APPLIED ELASTICITY

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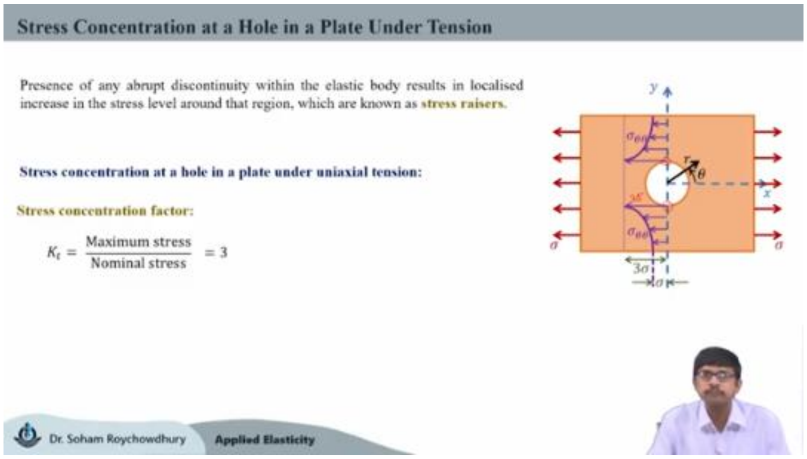
WEEK: 11
Lecture- 53

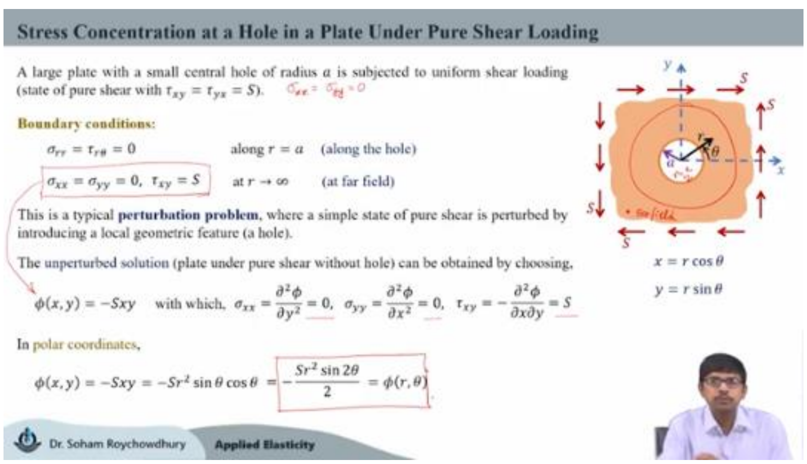
Welcome back to the course on applied elasticity. In today's lecture, we will continue our discussion on stress concentration problems, which we started in the last lecture.



So, let's proceed with the discussion on the stress concentration problem due to the presence of a hole within a plate under a uniaxial tensile field. This was discussed in the last lecture. Thus, the presence of a hole acts as a stress raiser, as does any kind of discontinuity within an elastic continuum.

It results in a localized increase in the stress level or stress values around that discontinuity region, which is known as the stress concentration effect, and the cause of this stress concentration is called a stress raiser. So, notches, holes, cracks, and abrupt changes in cross-section all act as stress raisers, resulting in stress concentration. So, we discussed a problem where an infinite plate or a very large plate has a small hole at its center, and the plate is subjected to a uniaxial tensile field of intensity sigma.

For that, we obtained the stress distribution due to the presence of the hole around the stress concentration region. And with respect to that obtained stress distribution, the hoop stress was found to be maximum, as shown in the figure, with the maximum value of the hoop stress occurring near this point of the hole. At these and these points, with $\theta = \pi/2$ to and $3\pi/2$, the maximum value of hoop stress there is 3σ, and thus we defined a quantity called stress concentration factor, which is the ratio of the maximum stress and the nominal stress. For the present problem of a plate with a hole subjected to uniaxial tensile stress, the stress concentration factor was obtained as 3.



Now, after this problem of a plate with a hole subjected to uniaxial tension, we are going to consider another stress concentration problem due to the presence of a hole within an elastic plate, which is subjected to shear loading. So, we are considering the problem of stress concentration in a plate with a hole which is subjected to pure shear loading, as shown in this figure. So, here a large elastic plate is considered, which has a small central hole at its center with radius a and the plate is subjected to a pure shear state of stress, as shown in the figure. So, for the plate, $\tau_{xy} = \tau_{yx} = S$, and σ_{xx} and σ_{yy}, these are 0.

So, this is a state of pure shear. No tensile load is acting, neither along the x-direction nor along the y-direction for the plate whereas it is subjected to a state of pure shear with τ_{xy}, τ_{yz}, and τ_{yx} equal to S. Now, this plate has a small central hole of radius a. And following the Saint-Venant principle, here also the effect of stress concentration should be visible near the hole. At the far field, the effect of the stress concentration due to the hole will not be present, will not be visible.

Now, to solve this problem, we will first write the boundary conditions. So, these are known as the far-field boundary conditions. So, σ_{rr} and $\tau_{r\theta}$ theta these two stress components are defined on the whole that is the stress concentration boundary condition and then we will also write the far-field boundary condition when r tends to infinity so for this problem, there are two boundaries: one is defined at r equals to a, this is the inner boundary the corresponding stress concentration at the inner boundary, that is r equals to a along the surface of the hole, is written here as within the hole surface, no normal stress or shear stresses are acting, we must have σ_{rr} and $\tau_{r\theta}$ to be 0 along r equals to a or along the hole, which is a free surface traction boundary condition for the hole.

Now, coming to the far field that is around the far field of the plate, much away from the stress concentration region. There, the far field is referred to with $r \to \infty$. So, the radius is much larger. We are considering the point somewhere here, let us say, which is the far field. Now, for this, σ_{xx} and σ_{yy} should be 0

τ_{xy} or τ_{xy} should be the applied pure shear loading S. So, the actual state of stress applicable for the plate without the hole should be the far-field boundary condition for the present stress concentration problem. Now, as the applied pure shear loading or pure shear state of stress was defined with respect to the rectangular Cartesian coordinate system xy. Thus, the far-field boundary conditions are also described in the rectangular Cartesian coordinate system xy, whereas near the hole. As it has a circular geometry, it is convenient to use the polar coordinate, and thus the whole.

Boundary condition at r equals to a, we are defining σ_{rr} and $\tau_{r\theta}$—these two polar coordinate stress components, polar normal stress and polar shear stress—to be zero. Along the whole, two sets of different coordinate systems are used: for the far-field, rectangular Cartesian coordinate boundary conditions are written; for the near-field around the whole. We are writing the boundary conditions in the polar coordinate. So, we need to map these two. Now, this is basically a perturbation problem where the state of pure shear is perturbed due to the presence of the hole.

So, if the hole was not there, the solution. Is basically this far-field boundary condition—this is the solution without any hole. Now, due to the presence of the hole, for a particular region around the hole here. We will have the solutions changing to something else, and that additional term is named as a perturbation term. So, with the actual solution—far-field solution. We are adding some perturbation, which is only valid within the stress concentration region, and using this perturbation approach, we are going to solve this problem.

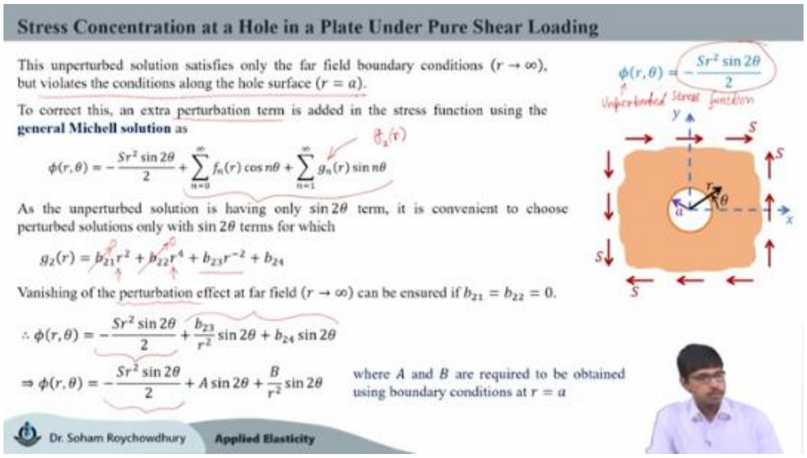
First, we are going to consider the unperturbed solution, which is for the far-field loading. Unperturbed means when the hole was not there, only a continuous elastic continuum or plate was there, which is subjected to pure shear loading S. For solving that, we choose the stress function $\phi(x, y)$ in Cartesian coordinates as $-Sxy$. So, $-Sxy$ is the stress function.

Using this stress function and the relation between the Cartesian stress components and ϕ, we can obtain σ_{xx} and σ_{yy} to be 0, which are $\frac{\partial^2\phi}{\partial y^2}$ and $\frac{\partial^2\phi}{\partial x^2}$, respectively, and τ_{xy}, the shear stress $-\frac{\partial^2\phi}{\partial x\partial y}$, to be S. Now, you can see σ_{xx} 0, σ_{yy} 0, τ_{xy} S matches our far-field boundary condition. Thus, this choice of stress function clearly depicts the far-field boundary condition, which is split with uniform pure shear, subjected to uniform pure shear without any hole.

Now, we will convert this particular stress function in terms of the polar coordinate variables r and theta because the stress concentration region is described, or it is convenient to describe the stress results within the stress concentration region, using polar coordinates. So, we cannot use two different coordinates; we will be going for coordinate transformation from x y coordinates. We will convert this particular ϕ, the far-field stress function or unperturbed stress function, into polar coordinates. So, for that, we can write $x = r\cos\theta$.

And $y = r\sin\theta$ from geometry, and with that, the ϕ can be written as $Sr^2\sin\theta\cos\theta$ or it can also be written as $-\frac{Sr^2\sin 2\theta}{2}$. So, this is the unperturbed stress function in the polar coordinate, which should be valid in the absence of the hole in the absence of the stress concentration, this should be the stress function in polar coordinate, which we should use to solve the problem.

Now, due to the presence of the hole, some additional terms are required to be added to this. So, we need to perturb this stress function, and extra perturbation terms are required to be added, which will capture the effect of stress concentration near the hole. So, for that, we are going to consider the perturbation. Now, this particular chosen form of $\phi(r, \theta)$ as $-\frac{Sr^2\sin 2\theta}{2}$ only satisfies the far-field boundary condition.



It violates the hole boundary condition near the stress concentration region, that is, at r equals to a. Now, to correct this, we are adding the perturbation term. So, one extra perturbation term is added to this unperturbed ϕ. This was the unperturbed stress function, which is valid only for the far field. Now, with that, we are adding the extra perturbation term. And for the choice of perturbation term, we are referring back to the general Michael solution for the polar coordinate problem. So, we know that the general Michael solution for the polar coordinate problem can have two types of terms. One is the cosine series term, which is written as $\sum_{n=0}^{\infty} f_n(r)\cos n\theta$, and the second term is the sine series term, which is written as summation $\sum_{n=1}^{\infty} g_n(r)\sin n\theta$.

Now, as the unperturbed solution is only having a $\sin 2\theta$ term—unperturbed means this one $-\frac{Sr^2 \sin 2\theta}{2}$—it is only having a $\sin 2\theta$ term as a function of θ, and hence it is convenient to choose the perturbation solution to have only a $\sin 2\theta$ term as far as the θ part is concerned. We should not choose $\sin\theta$, $\cos\theta$, $\cos 2\theta$, or θ^2, θ^3, these sets of terms. Then only these two sets of solutions will match each other.

It will be convenient to satisfy the boundary condition with the perturbed solution. So, we are choosing only the $\sin 2\theta$ term to be present in the entire perturbation solution. This total is perturbation. Now, the $\sin 2\theta$ term to be there means we are going to consider only $g_2(r)$. So, $f_n(r)$, all $f_n(r)$ are neglected because all of them will result in some cosine terms in which we are not interested.

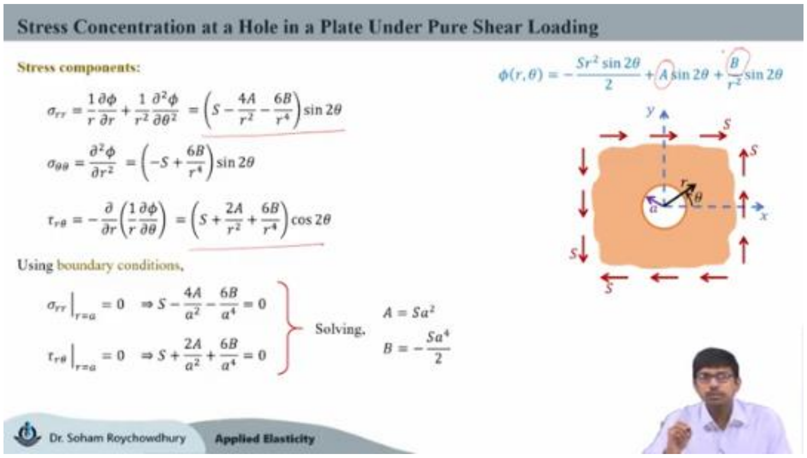
Now $\sin n\theta$ series can give $\sin 2\theta$ term with n equals to 2 corresponding g function is named as $g_2(r)$ and if you look back to the mickle solution discussed in the previous lecture where the $g_2(r)$, the general form of $g_n(r)$ was given, substituting n equals to 2 there, $g_2(r)$ would be $b_{21}r^2 + b_{22}r^4 + b_{23}r^{-2} + b_{24}$. These are the four constants involved.

Now, as the present solution, the present perturbation solution must vanish at the far field that we must ensure. So perturbation is localized. This perturbation as we want to have this perturbation effect only for the smaller values of r only near around the hole. So, for the large values of r at the far field, the effect of perturbation must vanish.

Now, if you look at the first term of $g_2(r)$, this is $b_{21}r^2$, which is proportional to r^2. If you keep on increasing the value of r as you are going to far field, the effect of these two terms $b_{21}r^2$ and $b_{22}r^4$ will keep on increasing. So, these two terms will be having more and more effect as you are increasing the value of r as you are going to far field which is not our objective. Our aim is to diminish the effect of the perturbation term as r value increases and that can be ensured only if we force these two terms to be 0.

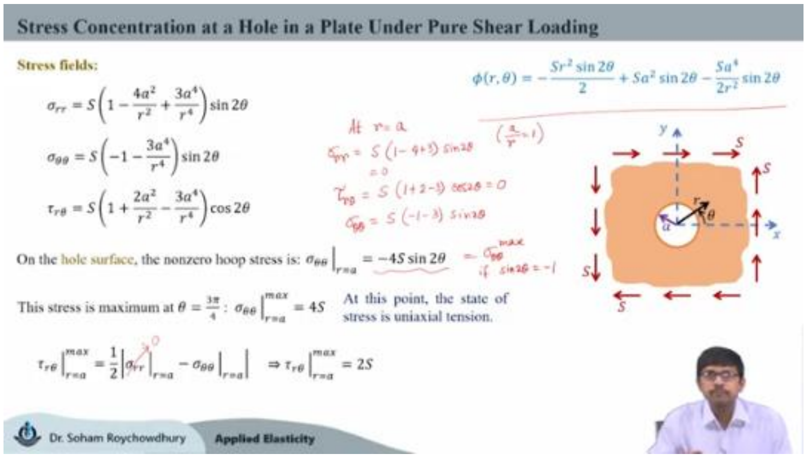
The last two terms would exist because they are either independent of r or proportional to 1 by r^2. So, this term b_{23}/r^2 would go to 0 as the r value goes to infinity, and hence the effect of this perturbation would be neglected, becoming 0 as you increase the value of r, as we are considering the far field, and hence the perturbed solution $\phi(r,\theta)$ is written like this: this part is the unperturbed solution $-\frac{Sr^2 \sin 2\theta}{2}$, which was already present due to the presence of the far-field pure shear state of stress. Then from $g_2(r)$, we only have two nonzero terms, b_{23} and b_{24}, both multiplied by $\sin 2\theta$, so this is the perturbation solution. which is $\frac{b_{23}}{r^2}\sin 2\theta + b_{24}\sin 2\theta$. So, the total perturbed stress function $\phi(r,\theta)$ becomes this.

It has three terms, and these two constants, b_{23} and b_{24}, can be renamed with two new constants, capital A and capital B. Thus, our perturbed stress function becomes $\phi(r,\theta) - \frac{Sr^2 \sin 2\theta}{2} + A\sin 2\theta + \frac{B}{r^2}\sin 2\theta$. Now, this perturbation field is valid for the entire region, both near the stress concentration region and the far field, where A and B are two unknown constants and those two can be determined by using the boundary conditions surface fraction boundary condition near the hole r equals to small a so the first term of ϕ was not satisfying those boundary condition. Now, we are adding these two perturbation terms so that these violation of the boundary condition at the hole can be avoided. So, using A and B, we will try to satisfy the whole boundary condition.



So, moving forward with this form of buttered pipe, we can find out the stress component σ_{rr} , $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ as this. σ_{rr} would be $\left(S - \frac{4A}{r^2} - \frac{6B}{r^4}\right)\sin 2\theta$, $\sigma_{\theta\theta}$ would be $\left(-S + \frac{6B}{r^4}\right)\sin 2\theta$, $\tau_{r\theta}$ would be $\left(S + \frac{2A}{r^2} + \frac{6B}{r^4}\right)\cos 2\theta$.

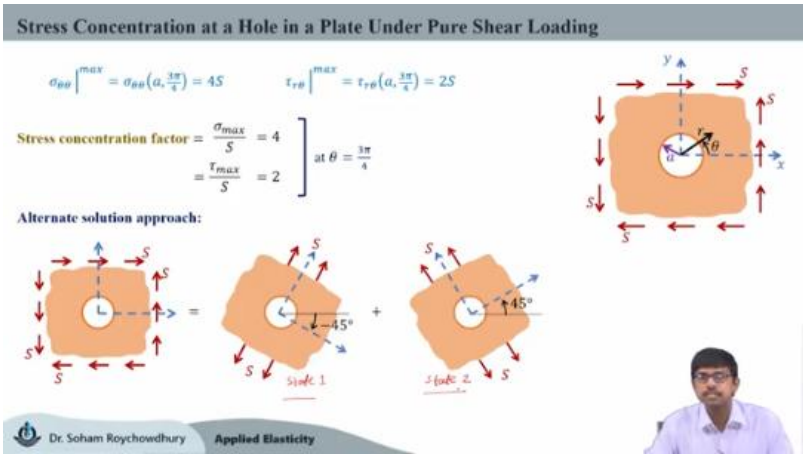
So, here both the normal stress component σ_{rr} and $\sigma_{\theta\theta}$ are functions of $\sin 2\theta$ whereas, the shear stress component $\tau_{r\theta}$ is function of $\cos 2\theta$. Now, with this stress component, we will try to see what the values of A and B are so that the boundary condition near the hole can be satisfied. So, considering the traction-free boundary condition near the hole, that is, at r equals a, we must have the normal stress $\sigma_{rr}|_{r=a}$ or along the hole to be 0. We also should have $\tau_{r\theta}$, that is shear stress, shear traction on the hole r equals to a to be 0. So, substituting this expression of σ_{rr} and $\tau_{r\theta}$, at r equals to a for all values of θ and equating them to 0 we will be having these two equations: $S - \frac{4A}{a^2} - \frac{6B}{a^4} = 0$ that is coming from the σ_{rr}, the normal stress being 0 at the hole. This equation we will get, and from shear stress being 0 at the hole, we will be getting $S + \frac{2A}{a^2} + \frac{6B}{a^4} = 0$. Now, solving these two equations simultaneously, these are the two algebraic equations involving two unknowns: capital A and capital B. Solution of this will give us the values of A and B as this: capital A would be small Sa^2, and capital B would be $-\frac{Sa^4}{2}$. Now, with these A and B values, we are able to complete the solution of this problem. So, now, these A and B values we can substitute here in the stress function. Also, we can substitute the A and B values in the stress component equations.



So, if you do so, the stress function ϕ would be obtained like this: this is the total perturbed ϕ, which is valid for the entire region, which satisfies the far-field stress condition, which also satisfies the stress boundary condition near the hole at r equals a. So, $\phi(r,\theta) = -\frac{Sr^2 \sin 2\theta}{2} + Sa^2 \sin 2\theta - \frac{Sa^4}{2r^2}\sin 2\theta$. This is the overall stress function. Using this stress function, if I find out the final stress field and final stress components, they will look like this. $\sigma_{rr} = S\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin 2\theta$. $\sigma_{\theta\theta}$, hoop stress, is $S\left(-1 - \frac{3a^4}{r^4}\right)\sin 2\theta$, then $\tau_{r\theta}$, the in-plane shear stress, is $S\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\cos 2\theta$.

So, this is the final state of stress for the plate with a hole subjected to a uniform pure shear state of stress. Now, on the whole surface at r equals to a, if we are interested in finding the stress distribution at r equals to a, substituting it in this state of state of stress, r equals to a means a by r. This ratio would be 1, and with that, we can get σ_{rr} as $S(1 - 4 + 3)$ 3)$\sin 2\theta$, which is 0. Similarly, $\tau_{r\theta}$, the last component, would be $S(1 + 2 - 3)$ 3)$\cos 2\theta$, which would also go to 0. So, along the whole, we have σ_{rr} and $\tau_{r\theta}$ to be 0, which are basically the traction-free boundary conditions of the whole, which our obtained stress field is satisfying.

So, the only non-zero stress component is the hoop stress along the whole, which would be $S(-1 - 3)\sin 2\theta$, which is $-4S\sin 2\theta$. So, this is the only non-zero hoop stress component on the whole surface r equals to a.



Now, for finding the maxima of this, this particular stress equals $\sigma_{\theta\theta}$ maximum. If this $\sin 2\theta$ term equals minus 1. If you are able to force $\sin 2\theta$ to be minus 1, then only the hoop stress along the whole surface would be maxima, and $\sin 2\theta$ can be minus 1 at $\theta = \frac{3\pi}{4}$. So, at the value $\theta = \frac{3\pi}{4}$, we can have the maximum value of hoop stress and that maximum value is nothing but $4S$. So, at this particular point, the state of stress would be uniaxial tension along the θ direction because σ_{rr} and $\tau_{r\theta}$ are 0. So, at this point of r equals to a, $\theta = \frac{3\pi}{4}$, we are having one uniaxial state of stress with non-zero normal stress along the theta direction equals to $4S$. Now, if you are interested to find out the maximum shear stress at the same point, τ_{max}, the maximum shear stress on the whole can be defined as half of the modulus of $\sigma_{rr} - \sigma_{\theta\theta}$, both evaluated at r equals to a or along the whole. So, τ_{max} for the in-plane state of stress, τ equals half of $\sigma_1 - \sigma_2$, where σ_1 and σ_2 refer to the two normal stresses.

So, $\tau_{r\theta}$ is half of $\sigma_{rr} - \sigma_{\theta\theta}|^{max}$, and all are evaluated at r equals to a. If I do so, σ_{rr} at r equals to a, this term is 0. So, hence, $\tau_{r\theta}|^{max}$ at r equals to a would be half of the $\sigma_{\theta\theta}|^{max}$ that equals to $2S$. So, the maximum value of hoop stress occurring on the whole surface equals $4S$, whereas the maximum value of the shear stress occurring on the whole surface is $2S$. For this particular stress concentration problem where a hole is present in a plate which is subjected to a state of pure shear of intensity S. So, $\sigma_{\theta\theta}|^{max}$ is $4S$ occurring at r equals to a, $\theta = \frac{3\pi}{4}$ and $\tau_{r\theta}$ at r equals to a, $\theta = \frac{3\pi}{4}$ is $2S$, which is the maximum shear stress. Now, here for this problem, we can define the stress concentration factor in two different ways.

The first one is with respect to hoop stress $\sigma_{\theta\theta}$. Another one is with respect to shear stress $\tau_{r\theta}$. If I define this with respect to hoop stress $\sigma_{\theta\theta}$, then that would be the maximum normal stress $\sigma_{\theta\theta}|^{max}$ divided by S, which is equal to 4. Whereas, if I define this with respect to τ, the shear stress, then it would be τ_{max}, that is $\tau_{r\theta}$ maximum, divided by S, which would be equal to 2. So, for the plates subjected to uniform shear and having a hole at the center, there are two possible ways of defining the stress concentration factor. If we define it in terms of normal hoop stress, then the concentration factor is 4. If we define the stress concentration factor in terms of the maximum shear stress, then the stress concentration factor would be equal to 2. And both of these two maximum values of the stress concentration factor occur at $\theta = \frac{3\pi}{4}$. Now, this is one possible way of solving this problem, where we are solving it by perturbing the stress field, the stress function which is used for the far field.

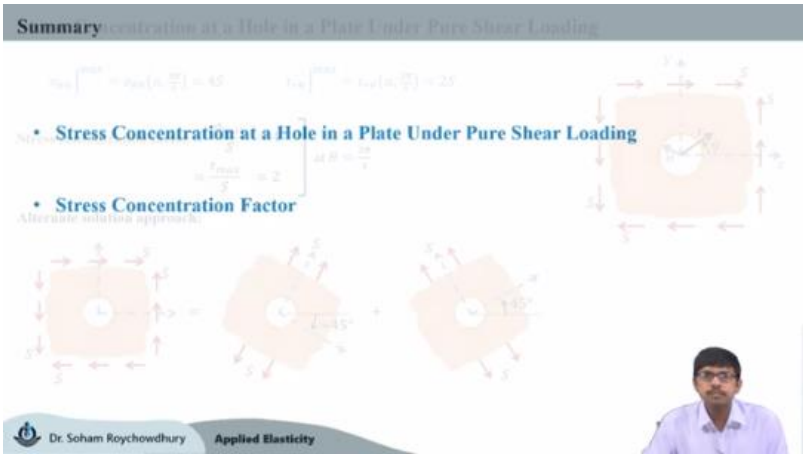
So, from the far field, we are choosing the stress function, then adding some perturbation in the stress function, which should be valid for the stress concentration region as well. Now, there is one alternate solution approach, which uses the principle of linear superposition. So, this is the required state of stress. One plate with a central hole subjected to a uniform state of pure shear. Now, this can be written as the summation of two different uniaxial problems.

So, if I consider state 1. So, state 1 and state 2, if both of them are solved separately and then superimposed, that will also give us the solution of the present problem. In state 1, we are considering a state of uniaxial tension of a plate with a hole where the direction of tensile stress is along a 45-degree angle from the x-axis. And then, for the other case, this is along a minus 45-degree angle from the x-axis. So, with that, if I apply the loading,

two uniaxial tensile loadings which are perpendicular to each other and then both of them are having the same intensity S. If these two states of stresses are solved separately and then their solutions are superimposed, we can get the solution for the present problem, that is, a plate with a hole subjected to uniform shear of intensity S. Now, the solutions of both state 1 and state 2 problems were discussed in the previous lecture.

where the plate with a hole subjected to uniaxial tension field was considered. So, the same solutions achieved or obtained can be obtained for state 1 and state 2, just with different directions of the hole, different orientations of the plate, and different directions of loading. So, for state 1 and state 2, the direction of loading is 90 degrees different, there is a change in angle of 90 degrees between the direction of loading between state 1 and state 2.

And then, using the principle of superposition, this problem can be solved using this alternate approach.



So, in the present lecture, we discussed the stress concentration problem of a plate with a hole at the center when the plate is subjected to pure shear loading. We also discussed the stress concentration factor for this particular problem. Thank you.