APPLIED ELASTICITY

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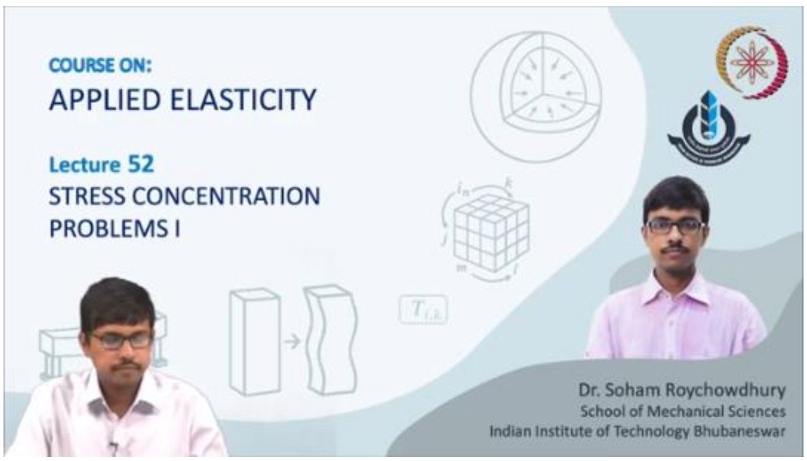
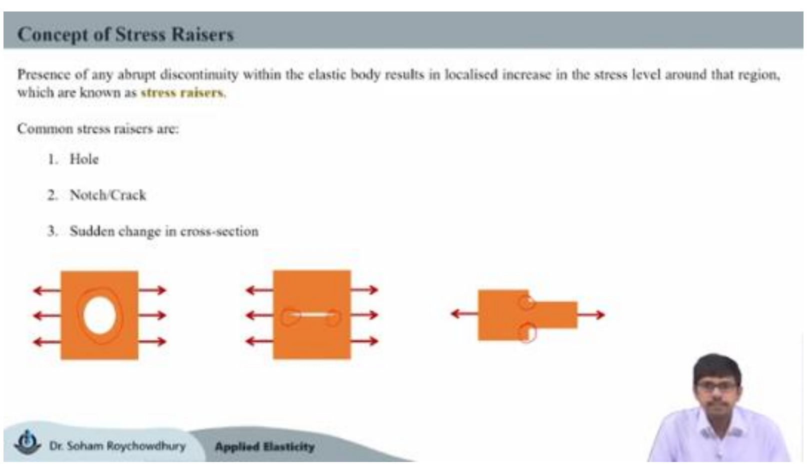
WEEK: 11
Lecture- 52



Welcome back to the course on applied elasticity. In today's lecture, we are going to talk about stress concentration problems. So, first we will start with the concept of stress raisers, where stress raisers are defined as the presence of any abrupt discontinuity within the elastic continuum, which causes a localized increase in the level of stress around that region where the stress raiser is present.

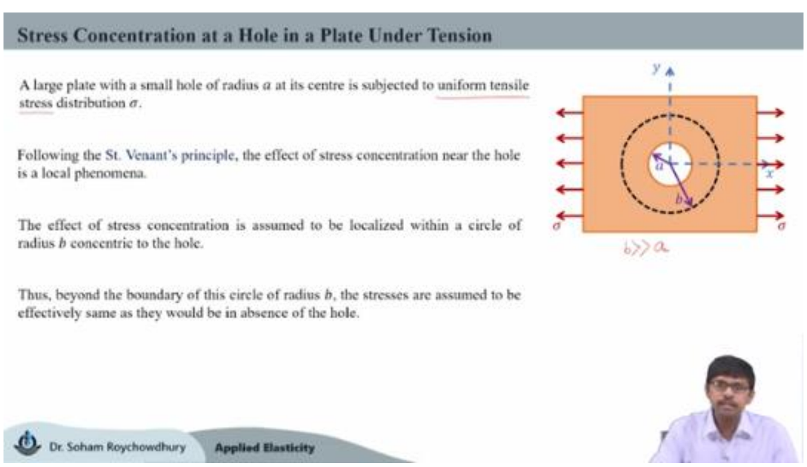


So, if you are considering a large elastic continuum, within that somewhere if there is any kind of abrupt discontinuity present, around that region there will be a high value of stress field generated. This effect is called the stress concentration effect, and those causes of stress concentration—that is, the abrupt discontinuities—are named as stress raisers. So, in different mechanical components, we can have various types of stress raisers present.

Some of them may arise unnecessarily, such as due to the generation of cracks within the body, or some may be there for geometric requirements, such as sudden changes in cross-section or the presence of notches. Or holes, which are required for fixing, let us say, nuts and bolts. So, some common stress raisers are the presence of holes within the body, which are often required to allow the fixing of fasteners. So, if you have a plate with a small central hole, the plate may be subjected to various types of far-field loading, which may be tensile, compressive, biaxial tension, or shear loading.

So, for such cases around the hole, there will be visible stress concentration. We may have notches or very small thin-line cracks. This is unwanted; somehow, the crack got generated due to material failure, and that would cause high values of stresses around the crack tips. There may be a sudden change in the cross-section of the element,

which is essential from the practical requirements, and due to that, near this region where the sharp change of the cross-section is present. A sudden change of the cross-section will also cause an increase in the stress level. So, this is also a stress raiser. So, near the hole, near the crack tips, or near the position of abrupt cross-section change, we will have high values of stresses, known as the stress concentration effect, and all these causes of stress concentration are named as stress raisers.



Now, here in this lecture, we are first going to talk about the stress concentration problem that is due to the presence of a circular hole at the center of a large plate when it is subjected to uniaxial tension. So, we are considering this large plate with a small central hole of radius a. So, a is assumed to be much smaller compared to all other dimensions of the plate.

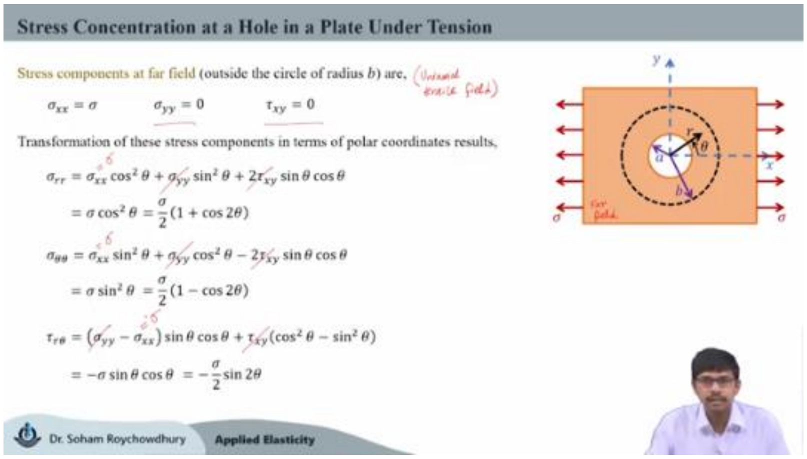
Now, the plate is aligned along the x and y axes and is subjected to a uniaxial tensile stress distribution of intensity σ along the x-axis. So, σ is the uniaxial tensile stress acting on the plate along the x-axis. Now, Following St. Venant's principle, the effect of the stress concentration due to the presence of this hole should be a localized phenomenon.

So, as we are saying, the hole is going to act as a stress raiser around the hole. So, near the region of the hole, in these regions, there will be high values of stresses generated, and following Saint-Venant's principle, that high stress will be concentrated only around the hole. If you go to some point far away from the hole, let us say somewhere here around the boundary of the plate.

There, no such effect of stress concentration or the change of the stress field due to the presence of a hole should be visible. So, let us consider another circle of radius b, which is concentric to the actual central hole, within which we are assuming that the effect of stress concentration is visible. So, due to the presence of holes, stress concentration would be there. We are assuming that the effect of an increase in stress is only there between these two circles: the an inner circle of radius a that is a hole and an imaginary outer circle of radius b. Beyond circle b, there would be no effect of stress concentration. So, beyond circle b, the stress fields would be the same as the stresses without the hole on the plate. So, within the circle of radius b, we can find out the effect of hole stress. Stress concentration due to the presence of a hole, and beyond surface b, the radius b, beyond the circle of radius b, the effect of stress concentration would be negligible.

So, in the case of the plate without any hole, whatever the stresses, the same stress would be visible. At the far field, that is, for all the points beyond the circle of radius b. Now, for all such problems, the value of a is normally very small. We are considering the The circle or the hole of a very small radius within the plate, and as compared to a, b is always larger. b is much greater as compared to a, but b is not as large as the dimension of the plate. The hole is very small, the stress tracer has a very small dimension, and thus the region b

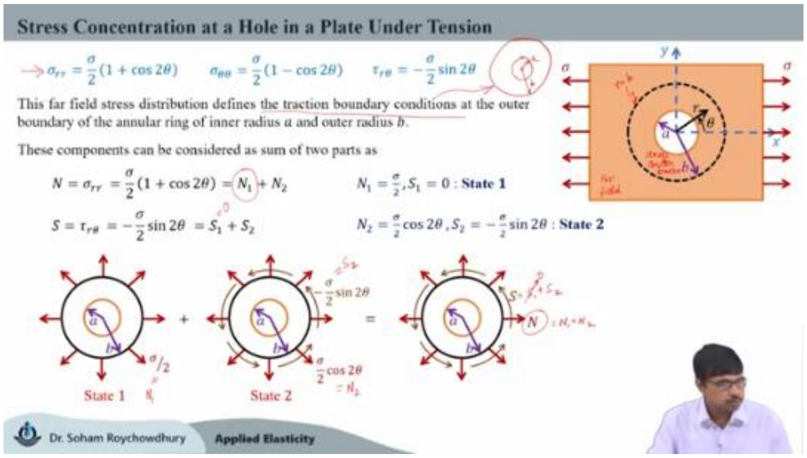
Within which the effect of stress concentration will be visible is much greater than a. However, b is not as large as the size of the plate.



Now, moving forward to the solution of this particular problem at the far field, Far field means outside the circle of radius b. We are first writing the stress distribution at the far field. So, this is called the far field. The far field region means which is Beyond the circle of radius b, where no effect of stress concentration due to the presence of a hole is visible.

So, here, the elements of the plate will just feel the uniaxial tension along the x-direction. So, σ x-axis σ is the given stress along the x-axis, whereas σ_{yy} and τ_{xy} would both be 0. So, this is a case of uniaxial loading at the far field. A uniaxial tensile field is observed for all points on the plate beyond the circle of radius b. Now, if we transform these stresses into polar coordinates—because within circle b, we need to use polar coordinates with the center at the center of the hole. Thus, it is convenient to solve this problem using polar coordinates. So, first, we are converting σ_{xx}, σ_{yy}, and τ_{xy} for the far field into polar coordinates, using the stress transformation equations for $2D$ σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$.

These can be written in terms of σ_{xx}, σ_{yy}, and τ_{xy}, where theta is the angle that the particular point is making. So, substituting σ_{yy} and τ_{xy} as 0, these terms vanish, and hence σ_{rr} would be just $\sigma\cos^2\theta$ or $\frac{\sigma}{2}(1+\cos 2\theta)$. So, σ_{xx} is σ, σ_{yy} is 0, and τ_{xy} is 0. With this substitution, we get σ_{rr} as $\frac{\sigma}{2}(1+\cos 2\theta)$, $\sigma_{\theta\theta}$ as $\frac{\sigma}{2}(1-\cos 2\theta)$, and $\tau_{r\theta}$ to be $-\frac{\sigma}{2}\sin 2\theta$. So, this is the far-field stress distribution expressed in terms of polar coordinates expressed in terms of the angular variable theta. which is valid beyond the circle of radius b.



Now, moving forward from the far field, we will now be coming to the field within that circle of radius b. Now, there must be continuity of the stress field at this circle of radius b, which is the junction of the stress concentration field and the far field.

So, outside the circle of radius b, we are having the far-field stress distribution. Within the circle of radius b, we are having the stress concentration region. So, the boundary condition is at this junction r equals to b, the circle should be the same for both the inner region as well as the outer region. So, for the outer region, we already know this is the far-field stress σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$, and hence if I substitute r equals to b and these equations being independent of r. these σ_{rr} , $\sigma_{\theta\theta}$, $\tau_{r\theta}$ directly define the traction boundary condition for this circle of radius b. So, if you consider the stress concentration region, the stress concentration region is having inner radius a.

Outer radius b. So, this is basically one annular zone. With inner radius a, outer radius b, and the boundary condition for this outer boundary r equals to b is defined by this far-field loading. So now our problem is basically reduced to the solution of this annular axisymmetric problem, annular axisymmetric zone. Which is subjected to these kinds of far-field boundary conditions. Now, as this far-field boundary condition is not axisymmetric,

the overall problem cannot be treated as an axisymmetric problem, though the geometry or this annular ring of inner radius a, outer radius b, is axisymmetric. So, hence, these components, these far-field stresses, we are going to divide into two parts. One is the axisymmetric part, another is the non-axisymmetric part, meaning the theta-independent part is axisymmetric, and the theta-dependent part is non-axisymmetric. So, let us write σ_{rr} and $\tau_{r\theta}$ in this particular fashion.

So, σ_{rr}, that is normal stress, is written as $N_1 + N_2$, where N_1 is $\frac{\sigma}{2}$ and N_2 is $\frac{\sigma}{2}\cos 2\theta$. So, N_1 is the θ-independent part of normal stress, and N_2 is the θ-dependent part of normal stress. Similarly, the shear stress S, $\tau_{r\theta}$ on this outer boundary of the annular ring of radius b, outer radius b, there we are once again separating S into two components: S_1 is θ-independent, and S_1 is θ-dependent.

Now, if you look at the expression of S, which is just $-\frac{\sigma}{2}\sin 2\theta$, it is completely dependent on θ. So, S_1 should be 0 here, as there is no θ-independent part present in S. Now, we will divide these annular ring states of stress into two parts. One is independent of theta, or the axisymmetric part. The other is dependent on θ.

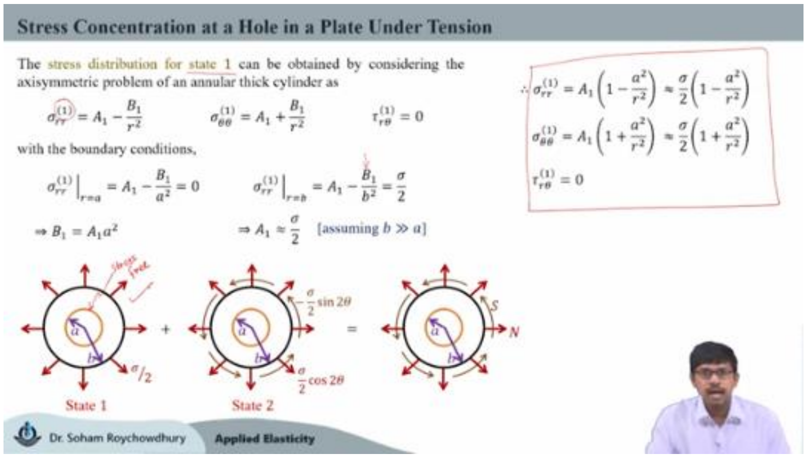
One contains normal stress N_1 and shear stress S_1. The other state contains normal stress N_2 and shear stress S_2 at the outer boundary. Hence, for state 1, We are considering the annular ring subjected to the boundary condition N_1 equals $\frac{\sigma}{2}$, $S_1 = 0$ at the outer boundary r equals b.

For state 2, the non-axisymmetric state, we are considering the normal stress N_2 to be $\frac{\sigma}{2}\cos 2\theta$ and S_2 to be $-\frac{\sigma}{2}\sin 2\theta$ acting over the outer boundary r equals b. So, here I am pictorially representing those two states of stress. So, here this figure is state 1. State 1 means it contains N_1, which is $\frac{\sigma}{2}$, the θ-independent part of N. So, $\frac{\sigma}{2}$ amount of radial stress, this is N_1, σ_{rr}.

That is acting at the outer boundary of this annular zone, which is an axis-symmetric problem. This is simply an annular cylinder subjected to a tensile stress distribution, a tensile radial stress distribution at the outer boundary, plus the second state, state two. which has N_2, the normal tensile stress as $\frac{\sigma}{2}\cos 2\theta$, and S_2, the shear stress as $-\frac{\sigma}{2}\sin 2\theta$. As you can see, both N_2 and S_2 are dependent on θ.

State 2 is a non-axisymmetric state, and if you superimpose these two, you will get the total field, where The entire stress concentration region is considered with an inner circle or inner radius a and outer radius b. The normal stress acting there is capital N, which is the summation of $N_1 + N_2$, and the shear stress acting there is capital S, which is the summation of S_1 and S_2, and S_1 is 0 here. So, this is directly S_2, which is $-\frac{\sigma}{2}\sin 2\theta$.

So, the solution for the stress concentration region we are going to obtain with the help of the principle of superposition where state 1 would be solved separately using the axisymmetric solution technique, state 2 would be solved separately by using a suitable stress function, and then they would be combined to find the net state of stress.



So first, let us start with state one, which, being an axisymmetric problem, is an annular thick cylinder subjected to external line loading radial load intensity of $\frac{\sigma}{2}$. So, for the thick cylinder problems, which we had discussed earlier, we know that the radial stress is some constant A_1 minus another constant $\frac{B_1}{r^2}$, hoop stress $\sigma_{\theta\theta}$ is $A_1 + \frac{B_1}{r^2}$ and $\tau_{r\theta}$ is 0. Now, A_1 and B_1 are the two unknown constants to be determined with the help of boundary conditions.

So, considering state 1, What are the boundary conditions? The inner equator at r equals to A, this is stress-free. So, no normal or shear stresses are going to act on the inner equator, whereas outer equator, the tensile stress sigma by 2 is acting. So, these are the two boundary conditions.

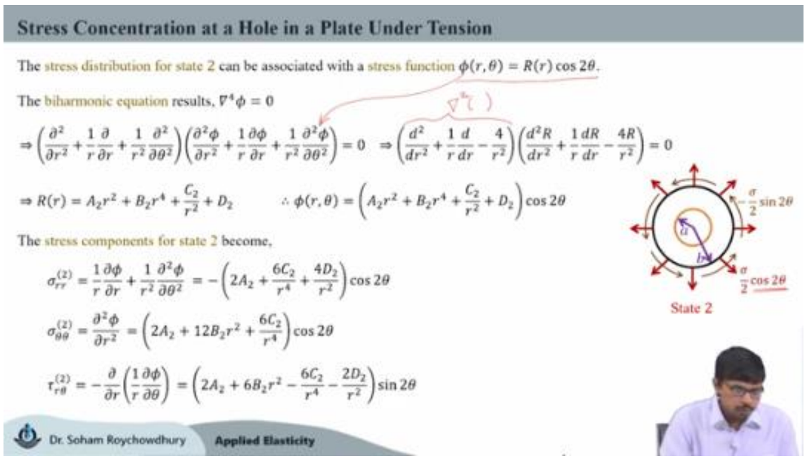
$\sigma_{rr}^{(1)}|_{r=a} = 0$ and $\sigma_{rr}^{(1)}|_{r=b} = 0$, the outer boundary is sigma by 2. For state 1, the annular cylinder subjected to axisymmetric tensile radial loading at the outer boundary, these are the two boundary condition. At $r = a$, σ_{rr} is 0. At r equals to b, σ_{rr} is $\frac{\sigma}{2}$.

You note that we are using this superscript 1 within brackets in all σ_{rr},$\sigma_{\theta\theta}$,$\tau_{r\theta}$—all these terms that refer to the stress field for the first state. This superscript within brackets (1) refers to the state of stress for state 1. If it is 2, then that would be for state 2, which we will consider after this. Now, using these two equations, from the first equation, we can relate b_1 as a_1 times small a squared.

Substituting that b_1 in the second equation and also using the assumption that small b is much larger compared to a. a is a very small hole, and b is the radius of the region within which the effect of stress concentration would be visible—that is larger, much larger compared to the size of the hole. So, the $(a/b)^2$ term is neglected or taken as 0.

With that, the constant a₁ is approximately obtained as $\sigma/2$. for the case of a small hole. So, substituting this a_1 and b_1 in the stress field, the stress field for the first state of stress would be obtained as $\sigma_{rr}^{(1)} = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right)$, $\sigma_{\theta\theta}^{(1)} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right)$, and $\tau_{r\theta}^{(1)} = 0$.

So, here we have solved the first state of stress. Now, we would proceed for the second one. So, this state of stress was axisymmetric. The first state is solved. Now, we will go for solving the second state so that their superposition can give us the total state of stress.

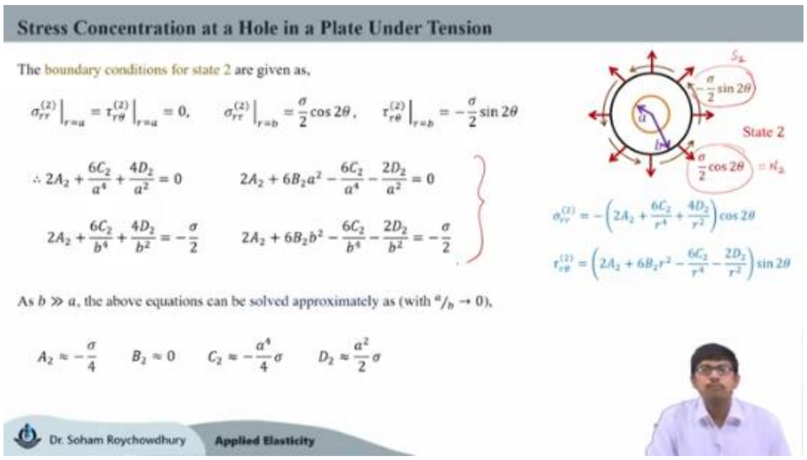


So, considering the second state of stress where we need to solve it using a stress function which must be θ dependent because this is a non-axisymmetric problem. So, we are going to choose the stress function as some capital R which is function of radial coordinate small r $\cos 2\theta$. Now, why $\cos 2\theta$? If you look at the σ_{rr} boundary condition, the second state is subjected to the radial stress $\frac{\sigma}{2}\cos 2\theta$ at the outer boundary. Because of this boundary condition, which our stress components must satisfy, that motivates us to select the stress function with a term of $\cos 2\theta$, because the θ dependent term in ϕ and σ_{rr} is normally same. So, that is why we are choosing ϕ to be function of θ as $\cos 2\theta$.

Because σ_{rr} was also having a $\cos 2\theta$ term, and capital R is some unknown function of r which we need to find out by using the biharmonic equation. So, for the present problem in the biharmonic expression, that is the biharmonic equation for the planar polar problems. And this is for the general non-axisymmetric case; $\frac{\partial}{\partial\theta}$ is non-zero for state 2, which is a non-axisymmetric state. So, here if we substitute this form of ϕ, which is capital R times $\cos 2\theta$, the $\cos 2\theta$ term will get cancelled, and the Laplacian operator would be $\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}$.

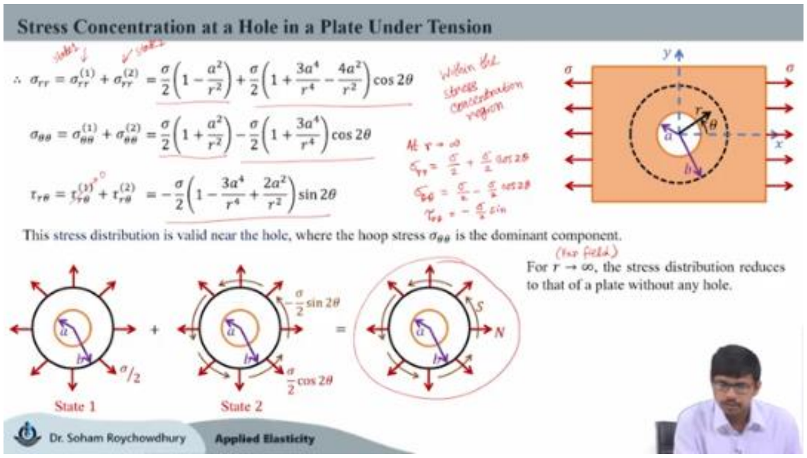
So, this would be the Laplacian operator for the present problem if we substitute ϕ equals to capital $R(r)\cos 2\theta$. So, from this, the general solution of capital R can be obtained as $A_2r^2 + B_2r^4 + \frac{C_2}{r^2} + D_2$. So, this is the radial function in the stress function phi, and thus the total stress function can be written like this, which is having only the cos 2 theta term as the theta-dependent one.

Now, from this, σ_{rr} , $\sigma_{\theta\theta}$, and $\tau_{r\theta}$, the stress components for state 2, the non-axisymmetric state, can be obtained like this, which involves these four unknown constants A_2, B_2, C_2, and D_2. Note that here in this stress field, I am using these two superscripts within brackets that refer to the state of stress for the second state.



Now, moving forward, we need to use the boundary conditions to solve for those four unknowns. So, the boundary condition for this state, state 2, it is written like this. At the inner boundary a, we have 0 normal stress and 0 shear stress, whereas at the outer boundary b of the annular ring, we have normal stress N_2 to be $\frac{\sigma}{2}\cos 2\theta$ and shear stress S_2 to be $-\frac{\sigma}{2}\sin 2\theta$. So, these give us 4 boundary conditions at r equals to a, both σ_{rr} and $\tau_{r\theta}$ to be 0 for the second state of stress. Also, for r equals to b, radial stress σ_{rr} is $\frac{\sigma}{2}(1 + \cos 2\theta)$ and shear stress $\tau_{r\theta} = -\frac{\sigma}{2}\sin 2\theta$. Now, using the expressions of σ_{rr} and $\tau_{r\theta}$, and putting them in these four boundary conditions, we will get these four equations involving four unknown constants A_2, B_2, C_2 and D_2.

So, by simultaneously solving these four equations, along with the assumption that b is larger than a, meaning a by b term is 0 or b by a term is infinity. With that assumption, we can get the 4 constants as follows. A_2 would be $-\frac{\sigma}{4}$, B_2 is 0, C_2 would be $-\frac{a^4}{4}\sigma$, and D_2 would be $\frac{a^2}{2}\sigma$. So, these are the 4 constants which can be evaluated by using the four boundary conditions for the second state of stress.

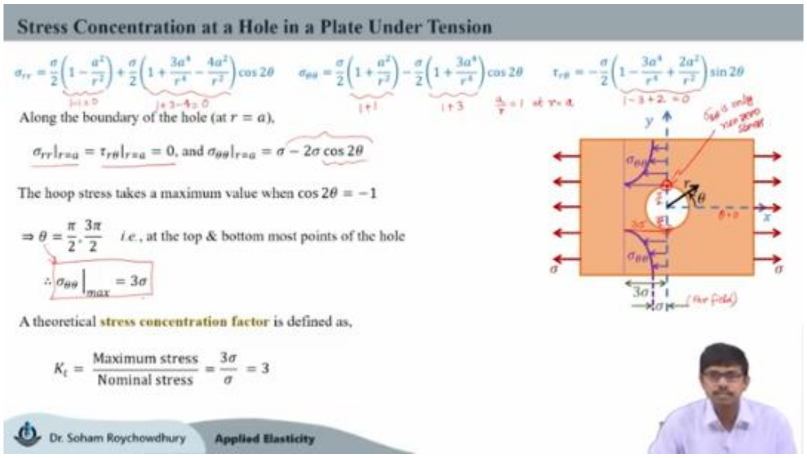


And finally, substituting these constants back, In the σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ equation, the total stress field for the superimposed solution, that is, for the final stress field within the stress concentration region, that can be obtained as the summation of the stresses for state 1 and the stresses for state 2.

So, the first row refers to state 1 stress components. The second column refers to state 2 stress components. Both we had obtained, and now if I add them, the first state σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$, these were axisymmetric. σ_{rr} for state 1 was $\frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right)$, this part. Then $\sigma_{\theta\theta}$ for state 1 was $\frac{\sigma}{2}\left(1+\frac{a^2}{r^2}\right)$, and $\tau_{r\theta}^{(1)}$, this was 0. So, no such term is there. For state 2, after substituting the obtained constants A_2, B_2, C_2, D_2, we can write σ_{rr} for state 2 like this, $\sigma_{\theta\theta}$ like this, and $\tau_{r\theta}$ like this, where σ_{rr} and $\sigma_{\theta\theta}$ are functions of $\cos 2\theta$, and $\tau_{r\theta}$ is a function of $\sin 2\theta$.

So, this gives us the total stress field within the stress concentration region. That is, within the annular cylinder of inner radius a (that is, the hole) and outer radius b, using the Saint-Venant principle. So, within the stress concentration region, these would be the stress fields generated. Now, if I compare the values of this, we can see that the $\sigma_{\theta\theta}$ or the hoop stress value is the dominant one within the stress concentration region.

And, as r goes to infinity—as we are going to the far field—r goes to infinity means far field. For that, many of these terms—wherever $\frac{1}{r^2}$ terms are there—all those will go to 0, and with that, we will have the stress distribution reduced to the state 1 case, that is, the case of the plate without any hole. So, for, as r tends to infinity, if I check σ_{rr} from the first term, it would be $\frac{\sigma}{2}$; from the second term, it would be another $\frac{\sigma}{2}$, and with that second term, it will be $\frac{\sigma}{2} cos\ 2\theta$. Then, $\sigma_{\theta\theta}$ here would be $\frac{\sigma}{2}-\frac{\sigma}{2} cos\ 2\theta$ and $\tau_{r\theta}$, this will be $-\frac{\sigma}{2}\sin 2\theta$. So, you can check these are exactly identical to the far-field stress distribution that we had obtained, that is σ_{xx}, σ_{yy}, and τ_{xy} are 0.



Now, moving forward with the obtained form of stress distribution in the stress concentration. Now, if I try to find out the stresses on the hole, that is at $r = a$. So, along the boundary of the hole where $r = a$, σ_{rr} and $\tau_{r\theta}$ would go to 0 with $r = a$. This a by r term will be equal to 1, and thus this particular term would be 1 minus 1. So, this would go to 0. This term would be 1 plus 3 minus 4. So, that would also go to 0, and hence total σ_{rr} would be 0 at r equals to a. Similarly, for the $\tau_{r\theta}$ expression, this term would be 1 minus 3 plus 2, which will once again go to 0.

So, $\tau_{r\theta}$ is also 0 along the hole. So, along the hole boundary, the only non-zero stress is $\sigma_{\theta\theta}$. $\sigma_{\theta\theta}$ is the only non-zero stress. And if I substitute that here, it would be 1 plus 3, and here it would be 1 plus 1.

That would result in $\sigma_{\theta\theta}$ at r equals to a to b. $\sigma - 2\sigma\cos 2\theta$ and this hoop stress $\sigma_{\theta\theta}$ is going to take a maximum value if this $\cos 2\theta = -1$. So, $\cos 2\theta$ can be -1 only if the value of $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

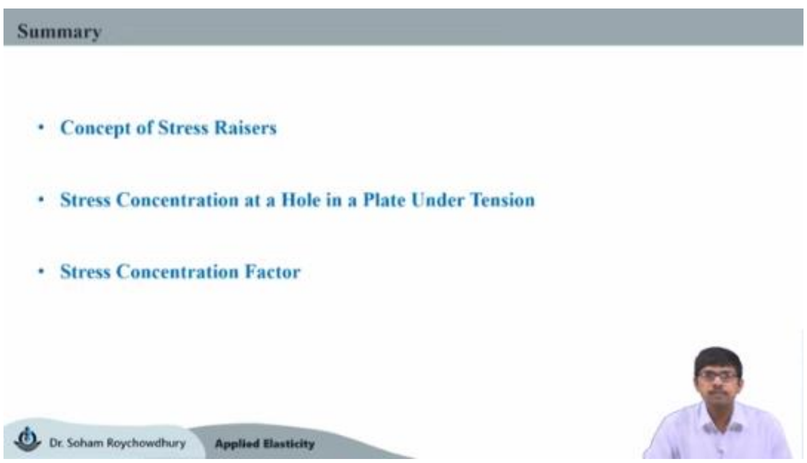
So, θ is measured from the positive x-axis. The positive x-axis is θ 0. So, if you consider $\theta = \frac{\pi}{2}$, that refers to this particular point. This point is $\frac{\pi}{2}$, and this particular point is $\frac{3\pi}{2}$. So, this refers to the topmost and bottommost points of the hole where the hole periphery boundary intersects with the plus or minus y-axis.

Those two are the critical points where the hoop stress on the hole is going to be maximum, for which $\cos 2\theta = -1$, and that maximum value of $\sigma_{\theta\theta}$ or hoop stress can be obtained like this, which is 3σ. So, at $\cos 2\theta = -1$, the second term would be -2σ times minus 1, that is plus 2 σ and adding the first σ term, the total $\sigma_{\theta\theta}|_{max} = 3\sigma$, which occurs at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, that is at the top or bottom point of the hole along the y-axis.

Now, if I plot this hoop stress, in the figure, now the hoop stress plot is there. You can see at the hole boundary, the hoop stress is maximum, which is equal to 3σ, whereas, at the far field, as you are going away from the hole boundary, at the far field, the hoop stress is equal to σ, that is, the given external tensile loading. So now, we define a quantity called stress concentration factor, which is defined as the maximum stress by nominal stress. Nominal stress means the average stress applied to the system, which was present in the absence of the hole, in the absence of the stress raiser. And maximum stress is the maximum stress generated in the same direction due to the presence of the hole or the stress raiser.

So here, the maximum stress is obtained as 3σ. If the hole was not there, then the entire plate was subjected to uniaxial tensile stress of sigma. So, the nominal stress is sigma. So, K_t can be obtained as $\frac{3\sigma}{\sigma}$, or 3. So, the stress concentration factor is 3. Hence, if the plate is subjected to uniaxial tensile loading

With a hole of intensity sigma, then around the hole boundary, the maximum stress generated would be 3 times the actual stress or nominal stress acting on the system. And this 3 is nothing but the stress concentration factor for the present problem.



So, in this lecture, we discussed the concept of stress raisers and then solved the stress concentration problem for a plate with a hole when it is subjected to uniaxial tensile loading. We also discussed or introduced the concept of stress concentration factor. Thank you.