Week 10

Lecture 49: Semi-infinite Domain Problems IV

Welcome back to the course on Applied Elasticity. In today's lecture, we will continue our discussion on semi-infinite domain problems. In the last lecture, we discussed the deformation or deflection of the elastic half-space when subjected to a normal line loading of intensity W. Now, we can consider an elastic half-space or a semi-infinite body that may be subjected to various types of loading.

Axial line load at tip

Shear line load at tip

Semi-infinite span distributed loading

Finite span distributed loading

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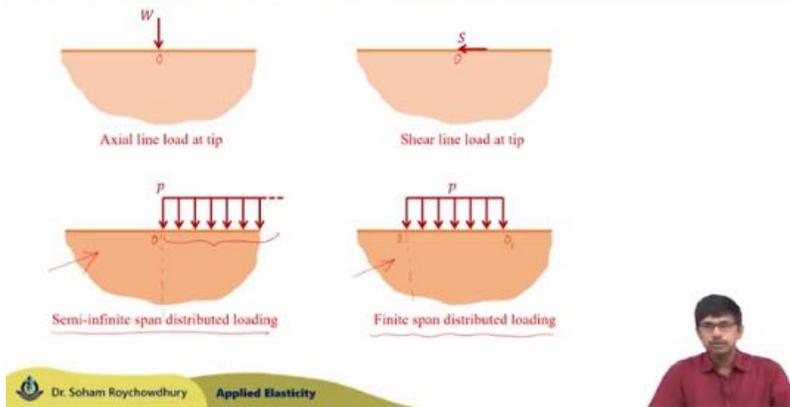
We considered two such line loading conditions: one being axial or normal line load at the tip, and another being shear line load at the tip at one particular point. Moving further, instead of the line loading acting at a specific point O, which we define as the tip or origin, apart from that, we may have distributed line loading acting on the semi-infinite body or elastic half-space. The span of that distributed normal line loading may be finite or semi-infinite as well.

If you consider the first case, the semi-infinite span distributed line loading, here we define our tip or origin at point O. Starting from that point, on one particular side for the entire elastic half-space, the normal distributed loading with intensity p, is acting, whereas for the second case, let us say this point is O. Starting from that point till another point O_1, for this finite span, the distributed normal line loading of intensity p is acting. So, the elastic half-space may be subjected to distributed normal loading over a semi-infinite span starting from the origin, the entire side is subjected to distributed line loading, as shown in these examples, or it may be subjected to normal distributed line loading of intensity p over a finite span from O to O_1, as shown in this figure.

Elastic Half Space under Uniform Distributed Loading Acting Over Semi-infinite Span

p: Uniform normal stress intensity for $y < 0$

Boundary conditions:

Along +ve y axis ($\theta = 0$): $\quad \sigma_{\theta\theta}(r,0) = \tau_{\theta r}(r,0) = 0$

Along −ve y axis ($\theta = \pi$): $\quad \sigma_{\theta\theta}(r,\pi) = -p, \quad \tau_{\theta r}(r,\pi) = 0$

To satisfy these boundary conditions, $\sigma_{\theta\theta}$ and $\tau_{\theta r}$ are required to be function of θ only (independent of r).

This can be ensured if $\phi(r,\theta)$ has only r^2 terms for radial variable r.

Choice of stress function:

$$\phi(r,\theta) = a_6 r^2 \theta + b_{21} r^2 \sin 2\theta$$

[From general Michell solution in which $\sin\theta$ and $\cos\theta$ terms do not have r^2 term, and $\cos 2\theta$ term would not satisfy the present boundary conditions]

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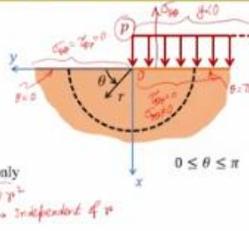
In today's lecture, we will discuss the solution of the Flamant problem, when it is subjected to distributed normal loading for semi-infinite span and also for finite span. First, let us consider the elastic half-space subjected to normal distributed loading acting over a semi-infinite span, that is, the first case.

Here, the elastic half-space is considered and starting from the origin O, the normal distributed load of intensity p is acting. I have marked the x and y axes. The x-axis is vertically downward, and the y-axis is the horizontal axis towards the left; it is positive. So, this entire region for which the distributed normal load is acting is for the negative y-axis. For $y < 0$, this distributed uniform normal loading of intensity p is acting over the elastic half-space.

θ is chosen to be measured from the positive y-axis in the counter-clockwise direction. Hence, the range of θ is defined from 0 to π. This particular positive y-axis refers to $\theta = 0$, and the negative y-axis refers to $\theta = \pi$. If I try to write the boundary condition on the surface, so here the surface is no more free; some part of the surface, that is the negative y-axis, is subjected to compressive loading of intensity p, while the positive y-axis, half of the surface, is free of any kind of normal or shear loading.

If you write the boundary condition for the positive y-axis, which is defined by $\theta = 0$, for this region, $\sigma_{\theta\theta}$ is 0, and $\tau_{\theta r}$ is also 0, because no normal traction, and no shear traction is acting on the positive y-axis part, $i.e.$, $\theta = 0$. So, $\sigma_{\theta\theta}(r,0)$ is 0, and $\tau_{\theta r}(r,0)$ is also 0.

Now, if you look at the negative y-axis, that is $\theta = \pi$, the right half of the problem, for that, shear traction $\tau_{\theta r}$ is 0. Shear traction, $\tau_{\theta r}$, for this region is 0, but $\sigma_{\theta\theta}$ is non-zero due to presence of this load p. So, $\tau_{\theta r}(r, \pi)$ is 0, but $\sigma_{\theta\theta}(r, \pi)$ is $-p$. Now, why minus? Because the applied load p is compressive, acting downward, and on this plane, $\sigma_{\theta\theta}$ direction by sign convention is upward positive. As the directions are different, so $\sigma_{\theta\theta}(r, \pi)$ should be equal to $-p$.

Now, if you look at the boundary condition, all these surface tractions are constant. They are either 0, three of them are 0 and one is non-zero, but a constant because p is a constant. If we want to have the constant stress, $\sigma_{\theta\theta}, \tau_{\theta r}$ must be constant or function of θ. They should not be function of r, as we want to make it independent of r. Stress components to be independent of r for such polar coordinate problems, we can have the stress function ϕ to be the function of or proportional to r^2.

If ϕ is proportional to r^2, only then, the σ_{ij} components would be independent of r, because the power of r in σ_{ij} is normally two orders less than the order of r in ϕ, by using the definition of stress components in terms of the partial derivatives of ϕ. Thus, to ensure constant σ components with respect to r, they may be dependent on θ, but there should not be any variation of $\sigma_{\theta\theta}$ and $\tau_{\theta r}$ with respect to r, only then, these boundary conditions can be satisfied. This can be ensured only if $\phi(r, \theta)$ has r^2 terms. No other function of radial variable r should be there in the stress function.

Hence, we choose our stress function like this, which is chosen from the general Michell solution. If you look at the Michell solution, which was discussed earlier, that had some terms, which were independent of θ, then θ-dependent term, then $\sin\theta$ term, $\cos\theta$ term, then series with $\sin n\theta$ and $\cos n\theta$. From all these terms, we are choosing the terms containing r^2.

From the terms without the sin or cos, these first terms are taken as $a_6 r^2 \theta$. Then, in the Michell solution, in $\sin\theta$ and $\cos\theta$ coefficients, no r^2 terms were there. So, no $\sin\theta$ or $\cos\theta$ term can be there. Then, considering n up to 2, we are choosing only one solution from that summation, which is $\sin 2\theta$ term multiplied with $b_{21} r^2$. From $\cos 2\theta$, we can choose one more $a_{21} r^2 \cos 2\theta$, but that term is not chosen here. If you choose that, later

we will see that to enforce these boundary conditions, we must force that constant to 0. $\cos 2\theta$ term, which has r^2 in its coefficient, cannot satisfy the present form of boundary condition and thus is not included in the stress function. So, we are choosing our stress function with these two terms: $a_6 r^2\theta + b_{21} r^2 \sin 2\theta$.

Moving forward, with respect to this stress function, the stress components σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ for the present problem can be obtained like this: σ_{rr} is $2a_6\theta - 2b_{21}\sin 2\theta$, $\sigma_{\theta\theta}$ is $2a_6\theta + 2b_{21}\sin 2\theta$, and $\tau_{r\theta}$ is $-a_6 - 2b_{21}\cos 2\theta$. These set of stress components, obtained with the help of the relation between stress function and stress components must satisfy all four boundary conditions.

If you look at the boundary condition, these were four boundary conditions. For the positive y-axis, that is $\theta = 0$, we have $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ to be 0. For the negative y-axis defined by $\theta = \pi$, we have $\sigma_{\theta\theta}$ to be $-p$ and $\tau_{r\theta}$ to be 0.

We know that the expression of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ has already been derived as a function of θ, and they are independent of r, which was our requirement. For the present form of ϕ, they are giving the stress field independent of r; thus, using those, for different values of θ, we can get these boundary conditions satisfied.

Let us substitute this $\sigma_{\theta\theta}$ with $\theta = 0$ in the first boundary condition. For $\theta = 0$, both the θ term and the $\sin 2\theta$ term in $\sigma_{\theta\theta}$ would go to 0, and thus the first condition is automatically satisfied.

Coming to the second condition, substituting $\theta = 0$ in $\tau_{r\theta}$ or $\tau_{\theta r}$, you would get $-a_6 - 2b_{21}\cos 0$. So, this would be $-a_6 - 2b_{21} = 0$, and through this, we can relate a_6 and b_{21} as $a_6 = -2b_{21}$.

From the third boundary condition, $\sigma_{\theta\theta}$ at $\theta = \pi$ is $-p$. Substituting $\theta = \pi$ in the $\sigma_{\theta\theta}$ expression, it would be $2a_6\pi + 2b_{21}\sin 2\pi$. Since $\sin 2\pi$ is 0, this term goes to 0, and this should equal $-p$. From that, we can get a_6 to be $-\frac{p}{2\pi}$. So, one of the constants we are able to get from this.

The last boundary condition, $\tau_{\theta r}$ at $\theta = \pi$, would also give us $-a_6 - 2b_{21} = 0$, or $a_6 = -2b_{21}$. So, you can see the second and fourth, these two boundary conditions, are giving us the same term. Substituting a_6 here, we can obtain b_{21} as $\frac{p}{4\pi}$. Both constants, which were present in the stress function, a_6 and b_{21}, we are able to obtain both of them.

Hence, substituting those in the stress function, $\phi(r,\theta)$ becomes this. ϕ is $-\frac{pr^2}{2\pi}\left(\theta - \frac{\sin 2\theta}{2}\right)$. This stress function is suitable for solving this problem where an elastic half-space is subjected to distributed normal line loading over a semi-infinite span.

Elastic Half Space under Uniform Distributed Loading Acting Over Semi-infinite Span

Stress function:
$$\phi(r,\theta) = -\frac{pr^2}{2\pi}\left(\theta - \frac{\sin 2\theta}{2}\right)$$

Stress fields:
$$\sigma_{rr} = -\frac{p}{2\pi}(2\theta + \sin 2\theta)$$
$$\sigma_{\theta\theta} = \frac{p}{2\pi}(-2\theta + \sin 2\theta)$$
$$\tau_{r\theta} = \frac{p}{2\pi}(1 - \cos 2\theta)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$
$$\tau_{\theta r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$0 \le \theta \le \pi$

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Moving to the stress components, substituting a_6 and b_{21} in these set of stress component equations, we can obtain the stress fields for the present problem like this: σ_{rr} would be $-\frac{p}{2\pi}(2\theta + \sin 2\theta)$; $\sigma_{\theta\theta}$ would be $\frac{p}{2\pi}(-2\theta + \sin 2\theta)$; and $\tau_{r\theta}$ is $\frac{p}{2\pi}(1 - \cos 2\theta)$. This

completes the present problem, which is the exact solution, as all the boundary conditions are exactly satisfied.

After this elastic half-space problem subjected to the distributed load over a semi-infinite span, let us move to the elastic half-space problem subjected to normal distributed uniform loading over a finite span. So, in this elastic half-space, the normal distributed loading p is acting over a finite span from O to O_1, and p is the intensity of the distributed normal loading.
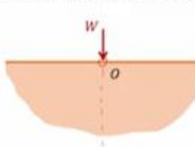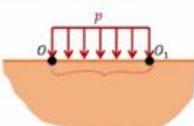

We will try to solve this problem with the help of the Flamant problem solution. What was that? The Flamant problem was elastic half-space subjected to a normal line loading of intensity W at a specific point O or the tip. For this, we obtained a purely radial stress distribution: $\sigma_{rr} = -\dfrac{2W}{\pi r} \cos \theta$, and $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ were 0.

Using this Flamant problem solution and the principle of superposition, we will try to obtain the solution of this problem, where the elastic half-space is subjected to distributed loading of fixed constant intensity p, acting over a finite span between these two points O and O_1.

Elastic Half Space under Uniform Distributed Loading Acting Over Finite Span
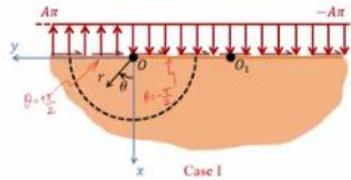
Choice of stress function:

$$\phi(r,\theta) = Ar^2\theta$$

Stress components:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 2A\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 2A\theta$$

$$\tau_{\theta r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -A$$

Boundary conditions:

$\theta = +\pi/2$ (+ve y axis): $\quad \sigma_{\theta\theta}(r,\pi/2) = A\pi, \quad \tau_{r\theta}(r,\pi/2) = -A$

$\theta = -\pi/2$ (−ve y axis): $\quad \sigma_{\theta\theta}(r,-\pi/2) = -A\pi, \quad \tau_{r\theta}(r,-\pi/2) = -A$

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Considering this particular case where O is chosen to be the origin. The x-axis is vertically downward from O, and the y-axis is the horizontal axis coinciding with the free surface of the elastic half-space. θ is measured from the positive x-axis, in the clockwise direction, positive as shown in the figure. The range of θ is from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ for the present problem, which is the same as the Flamant problem. This is the first case of the solution.

We will be solving two cases, then their superposition will give us the desired solution. For this first case, let us choose the stress function to be $Ar^2\theta$. Why r^2? If you have a stress function proportional to r^2, then the stress components would be independent of r; that is what we want. For that reason, we are choosing ϕ to be proportional to r^2, and it has only one θ-dependent term, which is $Ar^2\theta$.

With this form of ϕ, if we find the stress components σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$, σ_{rr} for this ϕ will be $2A\theta$, $\sigma_{\theta\theta}$ for this ϕ will be $2A\theta$, and $\tau_{r\theta}$ will be $-A$. Both σ_{rr} and $\sigma_{\theta\theta}$ are equal and are given by $2A\theta$, whereas the shear stress $\tau_{r\theta}$ is a constant, which is $-A$, independent of r and θ, which is what we want.

If we try to plot it for the free edges. If you look at the boundary conditions which these obtained stress fields are going to result. How is the boundary defined? The boundary for this problem is defined by $\theta = +\frac{\pi}{2}$, which is the positive y-axis, and $\theta = -\frac{\pi}{2}$, which is the negative y-axis. For the positive y-axis, $\theta = +\frac{\pi}{2}$. If you substitute that here in the σ expressions, then for these boundaries on the θ plane, we have $\sigma_{\theta\theta}$ as normal stress and

$\tau_{\theta r}$ as the shear stress. σ_{rr} does not exist on the positive or negative y-axis, *i.e.*, on the $\theta = \pm\frac{\pi}{2}$ planes. Only $\sigma_{\theta\theta}$ normal stress and $\tau_{\theta r}$ shear stress exist on those two planes. So, for the positive y-axis, substituting θ with $+\frac{\pi}{2}$, $\sigma_{\theta\theta}$ would be $A\pi$.

If we plot it here in the figure, we are going to have a positive $\sigma_{\theta\theta}$, positive normal stress, for the positive y-axis with amplitude $A\pi$, which is a constant amplitude. Starting from this, $\tau_{\theta r}$ equals a constant $-A$, and that I have drawn. $-A$ means it is along the negative y-axis, *i.e.*, it is along the negative r direction. So, this boundary condition at the positive y-axis gives us these stress fields as shown in the figure.

Now, we will go to $\theta = -\frac{\pi}{2}$, which is the negative y-axis. $\sigma_{\theta\theta}$ with $\theta = -\frac{\pi}{2}$, if you substitute here in this equation, that would be $-A\pi$. Hence, for the negative y-axis part, starting from point O, there will be normal stress of $-A\pi$ acting downward, that is compressive distributed normal loading of intensity $-A\pi$, and $\tau_{r\theta}$ is constant, which is $-A$. So, that will not be changing its sign.

If you look at this problem, that is the first case of the problem started with a choice of stress function as $Ar^2\theta$. The obtained stress components are resulting in a boundary condition like this, where we are having two types of forcing. The surface loading or surface traction boundary condition includes two types of forcing.

The first one is a uniformly distributed shear force of intensity $-A$. As it is $-A$, it would be acting along the negative radial direction for $\theta = +\frac{\pi}{2}$ surface and positive radial

direction for $\theta = -\frac{\pi}{2}$ surface. That is shown here. In the figure, you can see a shear load is shown going towards the negative y-axis, and the intensity of this shear load is $-A$. This is the first surface traction present for the problem.

Then, the next surface traction present in the problem is the uniformly distributed normal load of constant intensity $A\pi$, but its direction changes sign as you go from the positive y-axis to the negative y-axis. Towards the left of point O, we have a tensile normal load of constant intensity $A\pi$, and towards the negative y-axis, to the right side of point O, we have a compressive normal load of uniformly distributed intensity $-A\pi$, which is compressive in nature. So, a uniformly distributed normal load of intensity $A\pi$ is acting over the free surface, which abruptly changes its sign at the origin. This gives us the complete solution of case 1 with this kind of surface boundary condition as described in the figure, with these two possible forces or tractions acting on the free surface.

Coming to the second case, where instead of point O, we are choosing our origin to some other point O_1, which is the endpoint of the distributed loading. A load intensity p was acting over a finite span of the elastic half-space between O and O_1. The first case of the solution takes O as the origin, with which we had solved. Now, for the second case, we are choosing our origin at some other point O_1, and a stress function is chosen like this.

Our previous stress function, was ϕ, a function of r and θ. Now, we are choosing a new stress function ϕ_1, which is defined with respect to new polar coordinate variables r_1 and θ_1, having its origin at point O_1, and this stress function is defined as $-Ar_1^2\theta_1$. The previous stress function for case 1, $\phi(r,\theta)$, was $Ar^2\theta$. The new stress function centered at O_1 is $-Ar_1^2\theta_1$ - only the sign is changed - and with that, the second case of the solution can be obtained like this.

For the first case, the origin was at point O, the radial variable was r, and the circumferential variable was θ. For the second case, the origin is at O_1, the radial variable is r_1, and the circumferential variable is θ_1. For the first case, the stress function ϕ was $Ar^2\theta$. For the second case, the stress function ϕ_1 is $-Ar_1^2\theta_1$. The value of A is the same, but the variable is changed to r_1 and θ_1, and the sign of the stress function is changed.

As we change the sign of the stress function, the direction of all the free surface tractions will just be reversed. So, in this second case, at point O_1, which is the origin from which the positive y-axis starts, toward the positive y-axis, you have a compressive normal load of intensity $A\pi$, and toward the negative y-axis, you have a tensile normal load of intensity $A\pi$. For the entire surface, you have a shear traction of intensity $+A$.

Now, if I compare these two cases, case 1 and case 2, and try to go for a superposition, then the $+A$ shear traction and the $-A$ shear traction will balance each other. If you see the region beyond point O, towards the left of point O, let us divide it into three regions. One is this, one is this. This is span 1, this is span 2, and this is span 3. Here also, we are having span 1, then 2, then 3.

Now, in span 1 for case 1, tensile $A\pi$ distributed load acts while for case 2, compressive $-A\pi$ distributed load acts. They will cancel each other. The normal loading will be 0 for span 1. Similarly, for span 3, case 1 is showing distributed compressive load of intensity $-A\pi$; case 2 is showing distributed tensile load of intensity $+A\pi$. They would cancel each other.

Thus, in the superimposed solution for span 1 and span 3, there would be no normal loading. In the span 2, O to O_1, both the solutions from case 1 and case 2, result in compressive normal loading of intensity $A\pi$, and hence in the superimposed solution, we will have a total normal load intensity of $2A\pi$ between O and O_1, or span 2.

Elastic Half Space under Uniform Distributed Loading Acting Over Finite Span
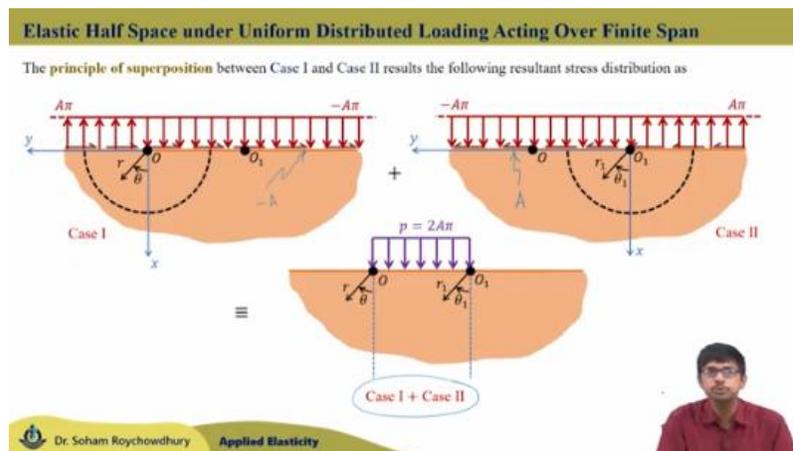
If you pictorially represent this principle of linear superposition between case 1 and case 2 types of loading, that would result in a stress distribution shown here, where between O to O_1, on this finite span, a uniform normal load of intensity $2A\pi$ would be acting. And the region left to O and the region right to O_1 would be free of any normal load. As for case 1, we had a shear load of intensity $-A$, and for case 2, we had a shear load intensity of $+A$; they would cancel each other. Thus, the superimposed solution will not have any kind of shear loading.

As we have already solved case 1 and case 2 problems separately with the stress function $Ar^2\theta$ and $-Ar_1^2\theta_1$, by superimposing them, we can get the solution of this half-space problem where the distributed normal load is acting over a finite span between O to O_1. If I compare this p with the $2A\pi$, the applied load intensity p for this problem is equal to $2A\pi$, and from that, we can obtain the value of A to be $\frac{p}{2\pi}$.

Thus, two stress functions to be used, $\phi(r, \theta)$ would be $\frac{p}{2\pi} r^2\theta$, which is described with respect to center O, and ϕ_1 would be $-\frac{p}{2\pi} r_1^2\theta_1$, which is described with respect to center O_1. The total stress function will be the superimposition or addition of these two, which will be $\frac{p}{2\pi}(r^2\theta - r_1^2\theta_1)$. Hence, using this ϕ, we can get the overall stress function, which would result in a stress distribution, and for that case, the elastic half-space will be subjected to the distributed line loading of intensity p within this finite span O to O_1.

The variables r, r_1, these two different radial variables, and θ, θ_1, two different angular or circumferential variables, can be easily related using the geometry once the span O to O_1 is given. This length, let us say L, will be a given length, that is, the span of the external loading for a given span of L. For any particular point, you can relate r and r_1, θ and θ_1 from geometry. Thus, with the help of this stress function, you can get the stress components for the elastic half-space problem subjected to normal loading acting over a finite span.

Summary

- **Elastic Half Space under Uniform Distributed Normal Loading**

    o **Acting Over Semi-infinite Span**

    o **Acting Over Finite Span**

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In this lecture, we talked about the elastic half-space problem subjected to distributed loading. We have considered two types of distributed loading. One is acting over a semi-infinite span, and the other is acting over a finite span.

Thank you.