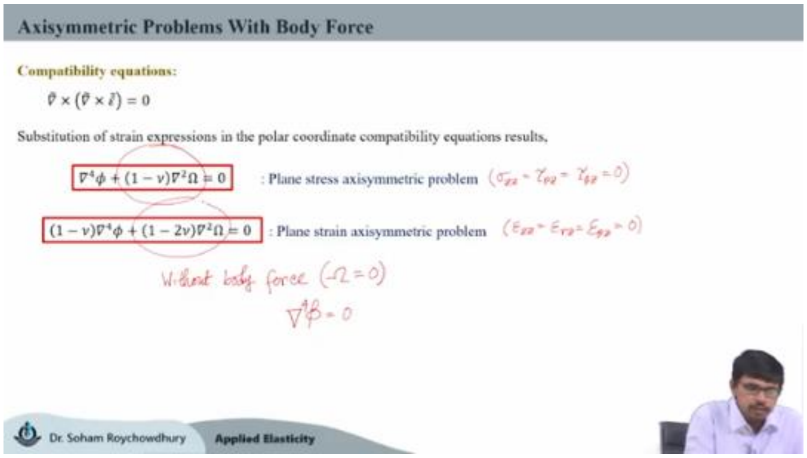
Now moving forward, these are the choices of the stresses in terms of the stress function ϕ and the potential function Ω, which takes care of the body force component. Now, from these stress components, using the constitutive equation, we are going to write the strain components.

So, the three normal strains— ε_{rr}, $\varepsilon_{\theta\theta}$, and ε_{zz}—are first written in terms of the normal stress components σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz}. So, ε_{rr} would be $\frac{1}{E}[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})]$. Then, $\varepsilon_{\theta\theta}$, the normal strain in the θ direction, would be $\frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})]$. And ε_{zz}, the axial normal strain, is $\frac{1}{E}[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})]$. So, these are obtained by using the constitutive equations for the linear elastic, homogeneous, isotropic solids involving the two material constants: E (Young's modulus) and ν (Poisson's ratio). Now, since two of the shear stresses are zero, the corresponding shear strains, $\varepsilon_{r\theta}$ and $\varepsilon_{\theta z}$, are also zero. The only non-zero shear strain, ε_{rz}, can be written as gamma r z divided by 2, which is $\frac{\tau_{rz}}{2G}$, where G is the shear modulus (modulus of rigidity).

Now, substituting all these stress components—σ_{rr}, $\sigma_{\theta\theta}$,σ_{zz}, and τ_{rz}, in terms of the stress function ϕ and the potential function Ω, we can write the strain components, all the non-zero strain components, like this. So, you can see that all three normal strains have one term in common: $(1-\nu)\nabla^2\phi$. This term, $(1-\nu)\nabla^2\phi$, is common for all three normal strains. Additionally, $(1-2\nu)\Omega$ is also common in all three normal strains.

All of them share a common factor of $\frac{1}{E}$. So, the only difference is the last term for ε_{rr}; the last term is $-(1+\nu)\frac{\partial^2\phi}{\partial r^2}$. For $\varepsilon_{\theta\theta}$, the last term is $-(1+\nu)\frac{1}{r}\frac{\partial\phi}{\partial r}$. And ε_{zz} has the last term as $-(1+\nu)\frac{\partial^2\phi}{\partial z^2}$.

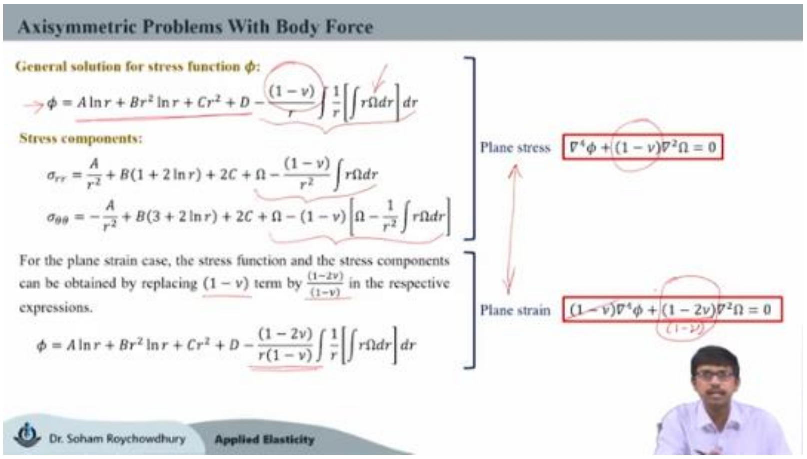
Whereas, the non-zero shear strain ε_{rz} is $-\frac{(1+\nu)}{E}\frac{\partial^2\phi}{\partial r\partial z}$. So now, with this, we have expressed all the strain components in terms of ϕ, the stress function ϕ, which is a function of r and z, also in terms of capital Ω, which is a function of r and z responsible for the body force.



Now, substituting these strain components in the compatibility equation, the strain compatibility equation, which is $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}) = 0$, and simplifying, we arrive at an equation like this, which equals $\nabla^4\phi$ (the bi-harmonic of ϕ) $+(1-\nu)\nabla^2\Omega = 0$ for the case of a plane stress axisymmetric problem. So, until the previous slide, we had obtained all six stress and all six strain components considering a $3D$ axisymmetric problem. Now, we are taking this $3D$ axisymmetric problem to be reduced to a two-dimensional axisymmetric problem, either with a plane stress assumption or with a plane strain assumption.

For the case of plane stress, we are considering the body to be small, for which σ_{zz} is τ_{rz} and $\tau_{\theta z}$. These three are taken to be 0, and with that assumption, the compatibility equation, only one compatibility is required to be satisfied; all the rest will be automatically satisfied. So, that compatibility would give us this equation, which is the governing equation for plane stress axisymmetric problem with body forces, $\nabla^4\phi + (1-\nu)\nabla^2\Omega = 0$. Similarly, if you go for a plane strain assumption, for which instead of the out-of-plane stress components, we have out-of-plane strain components, that is ε_{zz} then ε_{rz} and $\varepsilon_{\theta z}$, these three to be zero, then also except one, all the rest of the strain compatibility equations would be automatically satisfied and the only equation left to be satisfied is $(1-\nu)\nabla^4\phi + (1-2\nu)\nabla^2\Omega = 0$. This is the governing equation for the plane strain axisymmetric problem involving the body forces now. if you drop the body force, then capital Ω is 0. So, without body force, both equations would be reduced to a single equation of bi-harmonic of ϕ equals 0, with capital Ω being 0. The second term would vanish; the right-hand side being 0, we will only be left with bi-harmonic of ϕ is 0, which we had used for solving all our previous problems without body force.

If we have body force, the second additional term is required to be considered. This term has different factors based on either the plane stress assumption or the plane strain assumption.



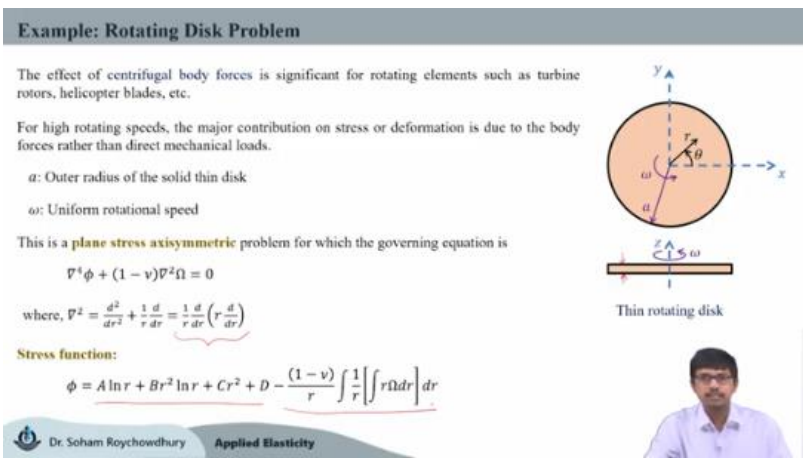
Now, the general solution of the stress function for this case is, For the first, we are considering the plane stress case. So, these would be the general solution: ϕ would be $A\ln r + Br^2 \ln r + Cr^2 + D - \frac{(1-\nu)}{r}\int \frac{1}{r}[\int r\Omega dr]dr$. So, this is the term; the last term is responsible for the body forces. For the axisymmetric $2D$ problems without body force, the first four terms were there, which we had earlier used for solving thick cylinder problems, compound cylinders, or annular disc problems without any body force. Now, as we are adding the body force, this extra term, which is dependent on omega, is coming.

Similarly, we can obtain the stress components σ_{rr} and $\sigma_{\theta\theta}$ for the plane normal stress components like this, where these last two terms in both σ_{rr} and $\sigma_{\theta\theta}$ are the extra terms responsible for the body force, which arise due to the last term of ϕ and are dependent on the potential function capital Ω. Now, this set of equations is valid for the plane stress problem, where the governing equation is $\nabla^4\phi + (1-\nu)\nabla^2\Omega = 0$.

Similarly, for the plane strain case, you can also obtain the equations. Instead of obtaining or calculating it once again, the easiest approach to derive the plane strain equation from the plane stress equation is to replace the $(1-\nu)$ term in the plane stress equation with $\frac{(1-2\nu)}{(1-\nu)}$. With that only, we can easily get the plane strain stress functions and plane strain stress components.

So, if you look at the ϕ expression for the plane stress, this was ϕ for plane stress, $1-\nu$ was appearing here. If you simply replace that $1-\nu$ here with $\frac{(1-2\nu)}{(1-\nu)}$, that would give us the ϕ stress function for the plane strain problem. Similarly, for σ_{rr} and $\sigma_{\theta\theta}$, you can obtain the corresponding plane strain stress expressions.

Stress component expression by this replacement of $1-\nu$ with $\frac{(1-2\nu)}{(1-\nu)}$. The plane strain governing equation, if you look back, is $(1-\nu)\nabla^4\phi + (1-2\nu)\nabla^2\Omega = 0$. So, if you cancel this $1-\nu$ term here, and divide the second term by $1-\nu$, then, if you compare the plane stress and plane strain case, the governing equation has only one difference, which is the coefficient of the $\nabla^2\Omega$ term. It changes from $1-\nu$ to $\frac{(1-2\nu)}{(1-\nu)}$. That is why this replacement in every equation, whether it is the stress function or any one of the stress components, this replacement will convert the corresponding equation from plane stress to plane strain.

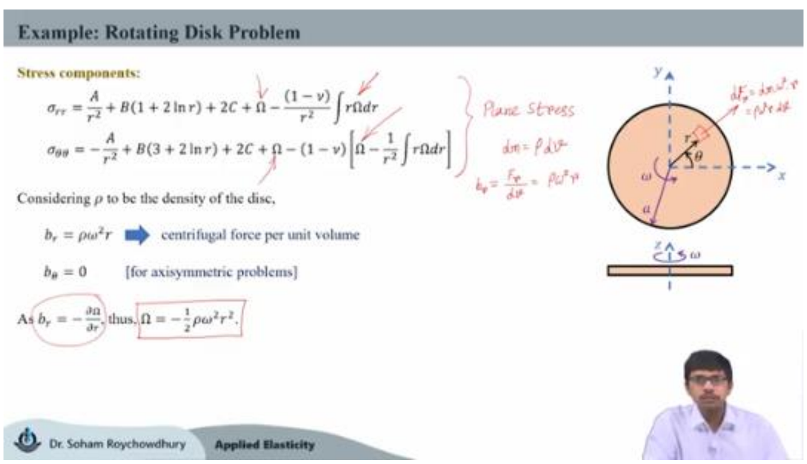


Now, considering a particular example involving an axisymmetric problem involving body force, which is the rotating disk problem. In many mechanical engineering applications, we come across rotating elements such as gears mounted on shafts, rotating pulleys, rotating turbine blades, helicopter blades, and different types of rotors. For all such rotating components, the effect of the body force would be there, which is the centrifugal body force effect acting in the radially outward direction.

Now, for high-speed components, which are rotating at significantly higher speeds, for that, this rotational component, which is responsible for the generation of the body force in the radial direction, that causes a significant amount of stress, displacement, or deformation of the body as compared to the corresponding direct mechanical loads. So, it's extremely important to consider the effect of the body forces for the high-speed rotary machine components where they may cause even more stress as compared to directly caused mechanical stresses. So, let us consider a thin disk. which is rotating with omega. We are considering a disk of radius a, rotating with a constant angular speed of omega about the z-axis. The z-axis is the axis of symmetry of the disk.

About which the body is rotating with a constant angular velocity of omega. As the body is thin, the thickness of the disc in the z-direction is assumed to be thin and small compared to the radius of the disc a. Hence, this can be modeled as a plane stress axisymmetric problem with the body force due to rotation. Now, this is a plane stress axisymmetric problem for which the governing equation is $\nabla^4\phi + (1-\nu)\nabla^2\Omega = 0$.

Now, we know that the Laplacian operator for the plane stress problem, the plane stress axisymmetric problem is like this: $\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$, which we can rewrite in this fashion: $\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$. Now, the stress function, the general stress function solution for the plane stress axisymmetric problem, we had obtained like this, where the first four terms were already there without body force. Due to body force, this extra term is coming. This was discussed in the last slide. And using this phi, the stress components were also obtained like this, where the last two terms are responsible for the body forces. And these are valid for the plane stress case of an axisymmetric problem.



Now, coming to the evaluation of this capital Ω, that is the potential function which will take care of the body forces. Here, the body force is coming due to the rotation of the disc. Now, considering rho to be the density of the disc, we are interested in calculating the centrifugal body force coming due to the rotation of the disc.

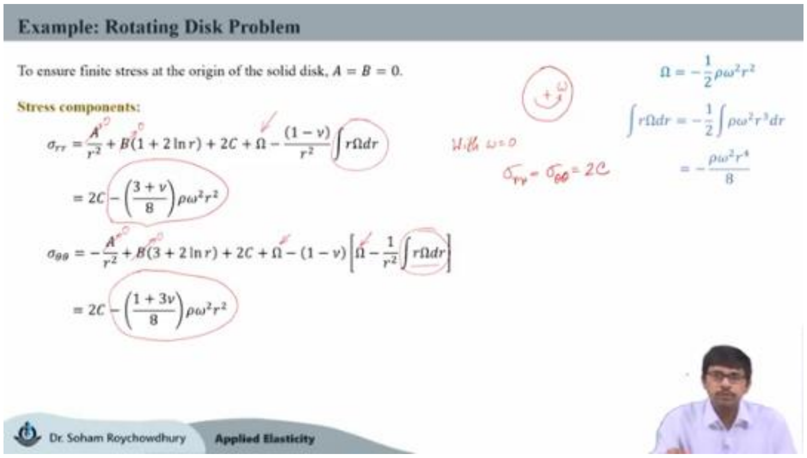
So, if ρ is the density of the disc, then the centrifugal force per unit volume acting along the radial direction can be written as $b_r = \rho\omega^2 r$. So, let us consider a small element here.

Now, on this particular element, which is at a distance r, let us say the volume of this element is dV.

So, the mass of the element dm would be ρdV. So, the centrifugal force b_r acting on this element will be the mass of the element $dm\omega^2 r$. Now, we are defining b_r. the body force in the radial direction as this centrifugal force or any body force per unit volume. So, if I divide this by volume, so b_r was $\rho\omega^2 r$ times the volume of the small element.

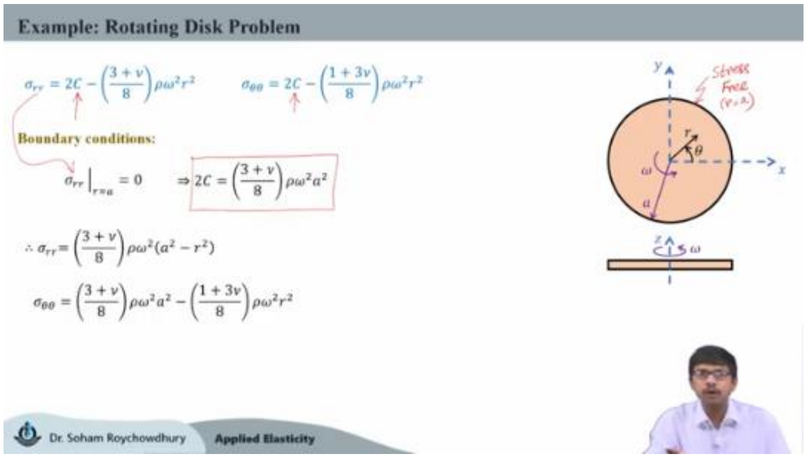
So, b_r would be F_r divided by the volume of the small element, which is $\rho\omega^2 r$. So, this is the body force per unit volume. And for a symmetric problem, the body force in the θ direction is 0. So, b_r, the body force in the radial direction per unit volume, which is caused due to the centrifugal effect of the rotating disc, is $\rho\omega^2 r$ and b_θ is 0. Now, by definition of capital Ω, the relation between capital Ω and b_r is such that b_r is $-\frac{\partial\Omega}{\partial r}$, and from that, we can obtain Ω of this form. If we have this potential function capital Ω as this $-\frac{1}{2}\rho\omega^2 r^2$, then $\frac{\partial\Omega}{\partial r}$ would be $\rho\omega^2 r$ and $-\frac{\partial\Omega}{\partial r}$ would be $\rho\omega^2 r$, which is nothing but b_r. So, our chosen form of capital omega, if we choose our potential function like this, that can represent the body force, the actual body force generated in the problem accurately, for which $b_r = \rho\omega^2 r$. Now, this form of ρ, this form of Ω, we can substitute here. in the expressions of σ_{rr} and $\sigma_{\theta\theta}$. Also, we can substitute this form of capital Ω in the expression of the stress function. Now, A and B are the two constants which we are having in the expression of the stress components.

So, you can see the first two terms, A and B, are two constants which, unless we force to zero, will cause infinite stress at the origin of the solid disk. So, this is something we have discussed multiple times while solving the solid axisymmetric problems, where the origin is contained within the domain. For such cases, if r equals to 0 comes under the domain, the first term, $\frac{A}{r^2}$, and the $\ln r$ term, that is the second term associated with B, they will go to infinity; they will have very large amplitude if r is going to 0.



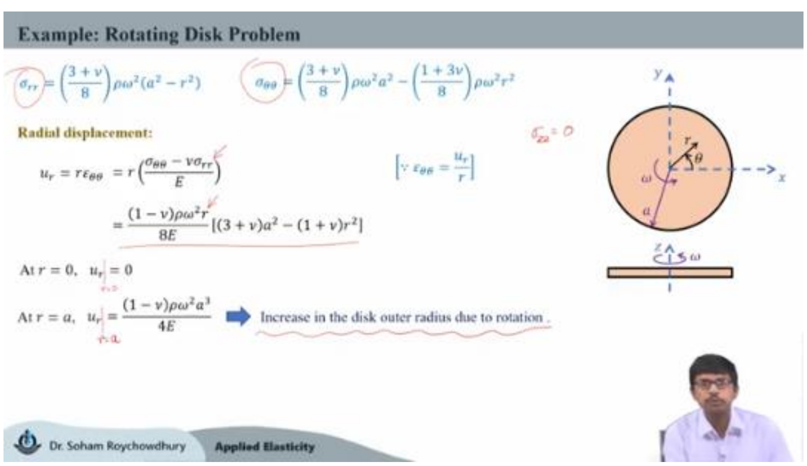
So, to ensure the solution to be bounded, we must have A and B to be 0, so these two constants would vanish, and then substituting capital $\Omega = -\frac{1}{2}\rho\omega^2 r^2$ in these particular expressions of σ_{rr} and $\sigma_{\theta\theta}$, we also need to evaluate the $\int r\Omega dr$. Evaluating this integral with Ω being $-\frac{1}{2}\rho\omega^2 r^2$, the integral becomes $-\left(\frac{\rho\omega^2 r^4}{8}\right)$. So, here, in place of these integrals, we will replace $-\left(\frac{\rho\omega^2 r^4}{8}\right)$. So, setting A and B to 0 and then substituting Ω and the integral of $r\Omega$ in the expression, the radial stress σ_{rr} is obtained as $2C - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^2$, and the hoop stress $\sigma_{\theta\theta}$ is obtained as $2C - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$.

So, these are the two stress components generated within the solid disk when it is subjected to uniform angular speed ω. So, if you see, the second term of σ_{rr} and $\sigma_{\theta\theta}$ contains the effect of ω. With ω equals 0, we would have σ_{rr} and $\sigma_{\theta\theta}$ equals $2C$, which is a state of stress where both normal stresses are non-zero and equal to a constant value. This was discussed for the case of a thick solid cylinder subjected to uniform external pressure. For that case, we obtained a stress function like this when omega was zero, which was a non-rotating case. Due to the effect of rotation, additional second terms are added in the expressions of stresses.



Now, moving forward, with the help of these two stress equations, if we write the boundary condition using which we will solve for this unknown constant C, A and B were 0, but C is still left, which we need to solve. Now, considering the outer boundary of the body, which is stress-free, So, a traction-free boundary condition is required to be imposed at r equals to a. No external pressure is acting here, as p_0 is 0.

So, σ_{rr}, the radial stress on the outer boundary r equals to a for this rotating disc, should be 0. And using that, we can write $2C$ equals $\left(\frac{3+\nu}{8}\right)\rho\omega^2 a^2$. So, using this expression of $\sigma_{rr}|_{r=a}$ and equating that to 0, we expressed this constant C in terms of $a, \rho,$ and ω like this. Now, substituting this $2C$ back into the stress equations, the final stress equation in the radial direction, radial normal stress σ_{rr}, would be $\left(\frac{3+\nu}{8}\right)\rho\omega^2(a^2 - r^2)$. And $\sigma_{\theta\theta}$, the hoop stress, will be $\left(\frac{3+\nu}{8}\right)\rho\omega^2 a^2 - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$. So, this gives us the complete stress distribution for a thin rotating disc rotating with constant ω.

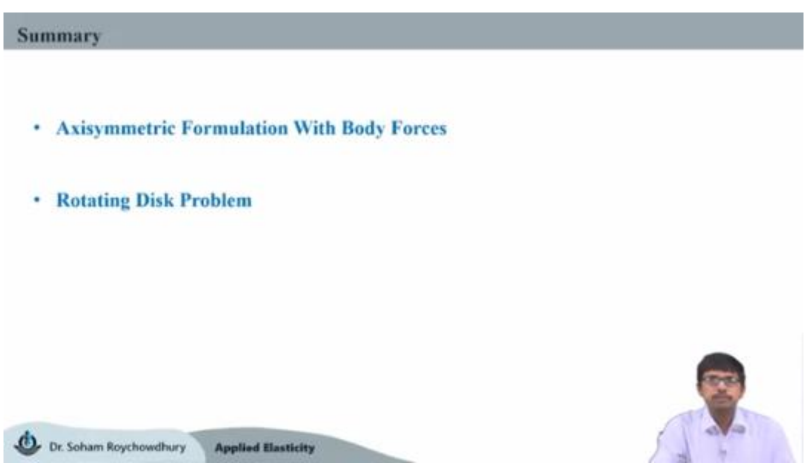


Now, moving to the calculation of the displacement, we are interested to see, due to the rotation, what is the change in the radius or diameter of the disc. Will there be any increase, and if so, how much would be the increase in the radius of the disc because of the rotation?

So, for that, we need to get the displacement component in the radial direction, u_r, that we will try to find out—the radial displacement of the disc at the outer periphery. So, u_r we can write as $r\varepsilon_{\theta\theta}$ because, for the axisymmetric problem, u_θ being 0, the circumferential strain $r\varepsilon_{\theta\theta}$ can be written as u_r / r, from which the radial displacement component u_r is 0. Written as $r\varepsilon_{\theta\theta}$.

Now, $\varepsilon_{\theta\theta}$, using the constitutive equation, we can write this as $\frac{\sigma_{\theta\theta}-\nu\sigma_{rr}}{E}$, and outside this, the small r is also multiplied. Note that, as it is a plane stress problem, we have σ_{zz} to be 0, thus that component is not present. In the expression in these expressions, otherwise, another term $-\nu\sigma_{zz}$ should be there if it is not a case of a thin disc. Now, substituting the expressions of $\sigma_{\theta\theta}$ and σ_{rr}, which are written here, if you substitute these here and simplify it, the expression of u_r, the radial displacement of the disc at any r, would be obtained like this. Now, considering the outer periphery, At r equals to a, we can obtain the increase in the radius of the disk as $\frac{(1-\nu)\rho\omega^2a^3}{4E}$, and at the centre, if you put r equals to 0, this term would go to 0. So, u_r is 0 at the centre. The centre point is not going to have any radial displacement, and that is obvious and expected because this must maintain axisymmetric geometry.

So, due to the presence of this r term, at r equals to 0, u_r is 0. And at r equals to a, that is at the outer boundary, we can obtain u_r as $\frac{(1-\nu)\rho\omega^2a^3}{4E}$, which is nothing but the increase in the disc's outer radius due to the rotation with speed omega. So, you can clearly see if omega is there, there will be a change in the radius if omega is very high. u_r is proportional to omega square at r equals to a. a significant effect of the higher speeds on the change in the radius of the rotating components.



So, in this lecture, we discussed the general axisymmetric formulation for the problems with body forces, and then we solved one example problem of a thin disc rotating with uniform angular velocity. Thank you.