APPLIED ELASTICITY

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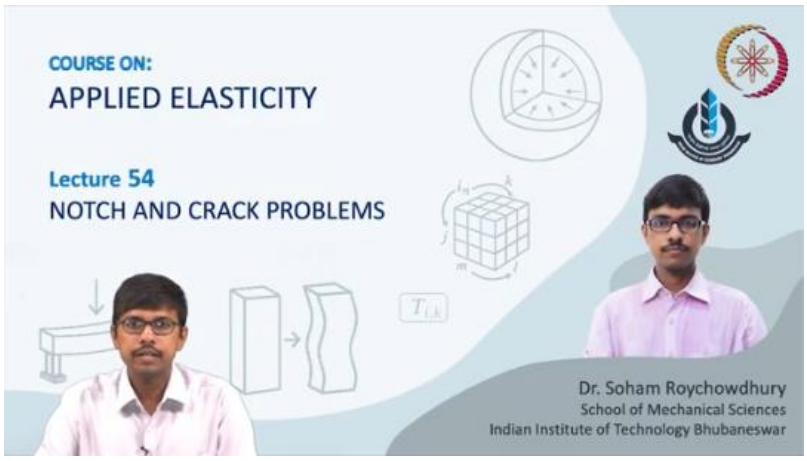
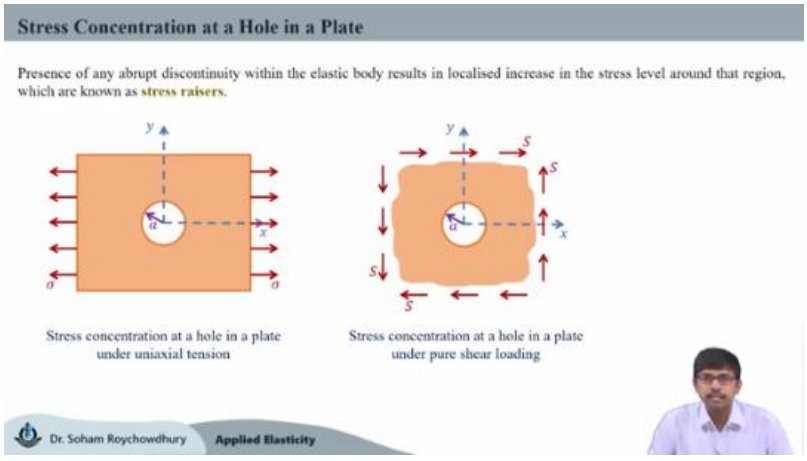
Lecture- 54



Welcome back to the course on applied elasticity. In today's lecture, we are going to discuss the notch and crack problems.

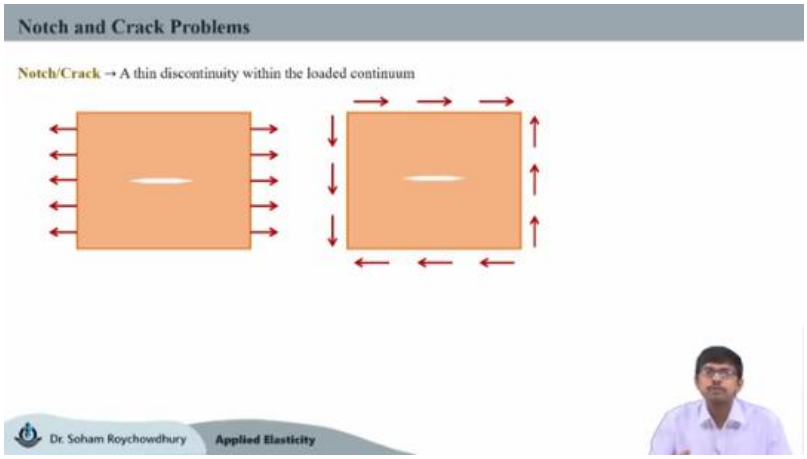


In the previous lectures—in two previous lectures—we talked about the concept of stress raisers and then considered two different stress concentration problems where a hole was present in a plate. So, stress raisers are discontinuities, abrupt discontinuities that can be present in any elastic continuum.

They result in a localized increase in the state of stress, the level of stress known as the effect of stress concentration, which occurs around the region of that discontinuity and such discontinuities are named stress raisers. So, one such discontinuity is the presence of a hole within the elastic continuum and the effect of stress concentration we have seen.

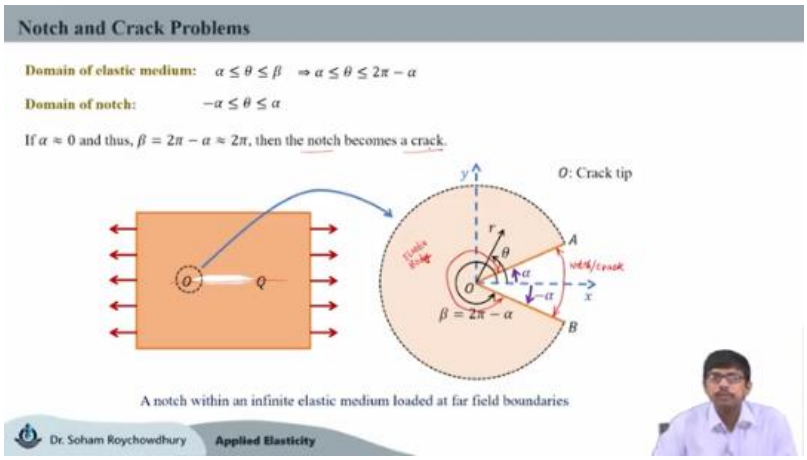
The measure of the stress concentration near the hole is the stress concentration factor, so we discussed and solved two problems of a plate with a hole at its center when it is subjected to either uniaxial tensile loading or subjected to pure shear loading. We solved these problems, obtained the stress fields near the hole, and also obtained the value of the stress concentration factor for both of these two stress concentration problems due to the presence of a hole in a plate.

Now, apart from the hole there may be notches or small thin cracks present within the elastic continuum, which may also lead to stress concentration.



So in this lecture, I'm going to discuss the effect of notches or cracks within the elastic continuum, and we will try to find out the stress field near the crack tip. So, considering a notch or crack problem, notches or cracks are thin discontinuities within the continuum, which is loaded at the far field. The dimension of the notch or crack is much smaller compared to the dimension of the plate. Now, let us consider a plate that has a thin notch within it, and this continuum or plate may be subjected to different types of loading, such as uniaxial tension, biaxial tension, pure shear, or even a combination of all these three different cases.

So, we are going to consider the solution of the stress field problem near the notch or crack tip.



Now, let us consider this particular problem where an infinite elastic continuum or a large plate is considered, which is loaded at the far field, and there is a notch present, spanning between O to Q within this elastic continuum. Now, let us zoom in at a region around one of the notch tip points, O.

So, if I zoom in on that particular region and redraw it, around the crack tip O, the figure will look like this. So, here the polar coordinate is used for the solution, and θ is measured counter-clockwise as positive from the positive x-axis. Now, θ varies between $-\alpha$ to α within the notch or the crack domain, whereas θ varies from α to β for the elastic domain.

So, this part is the elastic body, and this region—this opening portion—is the notch or the crack, the void part. So, θ varying between α to β defines the domain of the elastic medium or elastic continuum. Now, due to the symmetry of the notch about the x-axis, which we are assuming, we are assuming the notch to be symmetric about the x-axis, which may or may not be true.

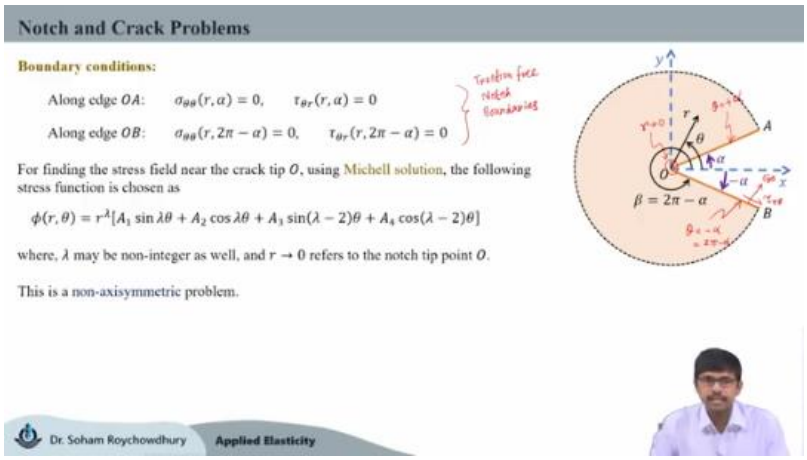
If it is not true, then two different values of α and β would exist. If it is a symmetric problem about the x-axis, about this horizontal line, if the geometry of the notch is symmetric, which can normally be assumed because these cracks or notches are very thin about their central line, about the central axis, we may assume it to be a symmetric geometry. And hence, for such cases, β can be written as $2\pi - \alpha$.

And using that, the range of θ for the elastic continuum domain can be written as α to $2\pi - \alpha$. θ varies between α to $2\pi - \alpha$, which is this particular region for the elastic domain. This is the domain of the elastic medium. And for the crack or the notch, θ varies between $-\alpha$ to α. This is the domain from $-\alpha$ to α for the notch.

Now, if the thickness of the notch is very small infinitely small, then we call that notch a crack. So, notches are components that are imposed due to the requirement of the geometry whereas cracks are the unintentional thin discontinuities generated within the continuum due to the initiation of failure. So, cracks are basically notches with almost zero thickness, and for such cases, the value of α will be very small.

So, if α tends to zero, then β, which is $2\pi - \alpha$, tends to 2π. For such cases, we can call the notches cracks. So, cracks are nothing but extremely thin notches with α tending to zero. And the span of the notch is almost zero, whereas the span of the elastic medium is from zero to 2π. Almost equals to zero to 2π. The range of θ for the elastic medium would be zero to 2π for thin cracks, whereas it would be almost zero for the cracks until that discontinuity is there. We are considering the crack; it is not a completely continuous body at the crack tip.

If we are going for the solution of the stress concentration near point O, our objective is to determine the stress fields, the stress components around the crack tip O.



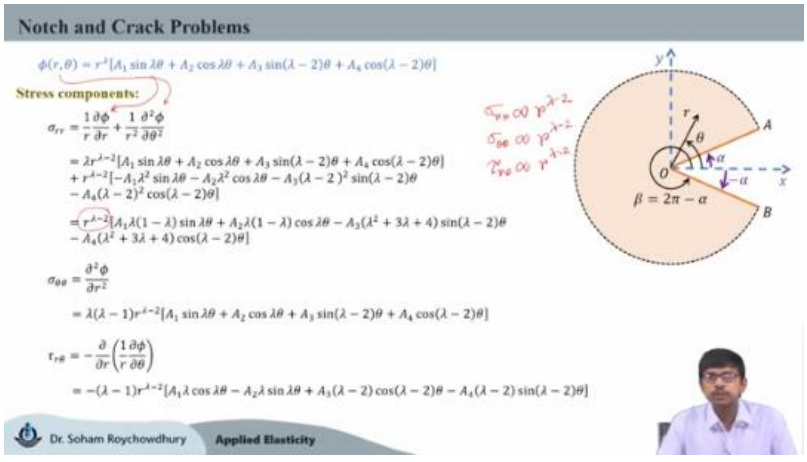
So, the first thing is to write the boundary conditions here. The crack or notch boundaries are defined at O to A line and O to B line, so O to A line is θ equals to plus α, and O to B line is θ equals to $-\alpha$.

These are the two boundaries on which we should have the surface tractions to be 0. So, the load is applied on the elastic continuum at the far field, far away from the notch or crack point O, the crack tip O. Hence, the crack or notch at the inner surface of the crack or notch is not subjected to any normal or shear stress. So, OA and OB, these are traction-free boundaries. And hence, the normal stress, sorry, the normal stress is $\sigma_{\theta\theta}$ here because these are θ planes.

So, the normal stress $\sigma_{\theta\theta}$. For these planes, this normal stress is $\sigma_{\theta\theta}$, and we will be having some $\tau_{r\theta}$ or $\tau_{\theta r}$, both of them should be 0. For θ equals to α plane, for θ equals to $-\alpha$ plane, $\theta = -\alpha$ plane can alternately be written as $\theta = 2\pi - \alpha$.

So, along edge OA, for $\theta = \alpha$ plane, we should have $\sigma_{\theta\theta}(r, \alpha) = 0$, also $\tau_{\theta r}(r, \alpha) = 0$. Similarly, for OB, the second free surface of the notch, you should have $\sigma_{\theta\theta}(r, 2\pi - \alpha) = 0$ and $\tau_{r\theta}(r, 2\pi - \alpha) = 0$. So, these are the traction-free boundary conditions, traction-free notch boundaries. As notch or crack boundaries are free of any kind of surface traction, we are going to have $\sigma_{\theta\theta}$ and $\tau_{\theta r}$ to be 0 for θ equals to α and θ equals to $2\pi - \alpha$ along line OA and along line OB.

Now, for finding the stress field near the crack-tip O, we will be using the general Michell solution, and we will be choosing, based on the boundary condition, motivated by this kind of boundary condition, the following form of ϕ, stress function from the general Michell solution, which is ϕ is chosen as $r^{\lambda}[A_1 \sin\lambda\theta + A_2 \cos\lambda\theta + A_3 \sin(\lambda - 2)\theta + A_4 \cos(\lambda - 2)\theta]$. So, here four unknown constants A_1, A_2, A_3, A_4 are associated with this particular stress function, and these r to the power λ. This λ may or may not be an integer, so λ may be an integer quantity or it may not be an integer quantity, with that constraint only. With that assumption only, the following or given form of stress function can satisfy the boundary conditions and also represent the notch or crack problems. At $r = 0$, r tending to 0, we are referring to the notch or the crack deep point, that is point O. So, point O is represented by r tending to 0.



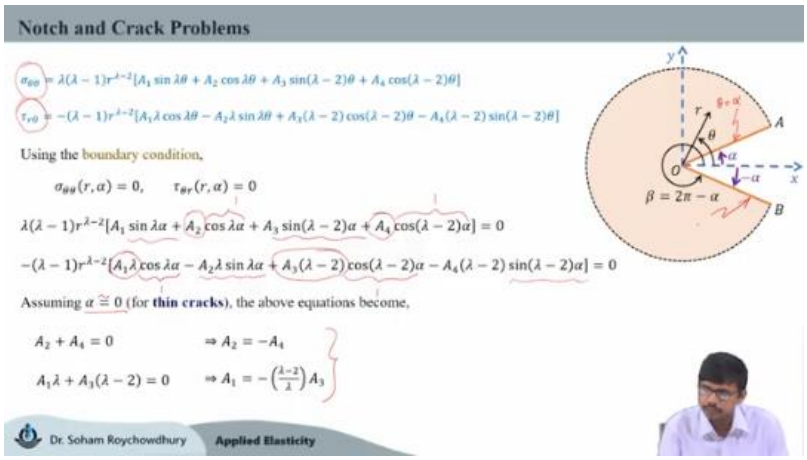
Now, moving forward. As you can clearly see, this stress function ϕ is dependent on θ. Sine and cosine θ terms are present. And hence, this problem—the solution of this problem—is a non-axisymmetric problem solution. Now, using this form of ϕ or the stress function, we can obtain the stress field σ_{rr}, $\sigma_{\theta\theta}$.

And $\tau_{r\theta}$, the normal and shear stress components. Near the crack tip O as σ_{rr} would be $(1/r)(\partial\varphi/\partial r) + (1/r^2)(\partial^2\varphi/\partial\theta^2)$, which. If ϕ is substituted in, it can be expanded in this form. So, this choice of stress function is substituted here and here, and then if I. Take the derivative with respect to r or θ.

As per the definition of σ_{rr} and simplifying this, some of the terms will be canceled, some terms can be clubbed together, and with that the final form of the radial stress distribution σ_{rr} would be like this. So, you can see outside the bracket we have a term r to the power lambda minus 2. So, σ_{rr} is proportional to $r^{\lambda-2}$ as far as the radial variable is concerned and then there is a function containing the $\sin\lambda\theta$, $\cos\lambda\theta$, $\sin(\lambda-2)\theta$, $\cos(\lambda-2)\theta$ term, which is θ dependent. Similarly, we can obtain $\sigma_{\theta\theta}$ by using $\frac{\partial^2\phi}{\partial r^2}$ and simplifying this, this would be the form of $\sigma_{\theta\theta}$, which is $\lambda(\lambda-1)r^{\lambda-2}[A_1\sin\lambda\theta + A_2\cos\lambda\theta + A_3\sin(\lambda-2)\theta + A_4\cos(\lambda-2)\theta]$. Now, here also, $\sigma_{\theta\theta}$ can be observed to be proportional to $r^{\lambda-2}$.

Coming to the last, that is the shear stress term $\tau_{r\theta}$, which can be obtained as $-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$. Substituting the form of ϕ here and simplifying, $\tau_{r\theta}$ can be obtained like this, which is once again, proportional to $r^{\lambda-2}$ for the radial variable part. So, $\tau_{r\theta}$ is $-(\lambda-1)r^{\lambda-2}$ into a function of θ containing sine and cosine terms. So, all three stress components,

All non-zero stress components for this notch or crack problem for the chosen form of phi are proportional to $r^{\lambda-2}$.



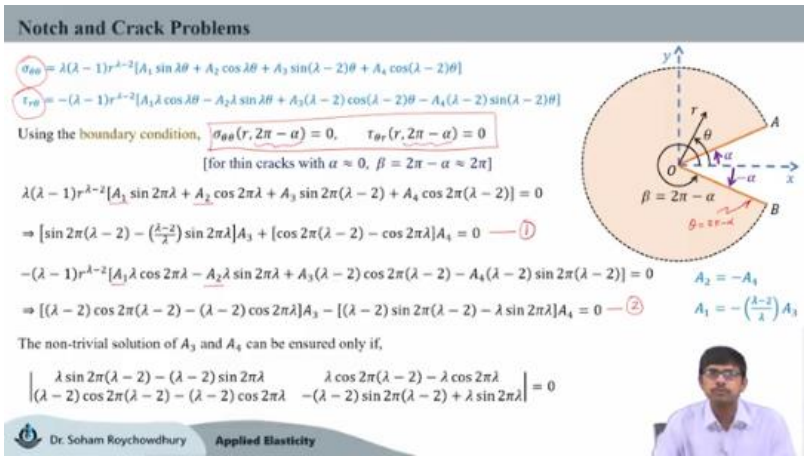
Now, moving forward with this obtained form of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, we will try to satisfy the boundary conditions. So, we had a total of four boundary conditions. Two boundary conditions were defined on the OA edge.

So, the OA edge, this edge is θ equals to α boundary. Two boundary conditions defined on this edge are $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ equals to 0 for any value of r when θ equals α. Using the obtained expressions of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, substituting those in these two boundary conditions with θ equals to α, we will get these two equations.

Now, using these two equations, we can solve for two of the constants in terms of the other two constants. For that, the first assumption we are making is reducing our notch problem into a crack problem. So, from now onwards, we will be solving for a crack problem, which is a very thin notch with α equals to 0. As we are assuming α equals to 0, $\sin\alpha$ or $\sin\lambda\alpha$, $\sin(\lambda-2)\alpha$, these terms will go to 0 for a very small value of α. α is not exactly 0; α is close to 0 for the thin cracks and span of the elastic continuum are close to 2π. Now, with α being very close to 0 or very small, the $sin\theta$ term, $sin0$, and $sin\alpha$ terms will go to 0, while $cos\alpha$ terms will be closer to unity. With that, the equation will be reduced to a form of $A_2 + A_4 = 0$, whereas the second equation would be $A_1\lambda + A_3(\lambda - 2) = 0$. So, this cosine term is unity here, and also this cosine term is unity. Whatever is there outside the bracket,

$\lambda, \lambda - 1, r\hat{}(\lambda - 2)$, these terms would get canceled as the right-hand side is 0. From the first equation, a_2 is one non-zero term, and A_4 is another non-zero term. So, this would be $A_2 + A_4 = 0$. From the second equation, $A_1\lambda$ is the first non-zero term, and $A_3(\lambda - 2)$ is the second non-zero term.

So, $A_1\lambda + A_3(\lambda - 2) = 0$. These are the two equations we have, through which two of the constants can be related and expressed in terms of the other two constants. We can write A_2 as $-A_4$ from the first equation, and from the second equation, we can write $A_1 = -(\lambda - 2)/\lambda$.



So, now, we have reduced the problem to a problem containing only two unknowns. And those two can be solved by using the other two boundary conditions which are defined on this plane. which is the θ equals to $2\pi - \alpha$ plane. So, if we go for those two boundary conditions for the OB edge, that is for θ equals to $2\pi - \alpha$ plane.

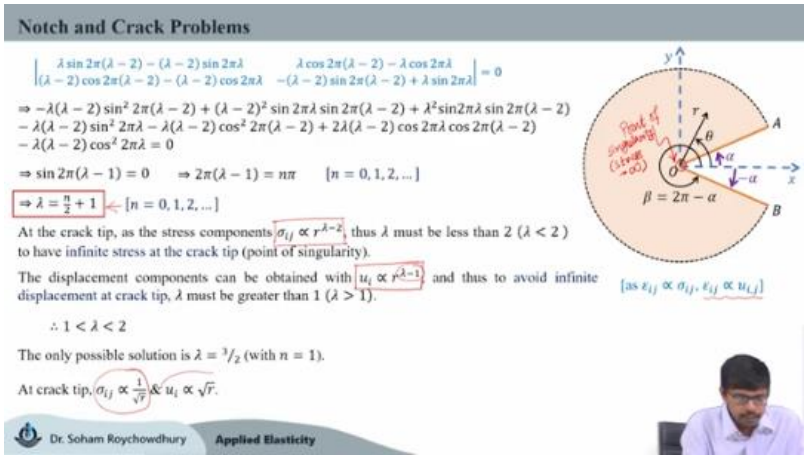
For θ equals to α plane, we have already solved those two boundary conditions, satisfied those two boundary conditions, and with the help of those two conditions, A_1 and A_2 are expressed in terms of A_3 and A_4, or alternately, A_3 and A_4 can be expressed in terms of A_1 and A_2. Now, coming to the two remaining boundary conditions prescribed on the notch or crack edge OB, which are normal and shear traction on OB is zero.

$\sigma_{\theta\theta}(r, 2\pi - \alpha) = 0$, $\tau_{r\theta}$ or $\tau_{\theta r}(r, 2\pi - \alpha) = 0$. Now, substituting these two expressions, these two boundary conditions in the form of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, and also assuming α to be very small. So, then this $2\pi - \alpha$ term will tend to 2π, β will tend to 2π. So, for such cases, if I write these two boundary conditions, they would be looking like this.

So, this first expression is for from $\sigma_{\theta\theta}(r, 2\pi - \alpha) = 0$ second expression is from $\tau_{\theta r}(r, 2\pi - \alpha) = 0$. Now, we already know A_1, A_2, A_3, A_4, all four of them are not independent, two of them A_1, A_2 are related or expressed in terms of A_4 and A_3 as $A_2 = -A_4$, $A_1 = -\left(\frac{\lambda-2}{\lambda}\right)A_3$. Now, if I replace this A_1 and A_2 in both the expressions in terms of A_3, A_4 and simplify this, we will be getting these two equations.

So, this is one equation, this is another equation. These are the two equations we are getting, which we are getting from these two boundary conditions on the crack face OB. Now these two equations 1 and 2 are two algebraic equations involving two unknown constants A_3 and A_4 and you can see the right hand side is 0 for both of these two equations 1 and 2. So, the trivial solution is A_3 0 A_4 0.

Now, if we have A_3 A_4 0 $A_1 A_2$ 0 would also be 0 entire ϕ stress function and then stress components everything would vanish. So, trivial solution is not the one which we are looking for. We must search for the non-trivial non-zero solution of A_3 and A_4 from equations 1 and 2. Now, if you have these two homogeneous coupled equations—coupled algebraic equations of two unknowns—the condition for a non-trivial solution is that the



The determinant of the coefficients of A_3, A_4 in these two equations must vanish. If this determinant is 0, then only we can ensure the non-trivial solution of A_3, A_4 exists. So, this determinant, the coefficient determinant, must be 0 for A_3, A_4 from equations 1 and 2. If I expand or simplify this determinant, then many of the terms will get canceled. So, simplifying this, we will get a very small or simple equation: $sin(2\pi(\lambda - 1)) = 0$.

If this condition is satisfied, then only we can have the non-trivial or non-zero solution of A_3, A_4 existing. Now, $sin(2\pi(\lambda - 1)) = 0$ means $2\pi(\lambda - 1)$ is simply $n\pi$, where n can take values from 0, 1, 2, 3, and so on going till infinity, and from that, we can cancel π from both sides and write λ. as $(n/2) + 1$, where n is 0, 1, 2, and so on.

So, the value of λ, which was the power of r in the equation of the stress function, ϕ, the stress function was written or assumed as r to the power λ times some function of θ. So, the power of the radial coordinate r in the stress function was λ, and that is now obtained as $1 + (n/2)$ and that is why, at the beginning, I told you for solving this kind of boundary condition, it is required to allow λ to be a non-integer. You can see clearly because of the presence of the $(n/2)$ term if you have n to be an odd number, then the value of λ will obviously be a non-integer, which should be allowed for the present problem. Now, if you recall the stress equations—σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ (or $\tau_{r\theta}$)— all three stress equations for the present problem, whatever expressions we obtained for them, it

was seen that all stress components are proportional to $r^{\lambda-2}$. So, all stress components are proportional to $r^{\lambda-2}$.

Now, if you consider the crack tip point O, this is the point of singularity. obviously the stress fields the stress values will tend to infinity for these particular point so at all the tip points for wedge tips as well we are going to have the infinite solution of the stresses which must be valid for the present problem because because that should be a singular point.

Now these can be ensured only if the value of λ is less than 2 So, in the expression of σ, we must have r in the denominator, then only at with r tending to 0, σ value will tend to infinity, will give us the singularity, we will get the singularity in the stress solution at the crack tip, if only if λ is less than 2, then only r will be in the denominator. So, λ the maximum value of λ can be 2, it must be lesser than that, cannot be more than 2.

Now, coming to the displacement, with respect to these stress components, which are proportional to $r^{\lambda-2}$, if we find out the strain and then the displacement components, you can check that the displacement components u_i that is u_r and u_θ, those would be proportional to $r^{\lambda-1}$. So, stresses and strains will have the same power of r if stress $\sigma_{ij} \propto r^{\lambda-2}$.

Strain components ε_{ij} would also be proportional to $r^{\lambda-2}$. Now, we know that the strains are nothing but $u_{i,j}$. ε_{ij} is $u_{i,j}$ from the strain-displacement relation. So, if you are taking the derivative of u, $\frac{\partial u_i}{\partial r_j}$, taking the derivative of u_i with respect to r once, we will be getting corresponding epsilon components. Thus, the order or power of r in the u expression must be $r^{\lambda-1}$.

Then only we can have σ_{ij} or ε_{ij} having $r^{\lambda-2}$ terms. And hence, u_i is proportional to $r^{\lambda-1}$. All the displacement terms are proportional to $r^{\lambda-1}$. Now, at point O, even though we are having infinite stresses, the displacement at the crack tip or point O must be finite. Otherwise, the crack will keep on propagating. If the displacement component of the crack tip is infinite, very large,

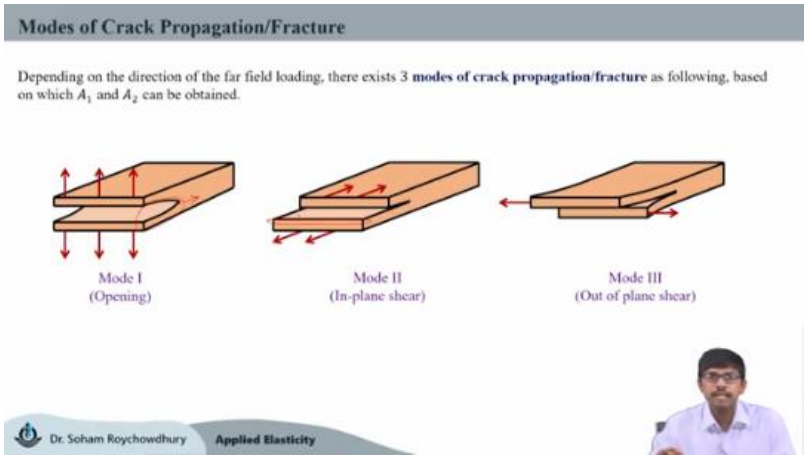
That means, at that point, the structure is going to have a large displacement, the crack is continuously propagating, which should be avoided. So, thus at the crack tip, the displacement must be finite, and to avoid that, these particular power $r^{\lambda-1}$, this $\lambda-1$ term should be positive. That can be ensured only if $\lambda > 1$. So, r should be in the numerator in the displacement equation, not in the denominator.

So, the range of λ is now fixed between 1 to 2: $\lambda > 1$ and $\lambda < 2$. And λ is also fixed by this equation, $\frac{n}{2} + 1$. So, hence combining all these, we can have only one possible choice of λ, which is $^3/_2$, occurring with $n = 1$. n is an integer number.

With $n = 1$ only, we can get λ to be $^3/_2$, which is within this admissible range of 1 to 2. Thus, λ being $^3/_2$, we can have $\sigma_{ij} \propto r^{\lambda-2}$, that is proportional to $\frac{1}{\sqrt{r}}$, and the displacement component $u_i \propto r^{\lambda-1}$, which is $u_i \propto \sqrt{r}$. So, the stress component $\sigma_{ij} \propto \frac{1}{\sqrt{r}}$, and the displacement component $u_i \propto \sqrt{r}$ for the present crack or notch problems. So, the stress field near the crack tip Substituting all these lambda values as 3 by 2 in all the equations: $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$. The stress field near the crack tip O can be obtained like this, which involves two unknown constants: A_1 and A_2. You can see all these stress components are proportional to $\frac{1}{\sqrt{r}}$ at r equals to 0.

All the stresses will go to infinity, which means there is a stress singularity present at the crack tip point O. Now, A_1 and A_2 are required to be obtained from the far-field loading condition. So, there will be a plate in which a small notch or thin crack is present, defined by this OQ. This figure is the enlarged view around point O. In the far field of the plate, there must be some kind of loading, which may be uniaxial tension, biaxial tension, or pure shear.

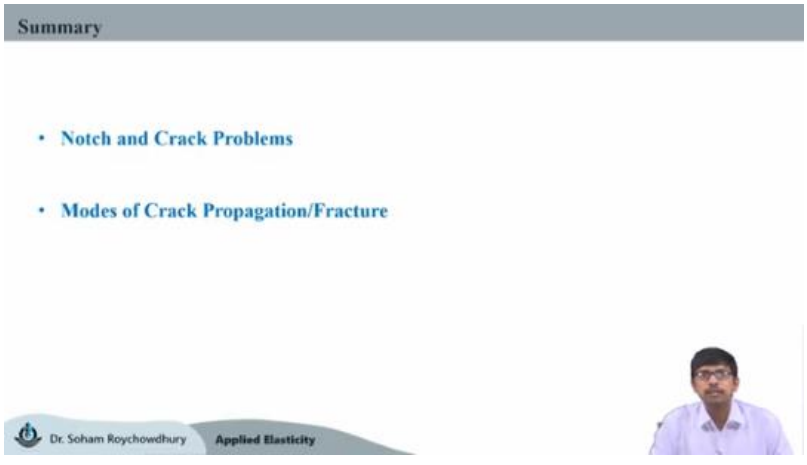
So, using that far-field loading, we need to find out these values of A_1 and A_2. Now, for different types of far-field loading, there will be different possible modes of crack propagation or fracture. And based on that, we can relate or define the respective stress intensity factor for these different modes of fracture, and A_1, A_2 can be related to those stress intensity factors. So, let us look into the different modes of crack propagation.



So, depending on the nature of the far-field loading and the direction of the far-field loading, we can define three different modes of crack propagation or three different modes of fracture as follows, based on which the A_1 and A_2 constants can be determined. So, the first mode is called Mode 1 fracture or opening mode, where this is the crack, and a uniform tensile load is applied which is perpendicular to the plane of the crack and perpendicular to the crack front. This is called the opening mode; the crack is trying to open up, and if this stress is higher than the limit, then the crack would propagate in this direction. Now, coming to the second mode,

This is the Mode 2 fracture, the second mode of fracture or crack propagation, known as in-plane shear. Here, this kind of in-plane shear loading is applied. So, the applied load is parallel to the crack plane. Also, it is perpendicular to the crack front. So, this is the crack front.

This line is the front of the crack front. So, applied shear loading is perpendicular to the crack front but parallel to the plane of the crack, which causes the mode 2 fracture. This is known as the shearing mode. And coming to the third one, this is the case or mode of fracture when it is subjected to out-of-plane shear. So here also, a shear load is present which is parallel to the crack front as well as the plane of the crack. This mode is called the tearing mode, where the top and bottom planes of the crack are going to get torn into two different out-of-plane directions. So, these are the three possible modes of fracture, depending on the far-field loading. Any one particular mode is required to be chosen, and based on that, different expressions for A_1 and A_2, the unknown constants in the stress component, stress field near the crack tip can be achieved.



So, in this particular lecture, we discussed the solution methodology for notch and crack problems, and we also discussed the three available modes of fracture or modes of crack propagation. Thank you.