# APPLIED ELASTICITY


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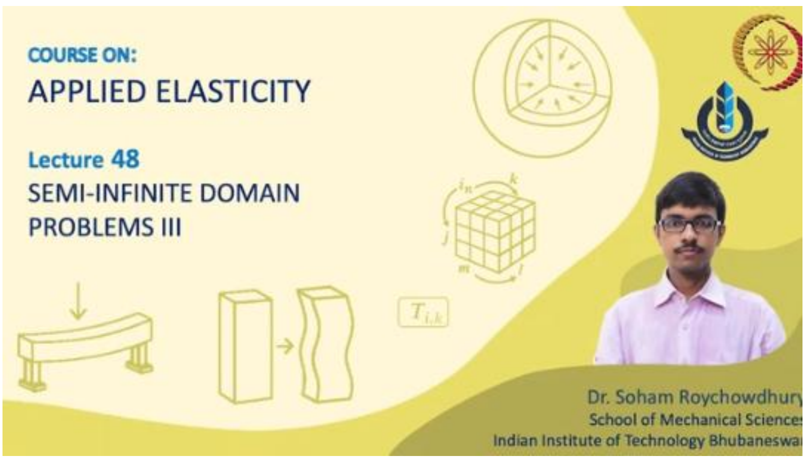
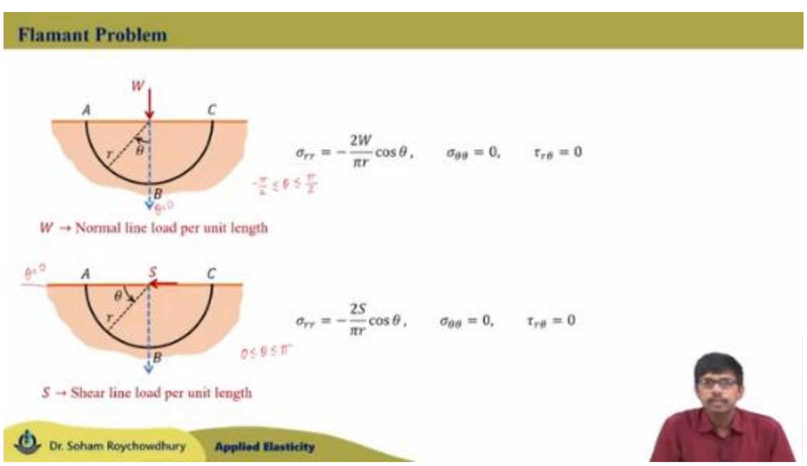
## Week 10

## Lecture 48: Semi-infinite Domain Problems III




Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the semi-infinite domain problems, which we started a few lectures back. In this particular lecture, we are going to talk about the Flamant problem.



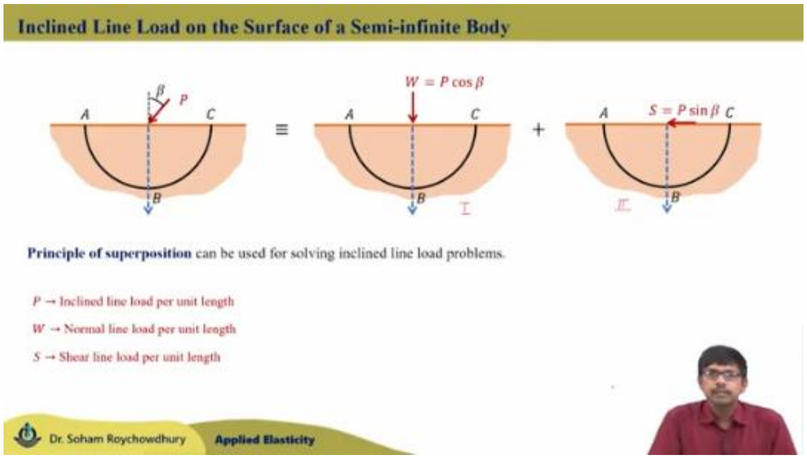
In the last lecture, we described the Flamant problem, which is nothing but the elastic half-space problem - a semi-infinite domain or elastic half-space - when subjected to line loading. We call this class of problem as the Flamant problem. Depending on the nature of the loading, whether it is a normal load or a shear load, we may have two types of Flamant problem.

The first case is when the semi-infinite elastic half-space is subjected to a normal line loading of intensity $W$ per unit length. For this particular case or this particular class of Flamant problem, we obtained the stress distribution like this. It was a purely radial stress field, described by $\sigma_{rr} = -\frac{2W}{\pi r}\cos\theta$, and the other two stress components, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, were both 0.

Apart from the normal compressive line loading, the elastic half-space, or semi-infinite domain, may also be subjected to a shear line loading of intensity $S$ per unit length, as shown in this figure. We also solved this problem using a similar approach to the previous one, and the stress fields were obtained like this. For the shear loading, the radial stress $\sigma_{rr} = -\frac{2S}{\pi r}\cos\theta$, and the other two stress components, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, equal 0. The difference between these two is the measurement of the angle $\theta$.

If you look at the first problem, you can see the $\theta = 0$ line was the vertical line from which $\theta$ was measured, and the range of $\theta$ for this problem was from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. However, for the shear loading problem, instead of taking the mid-vertical line as the $\theta = 0$ line, we have taken this line as $\theta = 0$.

This choice of the $\theta = 0$ line is basically governed by the direction of the line loading. We are taking the direction of the line loading as $\theta = 0$, and hence for the second problem, for the shear load, $\theta$ is varying between 0 to $\pi$. The range of $\theta$ is different for both of these two problems, even though the solution nature looks similar, but the measurement of $\theta$ is different for these two problems, and these two are the two different classes of Flamant problems.

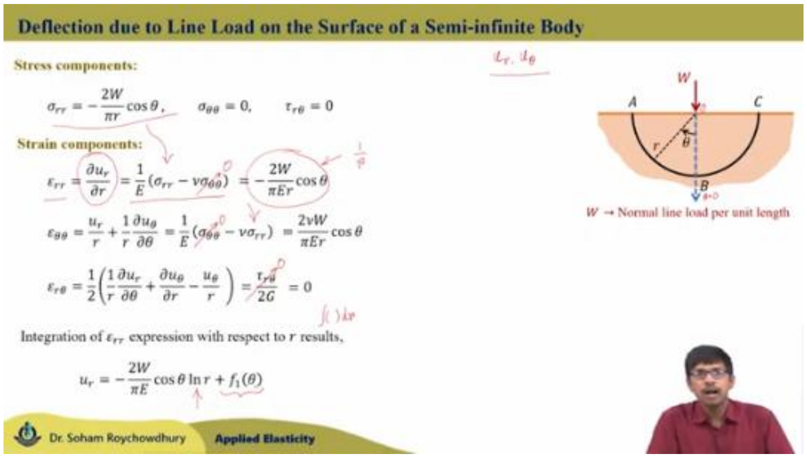



Now, in many real-life or practical applications, the elastic half-space is subjected to inclined line loading like this. $P$ is the intensity of the inclined line loading per unit length, which makes an angle $\beta$ with the vertical. For such cases, we need to use the previously obtained solutions of two Flamant problems: with only normal load and with only shear load, and then, using the principle of superposition for those two solutions, we can obtain the solution of this particular problem, that is, elastic half-space subjected to inclined line loading.

This inclined line loading $P$, which has an angle $\beta$ with the vertical, can be decomposed into two components. One vertical component downward, which is $P\cos\beta$ , and another horizontal component, which is $P\sin\beta$, as this. The principle of superposition is used to solve this kind of inclined line load problem, where this will be the superposition of this first case, where the vertical load $W$, that intensity is $P\cos\beta$, and then, the second case is equal to the shear load with intensity $S = P\sin\beta$.

We already know the solution of this $W$ and $S$. The normal line load intensity and shear line load intensity being $W$ and $S$, what will be the stress field generated within the elastic half-space? That we had discussed in the previous lecture. We had just discussed that in the last slide as well; the results were presented. Using those two, this case 1 and case 2, these two problems can be solved.

Now, if you have the inclined line load, then the solution of these two separate problems, with $W = P\cos\beta$ normal line load, and $S = P\sin\beta$ shear line load, if you superimpose that, we will get the total solution for this inclined line loading problem for the Flamant solution.




Moving forward to the displacement calculation for the Flamant problem. Till now, we only talked about the stress field generated within the Flamant problem or on the elastic half-space when it is subjected to line loading. Now, we are going to consider the calculation of the deflection of the elastic half-space when it is subjected to the normal line loading of intensity $W$ per unit length.

We are considering this elastic half-space where $W$ is applied at point $O$, which is the line load intensity per unit length acting along the vertically downward direction, and this vertical line refers to the $\theta = 0$ line. For this problem, the stress components were calculated as $-\frac{2W}{\pi r}\cos\theta$ is $\sigma_{rr}$, the radial stress, and the other two, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, were obtained as 0. This is a purely radial stress distribution for the elastic half-space subjected to normal line load.

From these stress components, let us obtain the strain components. Using the generalized Hooke's law, with the assumption of a linear elastic isotropic solid, we can obtain the strain components. Those strains can be related to the respective displacement components in the polar coordinate as this. The equations presented here are the strain-displacement relations in the polar coordinates, where the radial normal strains $\varepsilon_{rr}$ is $\frac{\partial u_r}{\partial r}$, $\varepsilon_{\theta\theta}$ is $\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$, and $\varepsilon_{r\theta}$, the tensorial shear strain, is $\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$.

Here, $u_r$ and $u_\theta$, these two are the displacement components along the radial and the angular or circumferential or tangential direction, respectively. Using the generalized Hooke's law for linear elastic isotropic solid, which is described by two material parameters: Young's modulus $E$ and Poisson's ratio $\nu$. For the planar problems, we can write $\varepsilon_{rr}$ as $\frac{(\sigma_{rr}-\nu\sigma_{\theta\theta})}{E}$.

Here, $\sigma_{zz}$ is not considered because we are talking about the planar problems. No stress along the $z$-direction is considered for this elastic half-space. So, $\varepsilon_{rr}$ would be $\frac{(\sigma_{rr}-\nu\sigma_{\theta\theta})}{E}$. Similarly, the other normal strain, $\varepsilon_{\theta\theta}$, would be $\frac{(\sigma_{\theta\theta}-\nu\sigma_{rr})}{E}$. And the shear strain $\varepsilon_{r\theta}$ can be related to the shear stress $\tau_{r\theta}$ as $\frac{\tau_{r\theta}}{2G} = \varepsilon_{r\theta}$.

Here, we know the expressions of $\sigma_{\theta\theta}$, $\tau_{r\theta}$, and $\sigma_{rr}$ for the present problem. As you know, $\sigma_{\theta\theta}$ is 0; these two terms will go to 0, and $\tau_{r\theta}$ is also 0. So, the only non-zero term is $\sigma_{rr}$. If you substitute these expressions here and here, then $\varepsilon_{rr}$ can be written as $-\frac{2W}{\pi Er}\cos\theta$. $\varepsilon_{\theta\theta}$ would be $\frac{2\nu W}{\pi Er}\cos\theta$, and $\varepsilon_{r\theta}$ is 0. We have written the strain components in terms of the applied loading $W$.

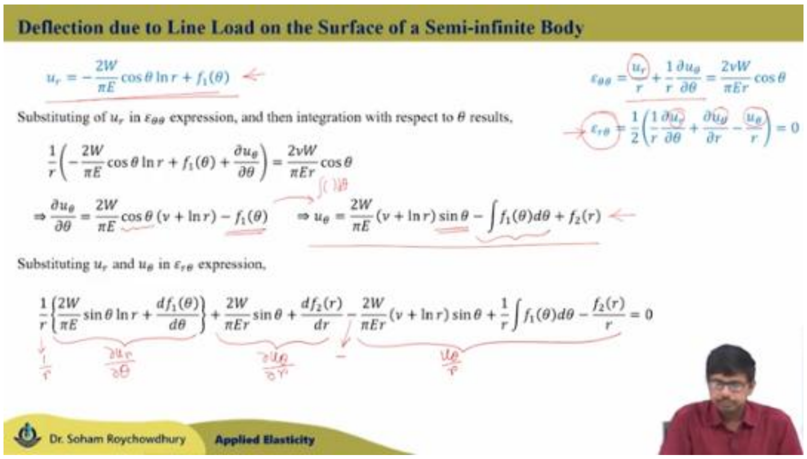
Now, our objective is to write the displacement components and get the expression for the displacement components $u_r$ and $u_\theta$. For that, we are going to use the strain-displacement relation. $\varepsilon_{rr}$ in terms of the displacement component $u_r$ can be written as $\frac{\partial u_r}{\partial r}$, and $\varepsilon_{rr}$ in terms of the applied line loading $W$ is obtained as $-\frac{2W}{\pi Er}\cos\theta$. So, $\frac{\partial u_r}{\partial r} = -\frac{2W}{\pi Er}\cos\theta$.

If we integrate this equation with respect to $r$ once, we will get an expression for $u_r$. Integrating this $\varepsilon_{rr}$ expression with respect to $r$, we can get the radial displacement component $u_r$. There was one $\frac{1}{r}$ term present here, which, upon integration with respect to $r$, results in this $\ln r$ term, and $u_r$ becomes $-\frac{2W}{\pi E}\cos\theta\ln r + f_1(\theta)$.

Note that we are integrating with respect to $dr$. As the integral is with respect to $dr$, the integration constant may either be a constant or any general function of $\theta$. Hence, this

$f_1(\theta)$ is taken to be that general function which arises due to integration with respect to $r$. So, $u_r$ becomes $-\frac{2W}{\pi E}\cos\theta \ln r + f_1(\theta)$. $f_1(\theta)$ is unknown at this stage, which we need to obtain.

Similar functions we had also seen for the plane stress or plane strain displacement problems, where the solution was done in the rectangular Cartesian coordinate system, where these functions were dependent on $x$ and $y$. Here, as we are solving the problem in polar coordinates, these functions are dependent on either $r$ or on $\theta$.




After obtaining $u_r$, let us proceed to find the other displacement component. For that, we will substitute this $u_r$ expression into the expression of $\varepsilon_{\theta\theta}$. From the strain-displacement relation, we know that $\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$. In terms of the applied load $W$, this was written as $\frac{2\nu W}{\pi E r}\cos\theta$.

Now, in this equation, $u_r$, the radial displacement component is replaced with the obtained expression of $u_r$. Hence, this equation would only have the term $\frac{\partial u_\theta}{\partial \theta}$ as unknown. From this substitution of $u_r$ in the expression of $\varepsilon_{\theta\theta}$, the left-hand side of this equation, we can derive an expression like this. From this, $\frac{\partial u_\theta}{\partial \theta}$ can be written in terms of $W$ as $\frac{2W}{\pi E}\cos\theta\,(\nu + \ln r) - f_1(\theta)$.

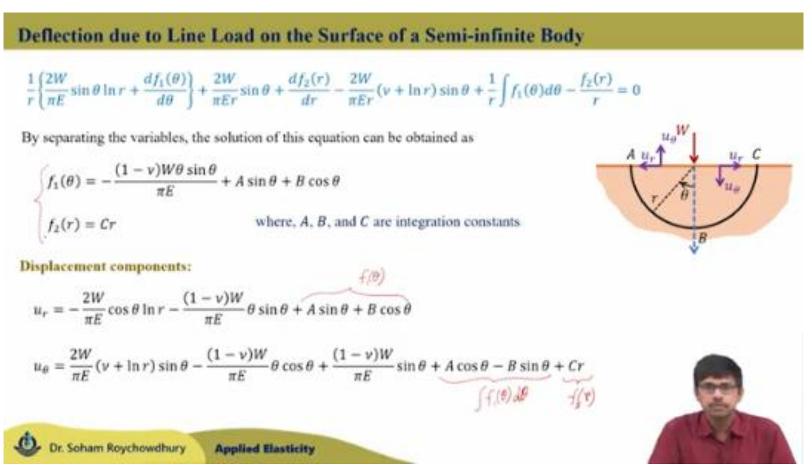
This is the expression of $\frac{\partial u_\theta}{\partial \theta}$. Integrating this expression with respect to $\theta$, we can obtain the initial expression of $u_\theta$, that is, the displacement along the tangential direction $\theta$.

Upon integration with respect to $\theta$, we can obtain $u_\theta$ as $\frac{2W}{\pi E}(\nu + \ln r)\sin\theta - \int f_1(\theta)d\theta + f_2(r)$. If you look at the first term of $\frac{\partial u_\theta}{\partial \theta}$, that was having a $\cos\theta$, and upon integral that is resulting this $\sin\theta$ term. $-\int f_1(\theta)d\theta$ integral, this term, is also present.

As we do not know the form of $f_1(\theta)$ at this stage, we are keeping this term, $\int f_1(\theta)d\theta$, as it is. As this integral is with respect to $d\theta$, due to integration, one integration constant would come, which may either be a constant or any general function of radial coordinate $r$. Here, we are taking that function to be $f_2(r)$ which is another unknown function of $r$ to be determined.

We have obtained expression of this $u_r$ here, involving one unknown, $f_1(\theta)$, and another expression of $u_\theta$ here, involving unknown $f_2(r)$. These two $u_r$ and $u_\theta$ expression must satisfy the third strain displacement relation which is defined for the shear strain $\varepsilon_{r\theta}$. Shear strain $\varepsilon_{r\theta}$ is $\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$, and that is equal to $0$ for the present problem. No shear strain is present as the shear stress $\tau_{r\theta}$ was $0$.

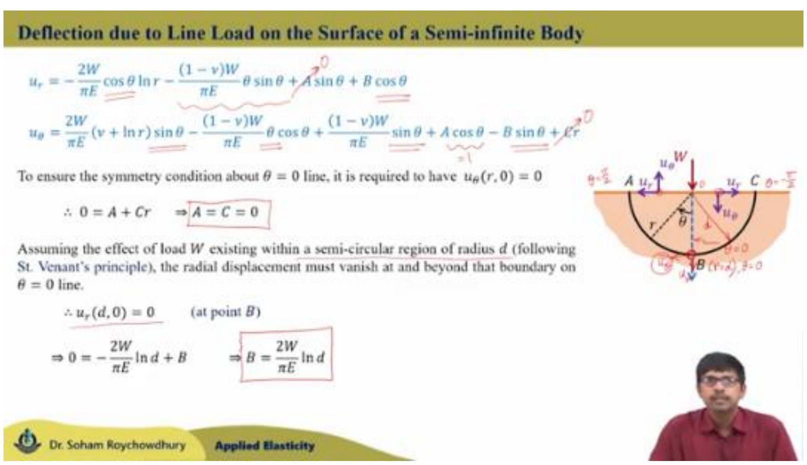
Here, $u_r$ and $u_\theta$ expressions are substituted as per the obtained expression. These $u_r$ and $u_\theta$, both of them are substituted in the expression of $\varepsilon_{r\theta}$ on the left-hand side and then equated to $0$. If we do so, we will get a big expression like this. So, the first term, this part is $\frac{\partial u_r}{\partial \theta}$. Then, the next term is $\frac{\partial u_\theta}{\partial r}$; this is $\frac{\partial u_\theta}{\partial r}$, and here $\frac{1}{r}$ was also there. Then, these set of last terms, these three terms are $\frac{u_\theta}{r}$ with a minus sign, and that equals $0$.



From this equation, if we combine the terms of $r$ and $\theta$ and separate them on both sides, so this expression of $\varepsilon_{r\theta}$, whatever we got, is a combination of $r$-dependent terms and $\theta$ -dependent terms. Writing all the $r$ terms on one side and $\theta$ terms on the other side by using the separation of variables, and then, equating that to a constant to be valid for all $r$ and $\theta$, we can get the explicit expressions for $f_1(\theta)$ and $f_2(r)$ as this.

$f_1(\theta)$ would come out to be $-\frac{(1-\nu)W\theta\sin\theta}{\pi E} + A\sin\theta + B\cos\theta$, whereas $f_2(r)$ would be $Cr$. After getting this $f_1(\theta)$ and $f_2(r)$ in terms of these integration constants $A$, $B$, and $C$ - these are pure constants, no longer functions of $r$ and $\theta$ - $A$, $B$, $C$ are integration constants to be obtained by using the boundary conditions.

Now, the displacement components, the expressions of $u_r$ and $u_\theta$, would become like this. Here, earlier we had some $f_1(\theta)$ term and in $u_\theta$, we had the integral of $f_1(\theta)$ term; then this was $f_2(r)$. So, those expressions of $f_1(\theta)$, $f_2(r)$, and $\int f_1(\theta)d\theta$ are replaced by using the obtained form of $f_1$ and $f_2$, and the total displacement components look like this. These describe the overall displacement field for the present Flamant problem subjected to the normal line loading of intensity $W$ per unit length.




Moving forward, we need to evaluate these constants $A$, $B$, $C$. For that, the first thing we are going to do is use the symmetry condition. This vertical line is described by $\theta = 0$. As the solution must be symmetric with respect to this mid vertical line or $\theta = 0$ line, we should have $u_\theta$, the displacement along this direction, and displacement along this radial direction is $u_r$.

You can see $u_r$ and $u_\theta$ are shown on $\theta = 0$ line, then also on the free surface, on both the sides; this is $\theta = \frac{\pi}{2}$, $O$ to $A$ part, and $O$ to $C$ part is $\theta = -\frac{\pi}{2}$. For all these sides, we have shown the positive $u_r$ and positive $u_\theta$ directions. Here, for $\theta = 0$ line, if you have a non-zero value of $u_\theta$, that will immediately break the symmetry because that will shift on one side of the mid plane. So, that should be forced to 0.

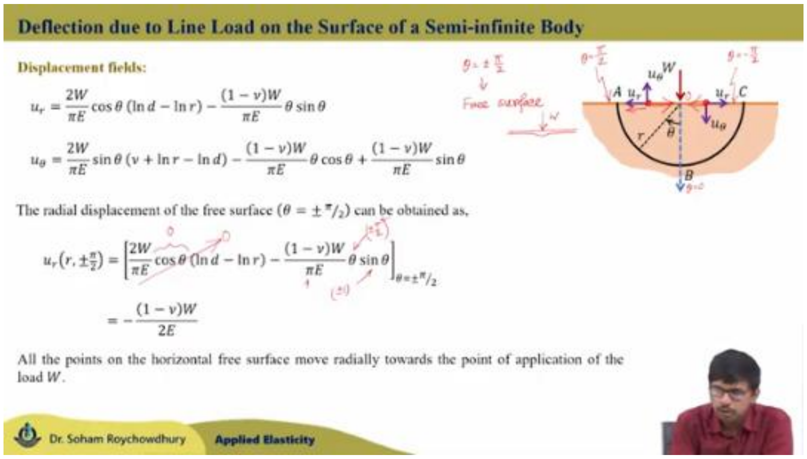
Substituting $\theta = 0$ in the expression of $u_\theta$ and equating $u_\theta$ to 0, we will end up with an expression like this. $\theta = 0$ in $u_\theta$, this $\sin\theta$ term will go to 0, this $\theta$ term will go to 0, this $\sin\theta$ will go to 0, this $\sin\theta$ will also go to 0, and this $\cos\theta$ will be equal to unity. Hence, $u_\theta(r, 0) = 0$ will give $A + Cr = 0$, which should be satisfied for all values of $r$, means $A$ and $C$, both these constants should be individually 0.

Two constants we had obtained: $C$ is 0, and $A$ is 0. For the remaining constant $B$, we will use St. Venant's principle. We had discussed that the effect of this line loading $W$ is valid within the semicircular region $ABC$. Let us consider the radius of this semicircular region is $d$. That is the region of interest, beyond which, we will call the solution not to be valid because that is far field while within that region, the obtained stress fields are valid.

Considering the radius of this semicircular region to be $d$, following St. Venant's principle, at this particular point $B$, which is at $r = d$ and $\theta = 0$, at this particular point $u_r$ must be 0, then only we can say beyond that semicircular region the effect of the stress distribution vanish. To ensure that we must have $u_r(d, 0)$ is 0, that is at point $B$.

On $\theta = 0$ line, point $B$ is the last point till which the effect of $W$ is visible. So, at point $O$, the effect is highest. As you move away from point $O$ towards point $B$, the effect diminishes, and finally, at point $B$, the effect vanishes. That is why at point $B$, we must have the radial displacement due to $W$, $u_r$ equal to 0.

The coordinates of point $B$ are denoted by $r = d$, $\theta = 0$. So, $u_r(d, 0)$ should be 0. Substituting that into the $u_r$ expression with $\theta = 0$, the second term $\theta \sin\theta$ would go to 0, $A$ is already 0, and $\cos\theta$ term would be 1. Thus, the expression $u_r(d, 0) = 0$ gives us $-\frac{2W}{\pi E}\ln d + B = 0$, and from that, we can obtain $B$ as $\frac{2W}{\pi E}\ln d$.




We have obtained all three expressions by using these boundary conditions, either through symmetry or St. Venant's principle, which states that the effect of line load $W$ is confined within a semicircular region, and we assumed the radius of the region to be $d$. After finding $A$, $B$, and $C$, if we substitute these into the expressions for $u_r$ and $u_\theta$, then the overall displacement fields will be as follows. This is the total displacement field in the radial and circumferential directions for the present problem due to the application of a normal line load of intensity $W$ per unit length. From this expression, we will examine the displacement of the free surface.

Here, the free surface is defined by two values of $\theta$. The vertical line is $\theta = 0$ line. The $O$ to $A$ portion is defined by $\theta = +\frac{\pi}{2}$, and the $O$ to $C$ portion is defined by $\theta = -\frac{\pi}{2}$. So, $\theta = \pm\frac{\pi}{2}$ defines the free surface of the elastic half-space. Our interest is to see how the free surface or the top surface of the elastic half-space will deform.

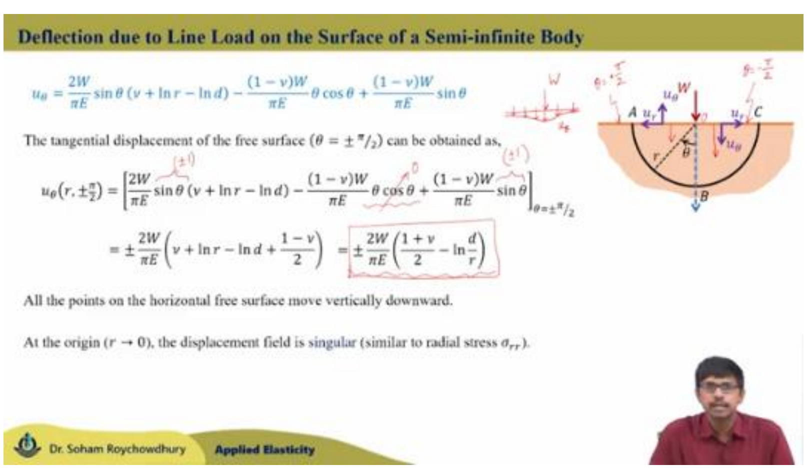
For soil-structure interaction problems, if a building stands, we try to note the radial and circumferential displacement of the soil around the base of the building. If you can find $u_r$ and $u_\theta$ around $\theta = \pm\frac{\pi}{2}$, then we can get the profile of the free surface. With the application of this load, initially being like this, we may expect the free surface to deform in this fashion, and that profile we want to obtain now.

For that, first we will go for the radial displacement, *i.e.*, $u_r$. Radial displacement for the free surface $\theta = \pm\frac{\pi}{2}$ can be obtained or written like this: $u_r$ for all values of $r$ can be anything, but $\theta$ is either $+\frac{\pi}{2}$ for $O$ to $A$ part, or $-\frac{\pi}{2}$ for $O$ to $C$ part for both sides of the mid-vertical plane. So, $u_r\left(r,\pm\frac{\pi}{2}\right)$ is the radial displacement of the free surface. The expression of $u_r$, I have written, is evaluated at $\theta=\pm\frac{\pi}{2}$.

At $\theta=\pm\frac{\pi}{2}$, $\sin\theta$ value should be either $+1$ or $-1$, whereas $\cos\theta$ value will be $0$. So, at $\theta=\pm\frac{\pi}{2}$, this $\cos\theta$ is 0 in the first term. The first term would vanish. In the second term, this $\theta$ will be $\pm\frac{\pi}{2}$, whereas this $\sin\theta$ will be $\pm 1$. Hence, if you simplify this, $u_r$ will only have one term. The entire first term will go to 0 because $\cos\pm\frac{\pi}{2}$ is $0$. $u_r$ would be $-\frac{(1-\nu)W}{2E}$. This $\pi$ and this $\pi$ would get cancelled, and as you have plus minus sign, both coming from $\theta$ and $\sin\theta$, they will also cancel. So, $u_r$ at free surface would be $-\frac{(1-\nu)W}{2E}$.

All the points on the horizontal free surface will be moving towards point $O$, towards the point of application of $W$. You can see for $\theta=+\frac{\pi}{2}$, $u_r$ is radially outward positive. As you are getting a negative value of $u_r$, it means these points will try to move towards point $O$.

Similarly, for $OC$, $\theta=-\frac{\pi}{2}$, here $u_r$ is positive radially outward. As we got $u_r$ to be negative on free surface, that point will try to move towards $O$. So, all the points on both the sides of the free surface will try to move towards the point of application of the load. That is the conclusion we are getting from this $u_r\left(r,\pm\frac{\pi}{2}\right)$ expression.



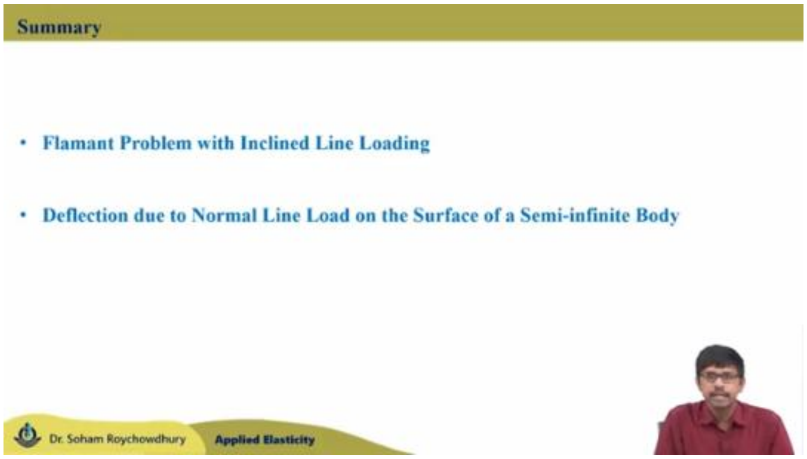
Moving to the circumferential displacement, $u_\theta$, for the free surface. The tangential or circumferential displacement $u_\theta$ for the free surface, defined by $\theta = \pm\frac{\pi}{2}$, can be obtained like this. Here, for $\theta = \pm\frac{\pi}{2}$, this $\cos\theta$ term would go to 0. This $\sin\theta$ term will be $\pm 1$. This $\sin\theta$ term will be $\pm 1$ as well.

Here you can see, in the solution of $u_\theta$ for the free surface, we will be having this plus minus sign, which have different directions and different magnitudes for $O$ to $A$ part than the solution from $O$ to $C$ part. Putting this $\sin\theta$ to be $\pm 1$, and $\cos\theta$ to be 0 for $\theta = \pm\frac{\pi}{2}$, if we simplify this, the expression of $u_\theta$ for the free surface would be obtained like this: $\pm\frac{2W}{\pi E}\left(\frac{1+\nu}{2} - \ln\frac{d}{r}\right)$.

If you plot it or if you check the variation of this $u_\theta$, we can see that the $u_\theta$, horizontal free surface, is going to move vertically downward for both the regions. Both for $O$ to $A$, defined by $\theta = +\frac{\pi}{2}$, and for $O$ to $C$, defined by $\theta = -\frac{\pi}{2}$, you will have $u_\theta$ to be downward. For $O$ to $C$, $u_\theta$ direction, by sign convention it is downward, and the value of $u_\theta$ for the $O$ to $C$ part would come out to be positive. So, the actual movement for this set of points, for $O$ to $C$, *i.e.*, $\theta = -\frac{\pi}{2}$, would be downward. For $O$ to $A$, $\theta = +\frac{\pi}{2}$, $u_\theta$ magnitude will come out to be negative, and by sign convention, $u_\theta$ is upward positive for this $O$ to $A$ part, and as $u_\theta$ magnitude would come out to be negative here, the points will also be moving in the downward direction.

Thus, as you are applying a load $W$, all the points will be moving in the vertically downward direction. None of the points are going to have upward movement, and that $u_\theta$ for the free surface is described by this particular form of equation.

If you look at the origin, at $r = 0$, you can see there is a discontinuity in the displacement field, resulting in an infinite value of the displacement at $r = 0$ or at point $O$, the origin, or at the point of application of the loading. Hence, the displacement field is singular at the point of application of the loading. A similar observation we had for the stress distribution. The radial stress $\sigma_{rr}$ was also having a singularity at the point of application of loading, which can be observed for the displacement fields as well.




In this lecture, we talked about the Flamant problem, *i.e.*, an elastic half-space subjected to inclined line loading, and then discussed the deflection components, and obtained $u_r$ and $u_\theta$ for the elastic half-space when it is subjected to normal line loading of intensity $W$ per unit length.

Thank you.