APPLIED ELASTICITY

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WEEK: 01
Lecture- 05

Welcome back to the lecture of applied elasticity. In the previous lecture we were talking about the tensor calculus. In this lecture also we will continue with the same topic of tensor calculus. So, in the tensor calculus last lecture we were discussing about the curl gradient divergence of vector and tensor quantities. Now, today first we are going to talk about the Laplacian.

Now, first Laplacian of a scalar field. So, how is it defined? for any scalar valued function $\phi(\tilde{x})$ the Laplacian is defined as Laplacian of ϕ equals to divergence of gradient of ϕ. So, if you are taking divergence of grad ϕ that is nothing, but the Laplacian of a scalar field where ϕ is a scalar. Now, we will try to express this Laplacian in terms of indicial notation.

from the definition of divergence of any vector field $\tilde{V}$ which we had discussed in the last lecture that is equals to trace of gradient of $\tilde{V}$. So, divergence of $\tilde{V}$ is defined as trace of gradient of $\tilde{V}$. Here $\tilde{V}$ is any vector and for the present problem for deriving the Laplacian of a scalar field we are interested to find out divergence of gradient of ϕ. Now, gradient of phi is a vector which is equivalent to $\tilde{V}$ for the present problem.

So, divergence of grad ϕ is nothing but trace of gradient of grad ϕ and gradient of ϕ can be written as ϕ i comma i. So, this we had discussed grad of ϕ is $\phi_{,i}$ that is nothing but $\partial\phi$ ∂x_i. ∂x_i. So, we are replacing this grad of ϕ as $\phi_{,i}$ then trace of gradient of $\phi_{,i}$ this will give rise to another partial derivative. So, this particular part the gradient of $\phi_{,i}$ is nothing, but $\phi_{,ij}$. So, this $\phi_{,ij}$ means we are having

two partial derivatives one with respect to x_i another is with respect to x_j $\partial^2\phi$ ∂x_i ∂x_j this is $\phi_{,ij}$. Now trace of this quantity means i and j both the indices will be kept same. So, thus the Laplacian of scalar function ϕ becomes $\phi_{,ii}$. So, this is the notation for expressing Laplacian of a scalar field. Now, moving to Laplacian of a vector field instead of ϕ now we are going to take a vector field $\tilde{V}$ which is function of position vector $\tilde{x}$ and Laplacian of this vector field $\tilde{V}$ is defined like this.

the Laplacian of $\tilde{V}$ is equals to gradient of divergence of $\tilde{V}$ minus curl of curl of $\tilde{V}$ through this identity we define the Laplacian of a vector field. Now, to simplify the right hand side we will be taking both the terms separately. First let us take the first term of the right hand side that is gradient of divergence of $\tilde{V}$. Now, divergence of $\tilde{V}$ can be written as $V_{j,j}$ where j is a dummy index divergence of $\tilde{V}$ is nothing, but a scalar and we are taking gradient of $V_{j,j}$ which is $V_{j,ij}$. So, gradient of any quantity is $\partial\ \partial x_i$ of that quantity partial derivative with respect to x_i.

So, here quantity on which we are using this gradient operator is $V_{j,j}$. So, its gradient is $V_{j,ij}$. Now, coming to second term of the right hand side that is curl of curl of $\tilde{V}$. Now, first expanding the external curl this one. So, we can say this is one complete vector and we are trying to expand the outer curl and curl of any vector can be written as minus e_{ijk} then that vector j comma k. So, here this is equals to minus e_{ijk} curl of $V_{j,k}$. Now, once again this part curl of V_j can be expanded with the help of another permutation symbol which I am writing as minus $e_{jmn}V_{m,n}$. So, already outside this minus e_{ijk} was present and then curl of $\tilde{V}$ is written as minus of $e_{jmn}V_{m,n}$ and we are taking this derivative comma k on the total quantity comma k is still remaining.

Now, if you cancel this negative sign. and simplify it it will be looking like e_{jkl} into $e_{jmn}V_{m,nk}$. Note that when you are taking the derivative of permutation symbol that will go to 0. So, comma k is only acting over $V_{m,n}$ and here we had also changed the location of the indices in the first permutation symbol here. Earlier this was e_{ijk} $i, j,$ and k were in cyclic order now I have rewritten it as e_{jki} where j, k, i are once again in cyclic order.

So, both of them are same now using the $e\ \delta$ identity the identity relating Kronecker delta and permutation symbol with that this first two terms the product of e_{jki} and e_{jmn}. can be written as $\delta_{km}\ \delta_{in}$ minus $\delta_{kn}\delta_{im}$ and entire thing is multiplied with the remaining term $V_{m,nk}$. Now, while simplifying this first $V_{m,nk}$ will be multiplied with this 2 Kronecker delta which will result m being replaced with k and n being replaced with i whereas, when you are multiplying $V_{m,nk}$ with the second Kronecker delta product n will be equals to k and m will be equals to i by definition of the product with Kronecker delta. And thus this will result the first term as $V_{k,ik}$ minus $V_{i,kk}$. So, this is the second term on the right hand side.

Now, in total if we combine both the terms and for that we are changing the name of the dummy index k to j. Now, combining both the terms that is this term this is first term on the right hand side and this term second term on the right hand side they are combined with a negative sign. So, Laplacian of $\tilde{V}$ $i\ th$ component of that is equals to $V_{j,ij}$ this one minus of this entire quantity that is minus $V_{j,ij}$ plus $V_{i,jj}$ and here this 2 terms will cancel each other and thus the ith component of Laplacian of $\tilde{V}$ Laplacian of a vector field is given as $V_{i,jj}$. So, $\partial^2 V_i\ \partial x_j^2$ this is equals to $\partial^2 V_i\ \partial$ So, $\partial^2 V_i\ \partial x_j^2$ this is equals to $\partial^2 V_i\ \partial x_j^2$.

Now, coming to few theorems related to the tensor calculus. The first one is the Green's theorem which in 2D takes a region R bounded by a closed curve C so as you can see in this figure we are considering this particular closed curve C which defines a bounded region R and within this bounded region a function P is defined to be a continuous function of both the independent variables x and y also we should have $\partial P\ \partial x$ and $\partial P\ \partial y$ the first order derivative of this function with respect to both the independent variables x and y to be continuous within this closed region R if that is so. at any point on this closed curve C, if you define $\tilde{n}$ vector to be the unit outward normal vector at any point on this closed boundary curve C, where n is defined as $n_x \tilde{e}_x$ plus $n_y \tilde{e}_y$, $n_x n_y$ are direction cosines for this unit normal n.

Now, with all this definition, the statement of the Green's theorem is like this. So boundary curve integral of the continuous function P multiplied with n_x integrated over the closed curve C is equals to $\partial P\ \partial x$ integrated over the total area of region R. Along with that the closed boundary curve integral of P and y is equals to area integral of $\partial P\ \partial y$ over the total closed region of R with area A. So, these are the two statements for Green's theorem in 2D. Now, if you are going for a three-dimensional problem instead of a planar region defined in xy plane, if you consider a closed volume capital omega bounded by a surface gamma in 3D and P and its derivatives P comma i are all continuous functions in the complete region.

In that case, the Green's theorem is like $P_{,i}$ derivative of the function P integrated over the entire region is equals to P dot n_i the dot product of P with the unit normal. into $d\Gamma$ integrated over the total boundary surface so with the help of green's theorem we can convert the volume integral to the boundary curve integral for any continuous function P provided its derivatives are also continuous now after green's theorem we are coming to gauss divergence theorem we are considering the similar region R bounded by a closed curve c where the volume of the region is capital omega and the boundary is surface boundary is defined as gamma for that if we choose any vector field $\tilde{v}$ the volume integral of divergence of $\tilde{v}$ over the total region is equals to surface integral of $\tilde{v}.\tilde{n}$ where $\tilde{n}$ is the unit normal defined as $n_x\tilde{e}_x$ plus $n_y\tilde{e}_y$ plus $n_z\tilde{e}_z$

this is the statement of divergence theorem for any vector v whereas instead of vector if you have any tensor $\tilde{\tilde{T}}$ in that case this Divergence theorem or Gauss theorem states that the volume integral of divergence of vector $\tilde{\tilde{T}}$ tensor $\tilde{\tilde{T}}$ is equals to $\tilde{\tilde{T}}$ acting over $\tilde{n}$, $\tilde{\tilde{T}}$ acting over the unit normal integrated over the entire closed boundary surface gamma. so in the indicial notation first the vector relation can be written as $v_{i,i}$ integrated over entire volume dV is equals to $v_j n_j$ this $v_j n_j$ is $\tilde{v}.\tilde{n}$ integrated over the surface boundary $d\Gamma$ whereas for the tensor field the indicial notation is like this $T_{ij,j}$ that is nothing but $\mathrm{div}\left(\tilde{\tilde{T}}\right)$ in the indicial form integral of that over the total volume is equals to $T_{ik}n_k$ integrated over the total surface gamma so these are the two statements for divergence theorem one is for any vector field $\tilde{v}$ another is for tensor field $\tilde{\tilde{T}}$ now coming to the next one that is stokes theorem

Now here Γ is considered to be an open two-sided surface bounded by a piecewise continuous curve C which is a closed curve. So here we are considering this closed curve C and that defines this surface Γ which is a open two-sided surface. Now $\tilde{v}$ being a vector field which is continuous within Γ its derivative is also continuous within Γ. So, $\tilde{t}$ defines the tangent at any point on the closed curve C and $\tilde{n}$ defines the unit normal at any point on the surface Γ. So, $\tilde{n}$ is the unit normal at any point on the surface Γ whereas,

For such cases with $\tilde{v}$ being a vector field which is continuous along with its first derivative being continuous in Γ the stokes theorem states that the closed curve integral $\tilde{v}$ dot $\tilde{t}ds$ integrated over closed curve C is equals to curl of $\tilde{v}$ dot $\tilde{n}$ integrated over the boundary closed surface gamma. So, $\tilde{v}$ dot $\tilde{t}ds$ integrated over closed C curve C is equals to double integral over gamma curl of $\tilde{v}$ dot $\tilde{n}$ this is the statement of stokes theorem. Using all these theorems stokes theorem greens theorem as well as divergence or gauss theorem with the help of that these theorems it is possible to convert the volume integral into surface integrals or area integrals it is possible to convert the surface integrals into line integrals for ah which are required for simplification of several tensorial expressions conveniently. Now after this we will be moving to few example problems dealing with the tensor calculus.

So, first problem is to show this particular identity to prove this. So, here the given is left hand side is $\tilde{v} \times (\tilde{\nabla} \times \tilde{v})$. We need to show that this is equals to half of gradient of $\tilde{v}.\tilde{v}$ minus $\tilde{v}.\tilde{\nabla}$ acting over $\tilde{v}$. Now, first taking the left hand side there are two vectors whose cross product is taken the first vector is $\tilde{v}$ second vector is curl of $\tilde{v}$. So cross product of any vector any two vector can be written as $e_{ijk}v_j$ and $\left(\tilde{\nabla} \times v\right)_k$ where curl of $\tilde{v}$ is the second vector now curl of $\tilde{v}$

can be further simplified with the help of permutation symbol as minus $e_{kmn}v_{m,n}$. So, here two permutation symbols we are having external one is e_{ijk} internal one is minus e_{kmn} and due to two cross products one is outside curl another is within curl these two permutation symbols are coming. Now, for simplifying this we need to use the $e - \delta$ identity and if you recall in the $e - \delta$ identity the first subscript for both the permutation symbols are same, but here these two are not same. The first subscript is i for the first permutation symbol, first subscript is k for the second permutation symbol. So, we need to go for some kind of manipulation with this

dummy indices with which we can match the first subscript of both the permutation symbol in order to use the $e - \delta$ identity. Now, first this is expanded minus $e_{ijk}\ e_{kmn}v_jv_{m,n}$. Now, I am rewriting this as second term e_{kmn} remains as it is whereas, the first term e_{ijk} this is written as e_{kij} both of them are same because i, j, k are in cyclic order in both of them. So, both of them are equals to 1. Now using the e delta identity this $e_{kij}e_{kmn}$ can be written as $\delta_{im}\delta_{jn}$ minus $\delta_{in}\delta_{jm}$. An entire thing is having a negative sign and it is post multiplied by $v_j.v_{m,n}$.

Now, by taking the product of 2 Kronecker delta for each of the term, we can get this as minus $v_jv_{i,j}$ plus $v_jv_{j,i}$, this is the left hand side. Now, coming to the right hand side, the first term of right hand side is half of gradient of $\tilde{v}.\tilde{v}$. Now, gradient of any quantity any scalar is basically that scalar comma i. So, gradient operator is written as $(\tilde{v}.\tilde{v})_{,i}$. Now, $\tilde{v}.\tilde{v}$ is nothing, but $v_j v_j$ dot product of 2 vectors j is a domain index and comma i means we are going to take the derivative with respect to $\tilde{x}_i$. So,

here we are applying the chain rule of differentiation we are taking derivative of first v j once and keeping one v j constant then we are taking derivative of second v_j keeping another v_j constant and you can see both the terms are same. So, then in that case half will get cancelled and it will become v_j v_j comma i. Now coming to second term on the right hand side $\tilde{v}.\tilde{\nabla}$ operator acting over $\tilde{v}$. Now $\tilde{v}$ dot delta is v_k ∂ ∂x_k gradient means ∂ ∂x_k operator that is acting over vi. Now this is equals to vk vi comma k this del del xk is acting over v_i and that is resulting $v_{i,k}$. Now if you

change this k to j name of the dummy index is changed this term is $v_j v_{i,j}$. Now, as right hand side is equals to the first term $v_j v_{j,i}$ minus the second term $v_j v_{i,j}$. If you do so, you will be getting exactly same as the left hand side term thus the Greven identity is proved. Now, moving to the next example, show that curl of gradient of ϕ, ϕ is a scalar that is equals to 0. Now, curl of any vector, gradient of ϕ is a vector, ϕ being a scalar grad phi will be a vector.

So, curl of any vector is defined as $-e_{ijk}$, then that vector j comma k and here that vector is grad of ϕ. Now, by definition of grad of ϕ this is equals to $\phi_{,j}$ thus $(\tilde{\nabla}\phi)_{j,k}$ becomes $\phi_{,jk}$ second order partial derivative of ϕ with respect to $\tilde{x}_j$ and $\tilde{x}_k$. So, $-e_{ijk}\phi_{,jk}$. Now, we are interchanging the dummy indices i and j just changing j to k and k to j. Interchanging these two it will become minus e_{ikj} $\phi_{,kj}$. Now this $\phi_{,kj}$ refers to partial derivative with respect to $\tilde{x}_k$ and with respect to $\tilde{x}_j$.

we can always interchange the order of the partial derivatives. So, instead of $\phi_{,kj}$ we can write this term as $\phi_{,jk}$ without having any other change in the remaining terms this is just by interchanging the order of the partial derivative. So, $-e_{ikj}\phi_{,jk}$ and then e_{ijk} being equals to minus of e_{ikj} we can write this term as $e_{ijk}\phi_{,jk}$. Now, if you compare this particular term $e_{ijk}\phi_{,jk}$ and this which is $-e_{ijk}\phi_{,jk}$. So, some minus of some quantity a is equals to same quantity and this can be true only if a equals to 0. Thus, curl of gradient of ϕ has to be 0. If you are asked to prove any such ah indicial notation which contains something equals to 0, there you need to prove that the indicial form of expression is equals to minus of the same form which will eventually result the quantity going to 0. Now, coming to next example $\text{div}(\tilde{u} \times \tilde{v})$ is required to be shown to $(\tilde{\nabla} \times \tilde{u}).\tilde{v} - \tilde{u}.(\tilde{\nabla} \times \tilde{v})$.

Now, starting with the left hand side $\text{div}(\tilde{u} \times \tilde{v})$ is $(\tilde{u} \times \tilde{v})_{i,i}$ divergence of any vector is vector i comma i. Now, $\tilde{u} \times \tilde{v}$ can be written with the help of permutation symbol as $e_{ijk}u_j v_k$, but due to presence of this comma i. here we are having that comma i acting over u_j and v_k. So, no need to take the differentiation with respect to $\tilde{x}_i$ for the permutation symbol that we are taking out. Now, if you expand it using the chain rule once the derivative with respect to i is taken on u j keeping v_k as it is and next the differentiation with respect to i is taken for v k keeping u_j as it is this is the left hand side expression. Now, coming to right hand side if you take the first term of RHS curl of $\tilde{u}$ dot $\tilde{v}$. So, dot product of any two vector can be written as the

first vector k component into second vector k component. So, I am writing this as curl of $u_k v_k$. Now, why to choose k as a dummy index? Why not i and j? That you need to choose very carefully, so that we can directly end up with the same expression of left hand side. If you carefully look at the first term of the left hand side, you are having a v_k term here.

Now, here this term dot $\tilde{v}$. on the first term of RHS will result v_k only if you choose the name of this 2 dummy indices to be k if you are choosing i later you have to go for replacement of the name of the dummy index to avoid that from beginning only you can carefully choose these names to be same as the name which is already appearing in the left hand side equation so curl of $u_k v_k$ and simplifying this curl of $\tilde{u}$ this will become minus $e_{kmn}u_{m,n}v_k$ and rewriting m with i and n with j this is equals to minus $e_{kij}u_{i,j}v_k$ using the property of the permutation symbol and interchanging the indices i and j this can be shown to be equals to $e_{kij}u_{i,j}v_k$.

So, you can see this term is same as the first term of the left hand side. Similarly, the second term on the right hand side $\tilde{u}\,.\,(\tilde{\nabla} \times \tilde{v})$ is written as $u_j(\tilde{\nabla} \times \tilde{v})_j$. expanding $(\tilde{\nabla} \times \tilde{v})_j$ as $e_{jab}v_{a,b}$ and that is multiplied with $-u_j$ and then doing the similar steps following the similar steps as the previous one this can be shown to be minus. $e_{ijk}u_j v_{k,i}$ you can see this term is same as the second term of the left hand side just the sign is different. Thus $\mathrm{div}(\tilde{u} \times \tilde{v})$ is equals to $(\tilde{\nabla} \times \tilde{u}).\tilde{v}$ minus $\tilde{u}.(\tilde{\nabla} \times \tilde{v})$ this identity is proved.

Now coming to the next example here we need to show $\mathrm{div}(\tilde{\nabla}\tilde{u})^T$ is equals to $\tilde{\nabla}[\mathrm{div}(\tilde{u})]$. Taking the left hand side divergence of $(\tilde{\nabla}\tilde{u})^T$ $(\tilde{\nabla}\tilde{u})^T$is a second order tensor. Divergence of any second order tensor $\tilde{\tilde{T}}$ can be written as $T_{ij,j}$. So, here we are writing this divergence of gradient of u transpose as $(\tilde{\nabla}\tilde{u})^T_{ij,j}$. Now, gradient of u_{ij}, this is equals to $u_{i,j}$ and as we are having a transpose here that transpose is also added.

Now, if we remove the transpose sign this i and j operate indices will be flipped. So, this is equals to $u_{j,ij}$ entire thing comma j. So, both the commas can be included in a single partial derivative with two different indices i and j thus left hand side becomes $u_{j,ij}$. Now, coming to the right hand side the $\tilde{\nabla}(\mathrm{div}(\tilde{u}))$ is equals to $\tilde{\nabla}(u_{j,j})$ as $\mathrm{div}(\tilde{u})$ is equals to $u_{j,j}$. And by applying the gradient operator $(u_{j,j})_{,i}$ will be coming which is same as $u_{j,ij}$.

So, you can clearly see both of these two terms the left hand side and right hand side are identical. So, thus this identity is proved. So, in total in this particular lecture we had introduced the concept of Laplacian of scalar and vector fields which was followed by the introduction to various theorems such as Green's theorem, Gauss theorem and Stokes theorem and finally, we had solved few example problems which are manipulation of the identities involving tensor calculus.

Thank you.