APPLIED ELASTICITY

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Week 10

Lecture 51: Semi-infinite Domain Problems VI
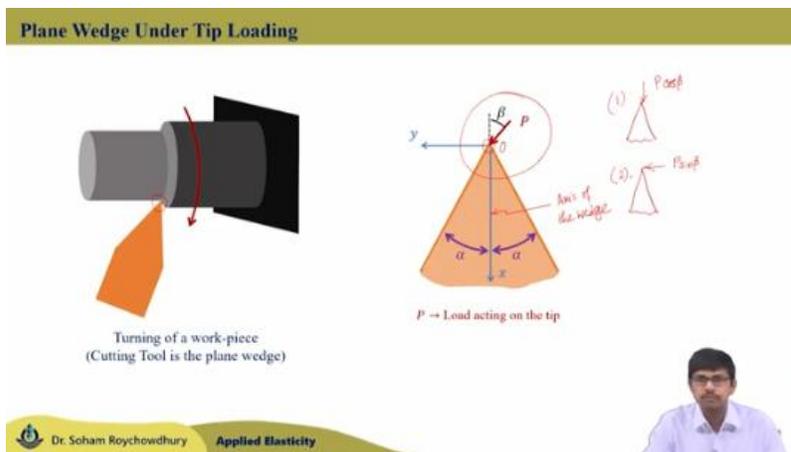

Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the topic of semi-infinite domain problems. We were discussing on the same topic for last couple of lectures which will be continued in this particular lecture as well.

In today's lecture, we are going to talk about the plane elastic wedge problems. As you know the plane elastic wedge can be considered as an elastic semi-infinite domain, where the span of that elastic domain is not π, which was the case of the elastic half space. The angular span or the range of θ was π or $180°$ for the case of elastic half space. For the case of elastic quarter plane, the angular range of θ was $\frac{\pi}{2}$. Now, for the wedge, this is just 2α. θ varies from $-\alpha$ to $+\alpha$, where we can say α is the semi-cone angle.

And often in different mechanical application, we can find this kind of elastic wedges are subjected to tip loading. This point, which is let us say point O, that is the tip of this elastic wedge, and this can be seen for different various applications, specially for the case of modeling of the cutting tools.

If you consider this left hand side figure, here, one cutting tool is shown which is turning one work piece in the lathe. While turning operation, this cutting tool, at the point of contact, that is, this point where the cutting operation is taking place, at that point, it will be in contact with the workpiece and thus, it would be subjected to a large amount of force. The cutting force will be imparted on the tool from the workpiece during the machining operation.

Due to that cutting operation, this load being applied on the cutting tool is basically a concentrated line loading, which is, let us say, having an intensity P. This is being applied near the tip of the wedge, which would cause heavy stress concentration, a very large value of stress around the tip of the tool. We need to have certain idea regarding the amount of stress developed in the tool, so that the failure of the tool can be avoided from excessive amount of cutting force coming at the tip during any of these machining operations Hence, we must know the stress distribution or stress field generated on these tools, which are being modeled as plane elastic wedge subjected to a tip load of magnitude P.

So, here we are considering one plane elastic wedge with the angle varying between $-\alpha$ to $+\alpha$. O is the tip point where the load P is acting. In general, P may be inclined to the axis of the wedge. This x-axis is called the axis of the wedge. Geometry is symmetric about this axis. Applied line loading at the tip may be along this axis, it may be

perpendicular to the axis, it may be inclined to the axis to any arbitrary angle β. One such case is shown here.

If we have an elastic weight subjected to tip loading P with an inclination angle β, that can be solved by solving the two individual problems and then superimposing those two solutions using the principle of superposition. What are those two individual problems? The plane elastic wedge subjected to axial loading. The load is applied along the axis of the wedge, and that load would be $P \cos \beta$. The second case may be the load applied in the direction which is transverse or perpendicular to the axis of the wedge, and let us say that equals $P \sin \beta$.

If we are able to solve these two problems, then we can superimpose those two solutions, and obtain the solution of this particular problem where the load is inclined to the axis of the wedge with an angle β. We will consider both of these two problems: the wedge subjected to axial load and the wedge subjected to transverse load perpendicular to the wedge axis, and we will try to find out the stress distribution.

Plane Wedge Under Axial Load at Tip

Following St. Venant's principle, a semi-circular region ABC is considered within which the effect of the applied normal loading on the stress distribution is dominant.

Boundary conditions: $\sigma_{\theta\theta}(r, \pm\alpha) = 0$ $\quad \tau_{r\theta}(r, \pm\alpha) = 0$

Vertical force balance results, $\int_{-\alpha}^{\alpha}(-\sigma_{rr}\cos\theta + \tau_{r\theta}\sin\theta)r d\theta = P = $ constant

To have constant P, σ_{rr} and $\tau_{r\theta}$ must be proportional to $^1/_r$, and thus stress function is chosen as

$$\phi(r,\theta) = r\Theta(\theta)$$

To satisfy biharmonic equation $\nabla^4\phi = 0$, $\phi(r,\theta)$ is chosen as

$$\phi(r,\theta) = r(A_1\cos\theta + B_1\sin\theta + C_1\theta\cos\theta + D_1\theta\sin\theta)$$

As the solution is required to be symmetrical about $\theta = 0$, it is required to enforce $B_1 = C_1 = 0$

$$\therefore \phi(r,\theta) = r(A_1\cos\theta + D_1\theta\sin\theta)$$

$P \rightarrow$ Axial load on the tip

$[-\alpha \leq \theta \leq \alpha]$

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First, starting with the plane elastic wedge subjected to the axial load at its tip. We are considering this wedge where the angular range of the wedge is defined from $-\alpha$ to $+\alpha$, α being the semi-cone angle. O is the tip at which load P is applied, which is along the axis of the wedge. The x-axis is the wedge axis.

As the load is applied at point O, following St. Venant's principle, the effect of the load or the stress distribution created by this load would be visible only near the tip point O. So,

let us consider a circular arc ABC within which the effect of the load is visible. Considering the polar coordinate (r, θ), where θ varies between $-\alpha$ to $+\alpha$, we have drawn one element where σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ stress fields are acting.

Coming to the boundary conditions, for this plane elastic wedge, there are two boundaries. One is $\theta = +\alpha$, and another is $\theta = -\alpha$. These two define our boundary, and both boundaries are free of any kind of normal or shear surface tractions. The load is acting only at the tip. This is a tip axial load acting on the wedge. Both wedge faces, $\theta = \pm\alpha$ planes, are free of any kind of surface traction. So, the boundary conditions are $\sigma_{\theta\theta}(r, \pm\alpha)$ is 0. Similarly, shear stress $\tau_{r\theta}(r, \pm\alpha)$ is 0. Both the normal stress and shear stress for $\theta = \pm\alpha$ planes, the two sides of the plane wedge, are equal to 0.

Coming to the force balance here, as the vertical force P is acting on the system, then for this region defined by ABC and O, the net force should be balanced. On the top, we have the downward load P acting along O to B, along the x-axis. Whereas, on the bottom, somewhere here, on a small element, we have two stress components. One is σ_{rr} at any angle θ, another is $\tau_{r\theta}$. If you are taking the component of $\tau_{r\theta}$ and σ_{rr} along the vertical direction. This angle being θ, σ_{rr} is having one downward component $\sigma_{rr} \cos\theta$, whereas $\tau_{r\theta}$ is having one vertically upward component, which is $\tau_{r\theta} \sin\theta$. On ABC, this circular arc region, we are having the upward force stress distribution $\tau_{r\theta} \sin\theta$ and downward stress distribution $\sigma_{rr} \cos\theta$.

Net stress acting along the x-axis in the vertical direction for the small element is equal to $-\sigma_{rr} \cos\theta + \tau_{r\theta} \sin\theta$ acting in the upward direction, As this is acting on the small element of $rd\theta$ length, the total force acting for the circular arc ABC would be integral of that for the total span of θ, which is $-\alpha$ to $+\alpha$. This total upward force acting on ABC region must balance the total downward force acting at point O, which is equal to P. This will satisfy the force balance but in the vertical direction for this plane elastic wedge within our region of interest, that is $OABC$, within which the effect of stress field will be visible. So, the final force balance equation becomes $P = \int_{-\alpha}^{+\alpha}(-\sigma_{rr} \cos\theta + \tau_{r\theta} \sin\theta)rd\theta$.

Now, as the applied P is a constant force and on the left hand side we have one r term present which is multiplied with both σ_{rr} and $\tau_{r\theta}$. Thus, this integral will result a constant term upon integration over θ from $-\alpha$ to $+\alpha$. Only these two stress components, σ_{rr} and $\tau_{r\theta}$, are proportional to $\frac{1}{r}$. This can be confirmed if the stress function ϕ is directly proportional to r.

If stress function is proportional to r, then the power of r in the stress components will be r^{-1} or $\frac{1}{r}$. So, we are choosing a stress function ϕ which is proportional to radial variable r and this is r times some Θ which is function of angular coordinate θ. Θ is the unknown function which we need to find out by using the biharmonic equation.

The biharmonic equation is $\nabla^4 \phi = 0$, and this chosen form of ϕ must satisfy the biharmonic equation. Substituting ϕ as $r\Theta$, the general solution for this Θ can be obtained like this. Thus, our stress function ϕ should be $r(A_1 \cos\theta + B_1 \sin\theta + C_1 \theta \cos\theta + D_1 \theta \sin\theta)$. We are having four terms.

If you try to recall one of our previous lectures for the elastic half-space subjected to concentrated normal loading, we had a similar form of stress function. Similar to that problem, this problem also has symmetry about the vertical axis O to B or about the x-axis. This is the axis of symmetry. Because the loading, the boundary conditions, and the geometry of the problem are all symmetric about this vertical axis, It is an axis of reflection symmetry, not rotational symmetry.

To ensure a symmetric solution about this axis of reflection symmetry, we must force two of the constants to be 0: B_1 and C_1, as these two terms would break that symmetry. For symmetry, at $-\theta$, $\sin -\theta$ should be the same as $\sin \theta$, but this is not true. We cannot have $\sin -\theta$ equal to $\sin \theta$. This is basically $-\sin\theta$. However, $\cos -\theta$ equals $\cos\theta$. So, the cos term is symmetric, but the sin term is not symmetric. All such asymmetric terms are dropped about $\theta = 0$ line, about the axis of reflection symmetry, and thus, B_1 and C_1 will vanish. We will have ϕ involving only two constants. ϕ would be $r(A_1 \cos\theta + D_1 \theta \sin\theta)$.

Plane Wedge Under Axial Load at Tip

$\phi(r, \theta) = r(A_1 \cos \theta + D_1 \theta \sin \theta)$

$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$
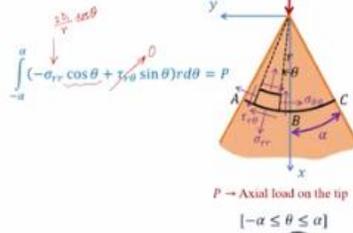
Stress components:

$\sigma_{rr} = \frac{2D_1 \cos \theta}{r} \qquad \sigma_{\theta\theta} = 0 \qquad \tau_{r\theta} = 0$

The vertical force balance results,

$\int_{-\alpha}^{\alpha}(-\sigma_{rr}\cos\theta + \tau_{r\theta}\sin\theta)r d\theta = P$

$P = \int_{-\alpha}^{\alpha}\left(-\frac{2D_1}{r}\cos^2\theta\right)r d\theta = -D_1\int_{-\alpha}^{\alpha}(1 + \cos 2\theta)d\theta$

$= -D_1\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\alpha}^{\alpha} = -D_1(2\alpha + \sin 2\alpha)$

$\Rightarrow D_1 = -\left(\frac{P}{2\alpha + \sin 2\alpha}\right)$

$P \rightarrow$ Axial load on the tip

$[-\alpha \le \theta \le \alpha]$

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Moving forward with this form of stress function, we can obtain the stress components σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ as this. σ_{rr} would be $\frac{2D_1 \cos \theta}{r}$, $\sigma_{\theta\theta}$ is 0, and $\tau_{r\theta}$ is 0. This gives rise to a purely radial stress distribution similar to the Flamant problem of an elastic half-space subjected to line loading.

Here, this unknown constant D_1 is required to be evaluated with the help of the vertical force balance. If we recall the vertical force balance, it was like this: $P = \int_{-\alpha}^{+\alpha}(-\sigma_{rr}\cos\theta + \tau_{r\theta}\sin\theta)r d\theta$. If I put the σ_{rr} and $\tau_{r\theta}$ expressions, as $\tau_{r\theta}$ is 0, the second term of this vertical force balance would go to 0, and here in σ_{rr}, we are replacing it with $\frac{2D_1 \cos \theta}{r}$.

If I do so, P would be written as $\int_{-\alpha}^{+\alpha}\left(-\frac{2D_1}{r}\cos^2\theta\right)r d\theta$. One $\cos\theta$ is coming from the σ_{rr} expression, one $\cos\theta$ was already there in the force balance equation, and thus we are getting a $\cos^2\theta$ term. If you look at this, σ_{rr} has a $\frac{1}{r}$ term and the integral already has an $r d\theta$ term. Thus, this r and $\frac{1}{r}$ would cancel, and hence, this integral is going to result in a constant term which equals P.

Writing this $2\cos^2\theta$ in terms of $1 + \cos 2\theta$, this integral P would be $-D_1\int_{-\alpha}^{+\alpha}(1 + \cos 2\theta)d\theta$, and now this is independent of r. Evaluating this integral and then substituting the limits of θ from $-\alpha$ to $+\alpha$, P would be obtained as $-D_1(2\alpha + \sin 2\alpha)$,

and hence D_1 would be $-\dfrac{P}{(2\alpha+\sin 2\alpha)}$. This is the only unknown constant present in the stress field for this plane elastic wedge subjected to the axial load at its tip, point O.
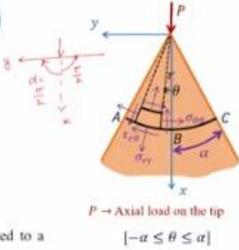

We can replace this D_1 back in the expression of σ_{rr} that is here and get the stress field. If I do so, substituting D_1 as $-\dfrac{P}{(2\alpha+\sin 2\alpha)}$ in the expression of σ_{rr}, the only non-zero radial stress field, we would be getting σ_{rr} as $-\dfrac{2P\cos\theta}{r(2\alpha+\sin 2\alpha)}$, and the other two stresses, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, are 0. Once again, this is a purely radial stress field.

This stress distribution can be reduced to the stress distribution of the Flamant problem, that is, a semi-infinite elastic half-space subjected to a normal line load of intensity P, if we take the value of α to be $\dfrac{\pi}{2}$ or $90°$. If I take the value of α to be $\dfrac{\pi}{2}$, this plane elastic wedge will have a geometry like this. This $\alpha = \dfrac{\pi}{2}$ and on this side also, it would be equal to $\dfrac{\pi}{2}$. So, θ varies between $-\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$.

With $\alpha = \dfrac{\pi}{2}$, this $\sin 2\alpha$ term would be $\sin\pi$ or 0, and hence this term would be 0, and this 2α would be π. So, for this particular case, σ_{rr} will be $-\dfrac{2P\cos\theta}{\pi r}$, and the rest, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, are obviously 0.

If you recall the Flamant solution of an elastic half-space subjected to a normal line loading of intensity P, this was the stress field: $-\dfrac{2P\cos\theta}{\pi r}$ was the radial stress, and the rest were 0. The solution of this plane elastic wedge subjected to an axial load at the tip can

be converged with the solution of the elastic half-space with replacing α by $\frac{\pi}{2}$. So, from wedge problem, we can always get the elastic half space solution back with a substitution of $\alpha = 90°$.

Plane Wedge Subjected to Tip Load Perpendicular to Wedge Axis

Boundary conditions: $\sigma_{\theta\theta}\left(r,\frac{\pi}{2}\pm\alpha\right)=0$ $\quad \tau_{r\theta}\left(r,\frac{\pi}{2}\pm\alpha\right)=0$

Horizontal force balance results, $\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\left(\sigma_{rr}\cos\theta-\tau_{r\theta}\sin\theta\right)rd\theta=-S$ $= Constant$

To have constant S, σ_{rr} and $\tau_{r\theta}$ must be proportional to $^1/_r$, and thus stress function is chosen as

$$\phi(r,\theta)=r\Theta(\theta)$$

To satisfy biharmonic equation $\nabla^4\phi=0$, $\phi(r,\theta)$ is chosen as

$$\phi(r,\theta)=r(A_1\cos\theta+B_1\sin\theta+C_1\theta\cos\theta+D_1\theta\sin\theta)$$

As the solution is required to be symmetrical about $\theta=0$, it is required to enforce $B_1=C_1=0$

$$\therefore \phi(r,\theta)=r(A_1\cos\theta+D_1\theta\sin\theta)$$

$S \to$ Transverse load on the tip

$\left[\left(\frac{\pi}{2}-\alpha\right)\leq\theta\leq\left(\frac{\pi}{2}+\alpha\right)\right]$

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After the plane wedge subjected to the axial tip loading, now we are going to consider a plane elastic wedge subjected to the tip loading which is transverse or perpendicular to the wedge axis as shown in this figure. Here, the shear load or transverse line load of intensity S is applied at the tip of the wedge at point O, which is acting along the y direction.

Similar to the previous case, we are considering one arc of ABC within which the effect of the stress distribution due to this load is visible, and beyond that, in the far field, we are not going to consider the effect of the stress due to S.

Here, we are choosing y-axis to be $\theta=0$ because $\theta=0$ line is normally chosen along the direction of application of the loading. As the load S is applied along y-axis, we are choosing that as our reference for the measurement of angle θ, with $\theta=0$ along y-axis. Here, this O to A plane is defined by $\theta=\frac{\pi}{2}-\alpha$ where α is the semi-cone angle for the wedge, and the ending plane O to C is defined by $\theta=\frac{\pi}{2}+\alpha$. Range of the angular variable θ is from $\frac{\pi}{2}-\alpha$ to $\frac{\pi}{2}+\alpha$.

Writing the boundary conditions on these two edges OA and OC, both are free of any kind of normal or shear stresses. $\tau_{r\theta}$, the shear stress, and $\sigma_{\theta\theta}$, the normal stress on both

OA and OC are 0, which are defined by $\theta = \frac{\pi}{2} \pm \alpha$ planes. So, $\sigma_{\theta\theta}\left(r, \frac{\pi}{2} \pm \alpha\right)$ is 0, and $\tau_{r\theta}\left(r, \frac{\pi}{2} \pm \alpha\right)$ is 0. Both the normal and shear stresses are 0 for the OA and OC edges of the plane wedge.

Now, for this case, we have to write the horizontal force balance equation because the load S is acting in the horizontal direction or along the y direction. Considering the force balance in the y direction because of the applied force S and the stresses generated on this circular arc ABC. We are considering this part of the wedge $OABC$, where S is applied on the tip along the positive y direction, and considering the normal stress σ_{rr} and $\tau_{r\theta}$. Here, the direction of $\tau_{r\theta}$ is towards the right (positive) because θ is measured from the y-axis, which is different from the previous case.

From this distribution, if you take the component of σ_{rr} and $\tau_{r\theta}$. θ is this angle here. So, horizontal component of σ_{rr} is $\sigma_{rr} \cos \theta$, and horizontal component of $\tau_{r\theta}$ (or $\tau_{\theta r}$) is $\tau_{r\theta} \sin \theta$. These two are acting in different directions, and in total, that is going to balance the applied load S.

If I write those components: $\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} (\sigma_{rr} \cos \theta - \tau_{r\theta} \sin \theta) r d\theta$. This gives us the net leftward force because $\sigma_{rr} \cos \theta$ is leftward. This should balance the applied force. The applied force being in the same leftward direction, we are adding the minus sign on the right-hand side. So, $\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} (\sigma_{rr} \cos \theta - \tau_{r\theta} \sin \theta) r d\theta = -S$, this is the force balance equation along the y direction or along the horizontal direction.

As S is a constant force, this force balance equation can give a constant upon integrating the left-hand side, only if these stress components, σ_{rr} and $\tau_{r\theta}$, are proportional to $\frac{1}{r}$. That $\frac{1}{r}$ will get cancelled with this r, and stress components to be proportional to $\frac{1}{r}$, stress function must be proportional to r. This is the same as the previous discussion of axial loading on the wedge. So, we are choosing the stress function ϕ as $r\Theta(\theta)$.

Θ can be obtained by using the biharmonic equation. And that would be having these four terms: $\Theta = A_1 \cos\theta + B_1 \sin\theta + C_1\theta \cos\theta + D_1\theta \sin\theta$. Using the symmetry of the problem about $\theta = 0$ line, which can be ensured by forcing B_1 and C_1 to be 0, the form of the stress function, which we will be using for this problem would come out to be: $\phi(r,\theta) = r(A_1 \cos\theta + D_1\theta \sin\theta)$.

Moving forward, using this, we can obtain the stress components in terms of this stress function as: $\sigma_{rr} = \frac{2D_1 \cos\theta}{r}$, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0. These stress components are exactly the same as the stress components of the previous problem.

Moving forward, using the horizontal force balance, that is $-S = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}(\sigma_{rr} \cos\theta - \tau_{r\theta} \sin\theta)rd\theta$. And then substituting this σ_{rr} and $\tau_{r\theta}$ in this equation, the second term would go to 0.

We will have $-S = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\left(\frac{2D_1}{r}\cos^2\theta\right)rd\theta$. From this, you can relate S with D_1 and obtain $-S$ as $D_1(2\alpha - \sin 2\alpha)$. Thus, D_1 would be $-\frac{S}{(2\alpha-\sin 2\alpha)}$.

$$\sigma_{rr} = \frac{2D_1 \cos\theta}{r} \qquad \sigma_{\theta\theta} = 0 \qquad \tau_{r\theta} = 0 \qquad D_1 = -\frac{S}{(2\alpha - \sin 2\alpha)}$$
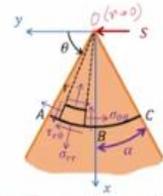
Stress fields:

$$\sigma_{rr} = -\frac{2S \cos\theta}{r(2\alpha - \sin 2\alpha)}$$

$$\sigma_{\theta\theta} = 0$$

$$\tau_{r\theta} = 0$$

With $\alpha = \pi/2$, this result can be used for a semi-infinite elastic half space subjected to a shear line loading.

$S \rightarrow$ Transverse load on the tip

$$\left[\left(\frac{\pi}{2} - \alpha\right) \le \theta \le \left(\frac{\pi}{2} + \alpha\right)\right]$$

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Substituting this expression of D_1 in the expression of ϕ and also in the expression of the stress function σ_{rr}, we would be getting the final stress fields like this. σ_{rr} is the only non-zero stress component. This is a purely radial stress distribution, where σ_{rr} is $-\frac{2S \cos\theta}{r(2\alpha - \sin 2\alpha)}$. The rest of the stress components are 0, and thus, it is a purely radial stress distribution.

Looking at this particular radial stress distribution field and comparing it with the previous field, where the wedge was subjected to the axial line loading at the tip, the stress fields are identical. Instead of P, here, we are just having S. But note that the measurement of θ is now changed. Earlier, $\theta = 0$ was along the x-axis; now, $\theta = 0$ is along the y-axis. With this change of θ, we can use the same form of solution. So, θ is always measured from the direction of the applied loading. The axis along which the load is applied is taken to be the $\theta = 0$ axis.

Now here also, at the tip of the wedge at point O, which refers to r tending to 0 point, that is a point of singularity because of the presence of the $\frac{1}{r}$ term in the σ_{rr} field. This stress distribution is singular at the tip of the wedge, and replacing $\alpha = \frac{\pi}{2}$, this result can be reduced to the result of the elastic half-space Flamant problem subjected to shear line loading of intensity S per unit length.

Similar to the normal load case, the results of the shear loading case on a plane wedge can be converted into the results or compared with the results of a half-space subjected to

shear line loading of intensity S if we substitute α with $\frac{\pi}{2}$, for which the range of θ would be from 0 to π.

In this lecture, we discussed or talked about two different elastic wedge problems. First, the wedge is subjected to the axial loading at the tip, and then it is subjected to transverse loading perpendicular to the axis of the wedge. And for both of these, we discussed or obtained the stress fields for this plane elastic wedge problem.

Thank you.