APPLIED ELASTICITY

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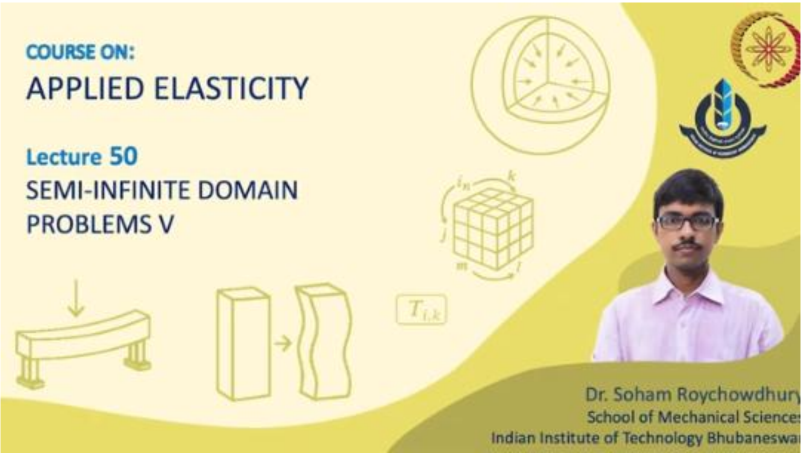
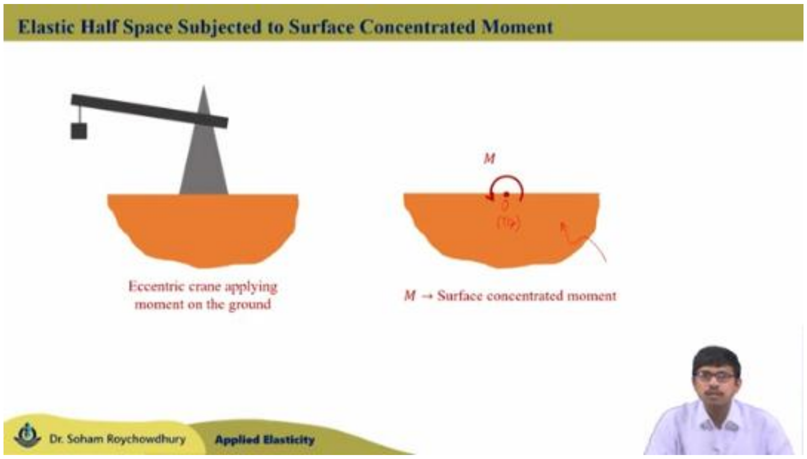
Week 10

Lecture 50: Semi-infinite Domain Problems V



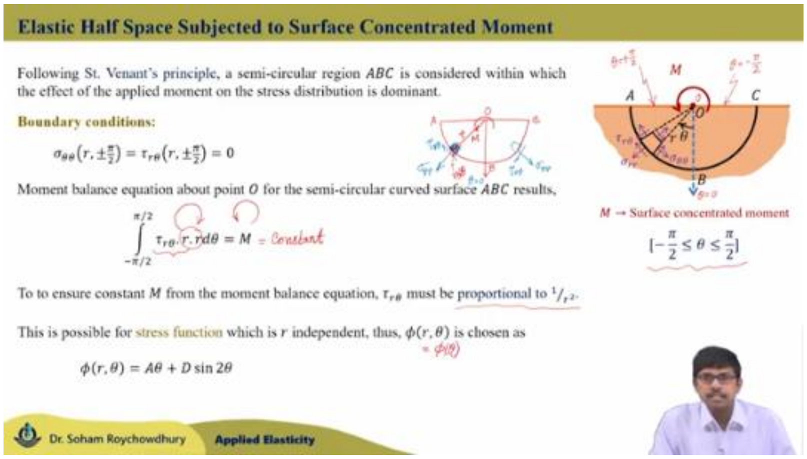
Welcome back to the course of applied elasticity. In this lecture, we are going to continue our discussion on the semi-infinite domain problems. In the last few lectures, we talked about the semi-infinite elastic domain problems when the semi-infinite elastic domain is subjected to different types of loading, such as concentrated normal line load, concentrated shear line load, and uniformly distributed line loading over a finite or a semi-infinite span. In this lecture, I am going to discuss the semi-infinite domain or elastic half-space subjected to a concentrated surface moment.



Earlier we talked about the deformation of the elastic half-space when it is subjected to either normal or shear or distributed pressure. Here, instead of the normal or transverse load, a surface moment is acting at a tip at a specific point on the free surface of the elastic half-space.

These kinds of problems are used to model the cases when we are applying some kind of load at the end of an eccentric crane, and where it is fixed to the ground, at that point, the ground will be experiencing some kind of moment. This can be effectively modeled as an elastic half-space problem where this elastic half-space is subjected to a tip moment at this point O. O is the tip point where a concentrated moment M is acting on the free surface. No shear or normal line loading is there.

Under the action of the surface concentrated moment, there should be a stress distribution generated at the base on the ground or on this elastic half-space. Our objective is to discuss that stress distribution for the semi-infinite elastic half-space when it is subjected to a surface concentrated moment of magnitude M.



Moving further, following the St. Venant's principle, which states the effect of the load is localized around the point of application of the load. Here, as M is applied at point O on the free surface of this elastic half-space, we will consider that the effect of stress distribution occurring due to this applied moment M is localized around a region near point O, which is bounded by the semicircular boundary defined by ABC. So, we are considering the semicircular boundary ABC, and following St. Venant's principle, the stress distribution caused due to surface moment M is restricted within the semicircular boundary ABC and the free surface.

Now, we need to write the moment balance equation. For that, let us use the polar coordinate, where r is the radial coordinate, theta is the circumferential or angular coordinate, and θ is measured from the mid-vertical line. This is the $\theta = 0$ line, O to B, from which θ is measured, varying from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. O to A, this is the $\theta = +\frac{\pi}{2}$ plane, whereas O to C is the $\theta = -\frac{\pi}{2}$ plane. So, the range of this angular variable θ is from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, which defines the entire elastic half-space.

Coming to the boundary conditions, there are two boundaries: one is between O to A, defined by $\theta = +\frac{\pi}{2}$, and another is O to C, defined by $\theta = -\frac{\pi}{2}$. Both these boundaries are free of any kind of normal or shear surface tractions. Thus, on these two theta planes with $\theta = \pm\frac{\pi}{2}$, we should have the normal stress component $\sigma_{\theta\theta}$ to be 0. We should also have the shear stress component $\tau_{r\theta}$ or $\tau_{\theta r}$ to be 0 for $\theta = \pm\frac{\pi}{2}$. This is the boundary condition which defines the free surface boundary of this elastic half-space.

Now, at point O, the external moment M is acting. Hence, considering this semicircular region, and then writing the moment balance equation. We are considering this semicircular region $OABC$, where at point O, this external bending moment M is acting. If you consider any point here. If you are considering a small element here at the boundary of this semicircular curve ABC, one σ_{rr} would be acting, which is the normal stress, along with that $\tau_{r\theta}$ would be acting, which is the shear stress on this semicircular boundary. As the geometry is symmetric, on this side also we will have σ_{rr} and then $\tau_{r\theta}$ acting.

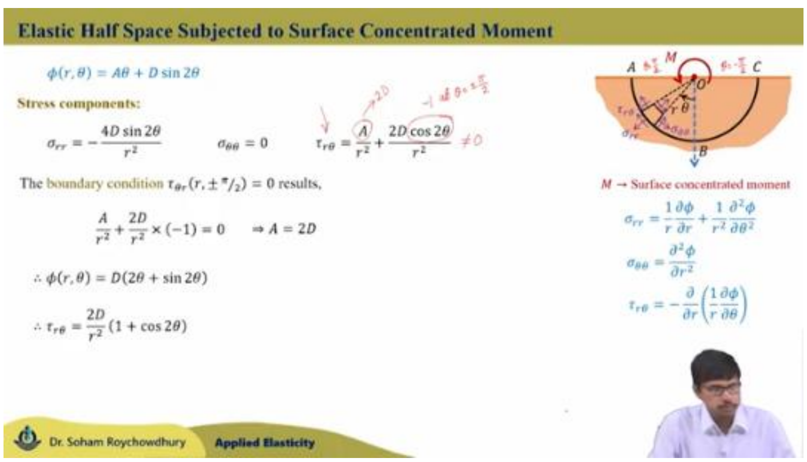
All these σ_{rr} components, if you are taking the moment of σ_{rr} about point O, they are not going to contribute anything. So, first of all, the body is symmetric about the vertical line, which is $\theta = 0$. Also, all the σ_{rr} passes through point O, so no moment can be caused by these normal σ_{rr} stress components. Only $\tau_{r\theta}$ would be causing a moment.

The moment created by $\tau_{r\theta}$ for this small element about point O is $\tau_{r\theta}r$, because r is the radius. The distance from here to here is r. So, $r\tau_{r\theta}$ is the moment created on the small element. Now, the length of this small element, this side length, is $rd\theta$. If I integrate this for $rd\theta$, with θ varying from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, I will get the total moment acting over the entire semicircular span ABC, and this is that moment: $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \tau_{r\theta} r^2 d\theta$.

Considering the direction of this $\tau_{r\theta}$, this moment acts in the clockwise direction, whereas the applied moment is in the anticlockwise direction, and these two should balance each other. Due to the application of this M, the stress distribution is generated, and in total, for the equilibrium of this semicircular region and the free surface, the elastic half-space, as shown in this figure, would be in equilibrium only if this counterclockwise moment given by M, the external moment, should balance the clockwise moment generated by the shear stress distribution $\tau_{r\theta}$ over the semicircular span. This is our moment balance equation, which is independent of σ_{rr}.

You can see the right-hand side of this equation is a constant quantity, a constant moment is applied. On the left-hand side, we are having a term $\tau_{r\theta}r^2$. Upon integration, this can only result in a constant if $\tau_{r\theta}$ is proportional to $\frac{1}{r^2}$. If $\tau_{r\theta}$ has a $\frac{1}{r^2}$ term, this r^2 will get cancelled with that $\frac{1}{r^2}$, and only then, this integral can result in a constant.

To have stress components proportional to $\frac{1}{r^2}$, the stress function ϕ must be independent of r because the power of r in the stress function is 2 orders higher than the power of r in the stress components, which we had discussed earlier. Hence, ϕ must be independent of r; ϕ would be a function of θ only. And that can be chosen as $A\theta + D\sin 2\theta$. Looking at the boundary condition, we are choosing ϕ as $A\theta + D\sin 2\theta$, which is independent of r. This form of choice would help us satisfy the boundary conditions.



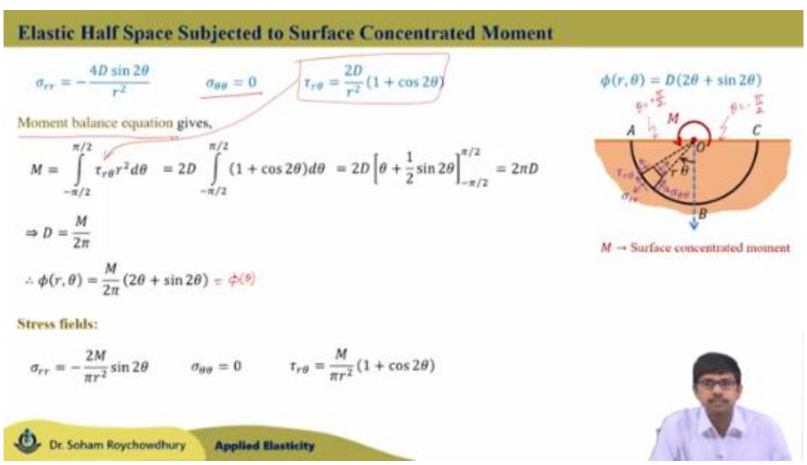
With this chosen form of ϕ, we can obtain the stress components σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$, two normal and one in-plane shear stress, as: $\sigma_{rr} = -\frac{4D\sin 2\theta}{r^2}$, $\sigma_{\theta\theta} = 0$, $\tau_{r\theta} = \frac{A}{r^2} + \frac{2D\cos 2\theta}{r^2}$.

If you recall the Flamant problem, which is an elastic half-space subjected to either normal or shear line loading, for that case, both $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ were 0; that was a purely radial stress distribution. However, here, when the elastic half-space is subjected to a surface concentrated moment, we are not getting a purely radial stress distribution because this $\tau_{r\theta}$, the shear stress, is non-zero. Both normal radial stress σ_{rr} and $\tau_{r\theta}$, which is the in-plane shear stress, would be present for this particular problem.

Coming to the boundary condition, we have two boundary conditions on the $\theta = +\frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ planes. On $\theta = \pm\frac{\pi}{2}$, the shear stress $\tau_{r\theta}$ must be 0. Substituting that in this $\tau_{r\theta}$ expression, the $\cos 2\theta$ term would be -1 at $\theta = \pm\frac{\pi}{2}$. 2θ is $+\pi$ or $-\pi$, cos of that equals -1, and from that, we can get a relation between these two constants A and D as $A = 2D$. Putting it back in the stress function, ϕ would be obtained as $D(2\theta + \sin 2\theta)$. This is the stress function with a single unknown D, which we need to evaluate by using the remaining boundary condition.

With this form of ϕ, we can also replace A in the $\tau_{r\theta}$ expression. This A is written as $2D$, and with that, the shear stress $\tau_{r\theta}$ can be written as $\frac{2D}{r^2}(1 + \cos 2\theta)$. This is the shear stress in terms of a single constant D.

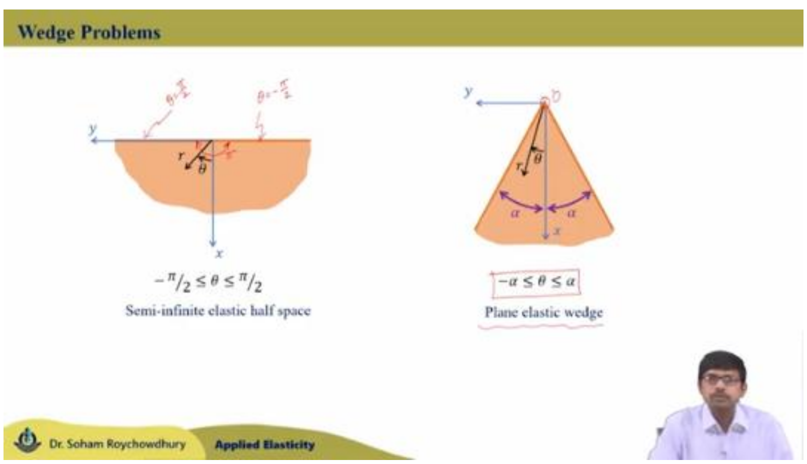
Coming to the second boundary condition, which was $\sigma_{\theta\theta} = 0$, that is automatically satisfied for both $\theta = +\frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ because $\sigma_{\theta\theta}$ itself is 0 for all values of θ. Hence, for finding D, we need to use the moment balance equation, which was $M = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \tau_{r\theta} r^2 d\theta$.



Substituting this expression of τ here in this equation, we will get M as $2D \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$. If we integrate the right-hand side with respect to θ, that would result in a simple π multiplied with the external constant $2D$. M will come out to be $2\pi D$, and thus the unknown constant D can be related to the applied moment as $D = \frac{M}{2\pi}$.

We can replace this D in the stress function, and in the stress distribution equation, and with that, the stress function, the final form of the stress function for this problem, would be $\frac{M}{2\pi}(2\theta + \sin 2\theta)$. This is the stress function, and note that this is a function of θ only, independent of r, which is necessary to get $\tau_{r\theta}$ to be proportional to $\frac{1}{r^2}$, which would be useful for satisfying the moment balance equation. And the stress fields: $\sigma_{rr} = -\frac{2M}{\pi r^2}\sin 2\theta$, $\sigma_{\theta\theta} = 0$, and $\tau_{r\theta} = \frac{M}{\pi r^2}(1 + \cos 2\theta)$.

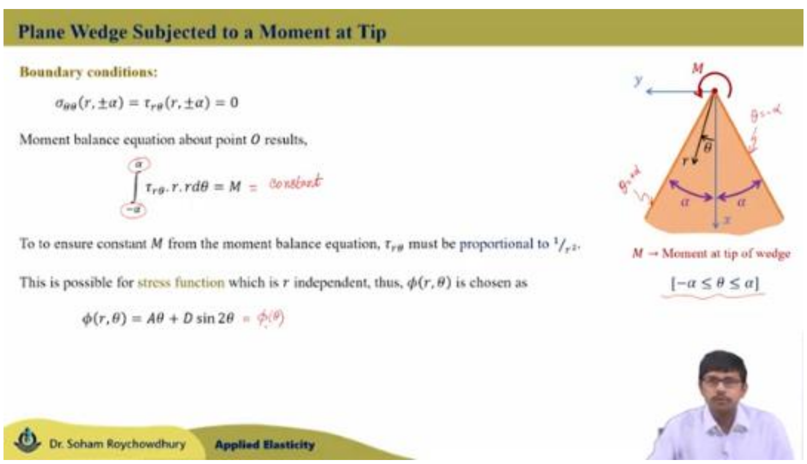
This is the complete solution of the elastic half-space problem when it is subjected to a concentrated moment, and this is the resulting state of stress, where both the radial normal stress and in-plane shear stresses are present. This is not a case of purely radial loading like the Flamant problem of an elastic half-space subjected to uniform line loading.



Moving forward, we will consider wedge problems. After this elastic half-space, wedge problems also fall under the category of semi-infinite domain problems for the elastic half-space. As shown in the figure, we are varying θ, the angle, from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$. This was the $\theta = +\frac{\pi}{2}$ plane. This was the $\theta = -\frac{\pi}{2}$ plane, which defines the boundary of the elastic half-space. Half of the region is semi-infinite elastic half-space, the remaining portion is free or void, from where we are applying the load.

Instead of this θ varying over a total angle of π. Here, the total range of θ is 180°, π, which is why we call it the elastic half-space. Instead of that, we may have a geometry like this, where θ varies between some $-\alpha$ to $+\alpha$, and such problems are called plane elastic wedge problems. Here, the elastic domain spans over $-\alpha$ to $+\alpha$, which is the range for θ. Within that, the elastic body exists and can be subjected to different types of external moments. We are considering this to be a planar problem, and these kinds of structures are called plane elastic wedges, the stress distribution within which can be analyzed for various types of loading using suitably chosen stress functions based on the boundary conditions.

We will once again solve this plane elastic wedge problem when it is subjected to a bending moment at its tip. The tip means the joining point of these two boundary lines, and α is the semi-cone angle for this elastic wedge.



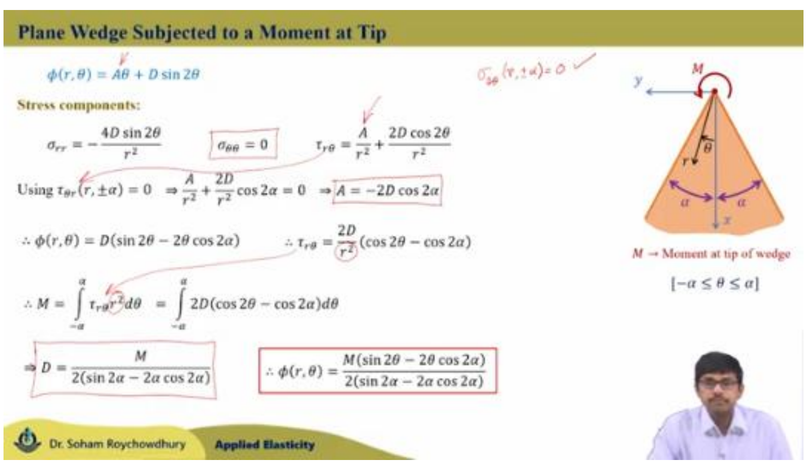
Considering a plane elastic wedge subjected to a tip moment M with θ varying between $-\alpha$ to $+\alpha$. Following a similar approach as the elastic half-space, the only difference is now the boundary conditions will be changed, and the range of θ would be changed.

Starting from the boundary conditions, here boundaries are defined as $\theta = +\alpha$ plane and $\theta = -\alpha$ plane, which are stress-free. Normal stress $\sigma_{\theta\theta}$ and shear stress $\tau_{r\theta}$ should be 0 for $\theta = \pm\alpha$. These are the traction-free boundary conditions for the elastic wedge, where α should be a known value from the geometry of the wedge.

The moment equation about point O would be exactly the same, apart from the limits of the integral. Previously, for the elastic half-space, the limits of this moment integral were the different range of θ, which was $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. Now, for the wedge problem, θ varies between $-\alpha$ to $+\alpha$. Thus, the moment would be $\int_{-\alpha}^{+\alpha} \tau_{r\theta} r^2 d\theta$. This is the only change in M.

Once again, $\tau_{r\theta}$ has to be proportional to $\frac{1}{r^2}$; only then, this integral will result in a constant bending moment M. To ensure that, we must have the stress function ϕ to be independent of r. So, the stress function ϕ is chosen as $A\theta + D\sin 2\theta$, which is only a function of θ and independent of r.

The choice of the stress function is the same because the nature of the loading is the same; only the boundary conditions and definition of the boundaries change from the previous elastic half-space problem subjected to a moment to this plane elastic edge subjected to a moment.



Moving forward, from this chosen form of ϕ, which is $A\theta + D\sin 2\theta$, we can obtain the stress components similar to the last case. σ_{rr} and $\tau_{r\theta}$ are the two non-zero stress components, whereas $\sigma_{\theta\theta}$ would be 0.

Using the boundary conditions, there were two boundary conditions: one was $\sigma_{\theta\theta}(r, \pm\alpha)$ is 0. Since $\sigma_{\theta\theta}$ is 0 for all values of θ, this condition is automatically satisfied. So, we need to enforce the second boundary condition, which is the shear stress $\tau_{r\theta} = 0$ for

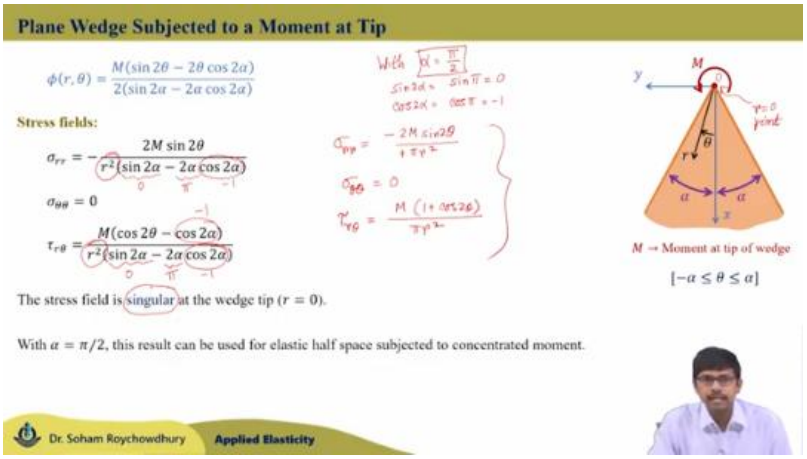
$\theta = \pm\alpha$. If you substitute $\tau_{r\theta}$ here in this boundary condition, we would get $\frac{A}{r^2} + \frac{2D}{r^2}\cos 2\alpha = 0$. Hence, a relation between the two constants A and D is obtained as $A = -2D\cos 2\alpha$.

Note that this relation for the elastic half-space was different, that was independent of α. Now, we are getting a relation between two constants where they are related through this semi-cone angle α. So, $A = -2D\cos 2\alpha$.

We can replace this A back in the expression of the stress function here and also in the expression of the shear stress τ. With that, the stress function would be obtained as $\phi = D(\sin 2\theta - 2\theta\cos 2\alpha)$, and the shear stress $\tau_{r\theta}$ would be $\frac{2D}{r^2}(\cos 2\theta - \cos 2\alpha)$. Now, we need to obtain D, the last unknown parameter, with the help of the moment balance equation.

The moment balance for this plane elastic wedge is M equals integral of $\tau_{r\theta}r^2 d\theta$, but the limits of the integrations are different from the elastic half-space; here, the integral would be from $-\alpha$ to $+\alpha$. Substituting this τ here, that is $\frac{2D}{r^2}(\cos 2\theta - \cos 2\alpha)$, this $\frac{1}{r^2}$ and the r^2 term in the integral would cancel each other. Thus, this would be just $\int_{-\alpha}^{+\alpha} 2D(\cos 2\theta - \cos 2\alpha)d\theta$. If I integrate this, then put the limits from $-\alpha$ to $+\alpha$, D can then be written in terms of the applied moment M as $D = \frac{M}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$. This is the constant D for the plane elastic wedge problem.

Moving forward, substituting this D in the stress function, the final form of the stress function would be like this: $\phi = \frac{M(\sin 2\theta - 2\theta\cos 2\alpha)}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$.



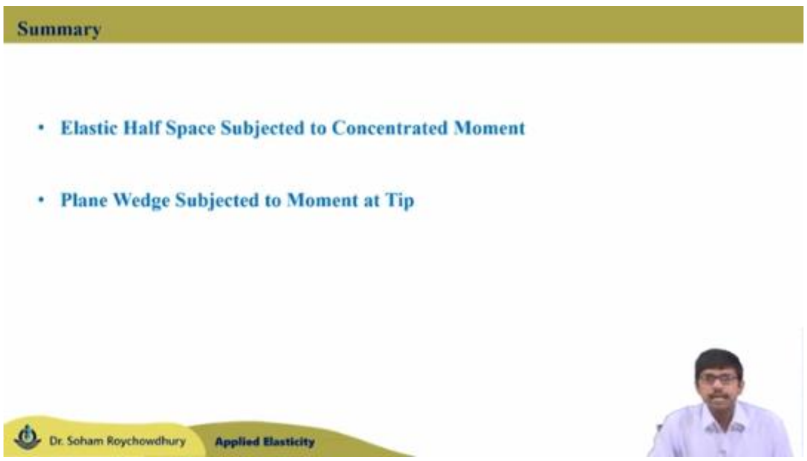
Using this final form of the stress function ϕ, we can obtain all three stress components, σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$, which would look like this. $\sigma_{\theta\theta}$ is 0, but radial stress σ_{rr} and in-plane shear stress $\tau_{r\theta}$ would exist for this plane elastic wedge when it is subjected to a tip moment M.

You can see that here, if we replace α with $\frac{\pi}{2}$, then these results can be matched with the results obtained for the elastic half-space. The elastic half-space is nothing but a plane wedge with the range of α varying from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. So, $\alpha = \frac{\pi}{2}$ for that case. If I put $\alpha = \pm\frac{\pi}{2}$, we should be able to get that.

With $\alpha = \frac{\pi}{2}$ for that case, $\sin 2\alpha$ would be equal to $\sin\pi$ which is 0. All these $\sin 2\alpha$ terms would go to 0 for the case of $\alpha = \frac{\pi}{2}$. Similarly, $\cos 2\alpha$ term would be $\cos\pi$, which would be -1. Hence, this $\cos 2\alpha$ term should be -1. Here also, this should be -1. Here also, this should be -1, and this 2α is nothing but π. If you replace this here, you can clearly see that the obtained stress function will be like this. σ_{rr} becomes $-\frac{2M\sin 2\theta}{\pi r^2}$. Then, $\sigma_{\theta\theta}$ is obviously 0 and $\tau_{r\theta}$ would be $\frac{M(1+\cos 2\theta)}{\pi r^2}$.

This would be the stress function obtained if I replace α with $\frac{\pi}{2}$, and these stress components would match with the case of elastic half-space subjected to the tip moment or surface concentrated moment of intensity M. So, if you are able to solve the plane elastic wedge problem, that result can always be extended to the elastic half-space problems, just by fixing the range of θ from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ or by setting $\alpha = \frac{\pi}{2}$.

Note one more thing that as this r^2 term is there in the denominator for both the non-zero stress components, thus, this solution has a singularity at the tip of the wedge, that is, at point O. This point O refers to $r = 0$, that is, a point of singularity where we will have infinite stress values for both radial stress σ_{rr} and shear stress $\tau_{r\theta}$. Hence, this solution has a singularity at that particular tip point.



In this lecture, we discussed the elastic half-space subjected to a surface concentrated moment. Then, we defined the plane elastic wedge problems and solved one plane elastic wedge problem subjected to a tip moment.

Thank you.