APPLIED ELASTICITY

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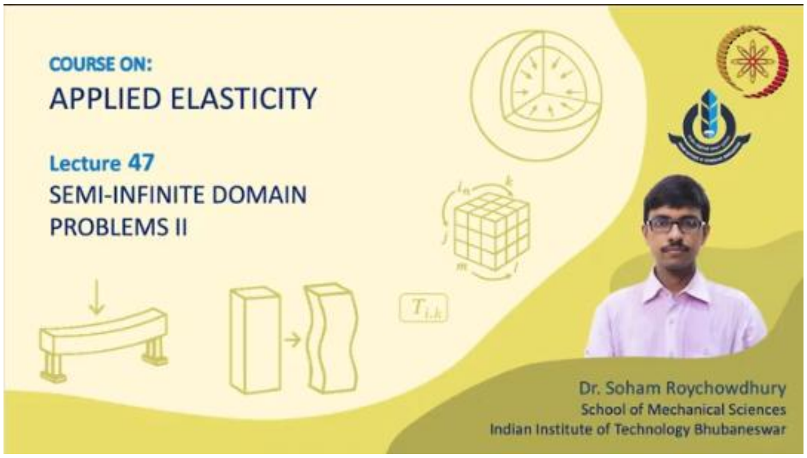
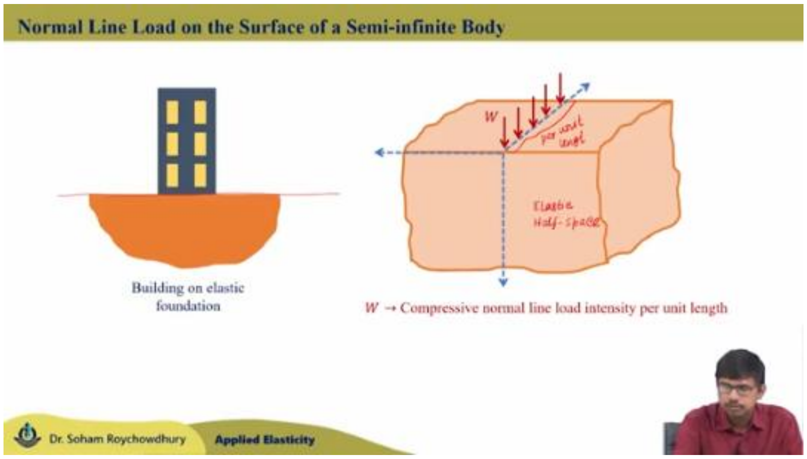
Week 10

Lecture 47: Semi-infinite Domain Problems II

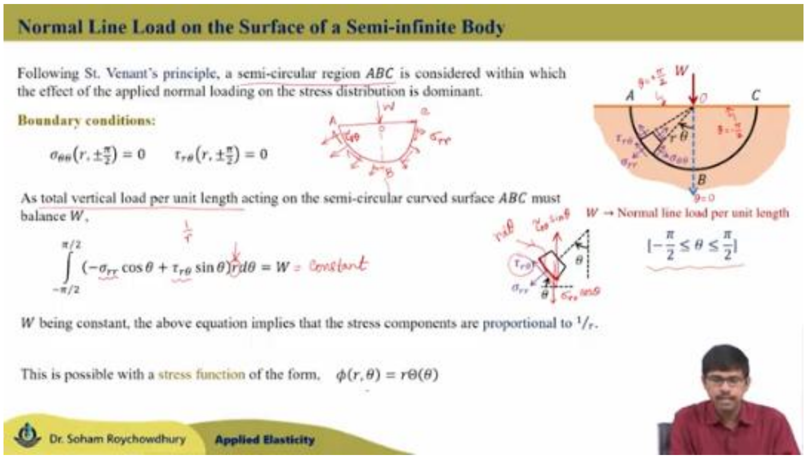


Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on semi-infinite domain problems. In the last lecture, we introduced different types of semi-infinite domain problems and then solved one such problem, which was a quarter plane subjected to end shear loading at one of its edges, the vertical edge.



In this lecture, we are going to talk about the normal line loading acting on the surface of a semi-infinite body, which is also known as the elastic half-space. This problem is an elastic half-space subjected to line loading. This is a typical example of a soil-structure interaction problem where a building stands on a foundation, and we are interested in finding the stress distribution on the elastic foundation near the base of the building. This can be modeled as an elastic half-space extending below the soil.

This elastic half-space exists below the ground level infinitely, and above the ground level, it does not exist, which is why it is named an elastic half-space. So, we call this an elastic half-space, which is one semi-infinite body. And that is subjected to this line loading, a normal compressive line loading of intensity W per unit length. W is defined per unit length, and along the width direction, it is acting per unit length. It is acting in the downward direction, is compressive in nature, and defined per unit length along this direction. We will try to find out the stress distribution within the elastic half-space when it is subjected to this kind of compressive line loading of intensity W per unit length.



Considering the front view of the structure, the elastic half-space is subjected to the normal compressive line load of intensity W. Here, following St. Venant's principle for our solution, we are going to consider this semicircular region ABC within this semi-infinite elastic half-space.

Within this semicircular region ABC, the effect of W will be visible because St. Venant's principle states that the effect of stresses would always be localized at the point of loading near the boundary condition. At the far field, it would not be affecting the far field stresses. Hence, the effect of stress concentration or the stress distribution generated due to application of W is assumed to be confined within this semicircular region, defined by the semicircular arc ABC and the top horizontal face on which W is acting.

Considering this polar coordinate with origin at O, that is point of application of W and considering this vertical line that is the line of loading W to be $\theta = 0$ line, and taking θ to be clockwise positive here. θ is shown in that diagram. In this direction, θ is taken to be positive.

Then, we are taking a small element, then σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$, the stresses are shown on that small element. Range of θ is from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. θ is equal to 0 on the vertical line, that is, on the line of application of W. This particular part O to C, here, θ is equal to $-\frac{\pi}{2}$, and for this part O to A, θ is equal to $+\frac{\pi}{2}$. So, range of θ is in between $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, that defines this elastic half space.

Coming to the boundary condition, both $\theta = +\frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$, the complete free surface towards the left of point of application of W and towards the right of point of application of W, should be free of normal and shear stresses. Only at point O, the normal load W is acting. No surface traction is there between O to A, and no surface traction is there between O to C. Thus, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, both normal stress and shear stress should be 0 for all values of r with $\theta = \pm\frac{\pi}{2}$.

$\theta = \pm\frac{\pi}{2}$ defines the free surface. On the free surface, normal traction is 0. So, $\sigma_{\theta\theta}$ is 0. On the free surface, shear traction is 0. So, $\tau_{r\theta}$ is 0. These are the traction-free surface boundary conditions available to us.

Coming to the vertical load balance. If you consider the effect of the load being confined to this semicircular region, you can think that this is the elastic half-space semicircular region which is subjected to W at point O, and on the semicircular boundary, two stresses are acting. One is σ_{rr}, acting like this, and another is $\tau_{r\theta}$, which is acting like this over the entire semicircular region.

Now, the semicircular region of interest, following St. Venant's principle, should be in equilibrium under the action of W on the top face, and the σ_{rr} and $\tau_{r\theta}$ resultant force in the vertical direction acting on the semicircular boundary ABC. Thus, the total vertical load balance per unit length for the semicircular curved surface ABC would result in this equation.

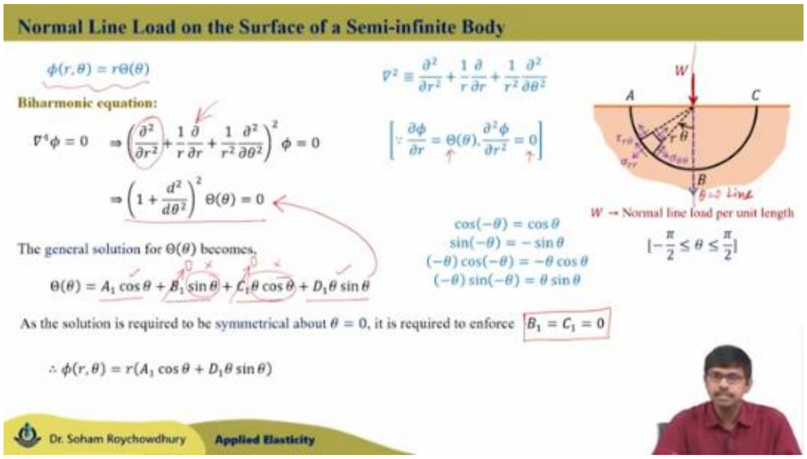
Considering this small element as shown in this small figure, σ_{rr} and $\tau_{r\theta}$ both will have some vertical component. The vertical component of σ_{rr} in the downward direction would be $\sigma_{rr}\cos\theta$, whereas, vertical component of this $\tau_{r\theta}$ in the upward direction it would be $\tau_{r\theta}\sin\theta$. So, the net vertical force acting on the small element is $\tau_{r\theta}\sin\theta - \sigma_{rr}\cos\theta$ acting in the upward direction, and the area of this particular element, with this length equal to $rd\theta$.

If you integrate it over θ from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, that will give you the total vertical load acting per unit length of the semicircular arc ABC and that must balance the applied external load W. This resulted stress distribution integrated over ABC in the upward direction must balance the downward acting load W. So, this equation: $W = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (-\sigma_{rr} \cos\theta + \tau_{r\theta} \sin\theta) r d\theta$, must be satisfied for the semicircular boundary ABC. This equation will help us to solve for the unknown stress components.

Now, right hand side of this equation is a constant. W is a given constant acting in the downward direction and left hand side, we are having one r here, integral is over θ from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. Left hand side being a function of r within the integral, if we want to have the right hand side to be constant, this σ_{rr} and $\tau_{r\theta}$ both should be proportional to $\frac{1}{r}$. Only then will that $\frac{1}{r}$ cancel this r and right hand side after integral can come out to be a constant.

So, W being a constant, above equation implies that the stress component σ_{rr} and $\tau_{r\theta}$ must be proportional to $\frac{1}{r}$. That can be confirmed only if the stress function is proportional to r. Normally in the polar coordinate, the order of r in the stress function is two orders more than the power of r in the stress components. So, if ϕ is dependent on r^2, stress fields will be constant, which was the case for the previous problem discussed for the quarter plane subjected to shear loading.

If we want stress function to be proportional to r, then the stress components will be proportional to $\frac{1}{r}$. The order of r would be 2 orders less in the stress components as compared to the order of r present in the stress function ϕ. Hence, we should have a stress function proportional to r. ϕ should be r times some unknown Θ function, only then σ_{rr}, $\tau_{r\theta}$ would be proportional to $\frac{1}{r}$.



Now, using this ϕ equals r times some Θ, which is unknown function of θ, substituting that in the biharmonic equation. Biharmonic equation is like this: $\nabla^2\phi = 0$.

For the present problem, ϕ being this, $\frac{\partial\phi}{\partial r}$ is nothing but this $\Theta(\theta)$ function, and $\frac{\partial^2\phi}{\partial r^2}$ is 0. Now, $\frac{\partial^2\phi}{\partial r^2}$ being 0, this first term would go to 0. Substituting this $\frac{\partial\phi}{\partial r}$ here, in this term, then simplifying it, the biharmonic equation would have a form like this: $\left(1 + \frac{d^2}{d\theta^2}\right)^2$, this operator is acting over $\Theta(\theta)$ function and is equal to 0. There would be a $\frac{1}{r}$ term common, but as the right-hand side is 0, I had cancelled that.

This equation will give us the solution of this $\Theta(\theta)$ function as $A_1\cos\theta + B_1\sin\theta + C_1\theta\cos\theta + D_1\theta\sin\theta$. This is the general solution of Θ, which satisfies this. Hence, if we choose our stress function ϕ to be $r\Theta(\theta)$, we will be getting the biharmonic equation satisfied.

Coming to the symmetry of the problem. With respect to this vertical axis, central midline, line on which W is applied, the problem is symmetric. Loading, boundary condition, geometry, material property, everything is symmetric. Hence, the stress function must also be symmetric with respect to this vertical line, which is defined by $\theta = 0$. About $\theta = 0$ line, ϕ or stress component must be symmetric.

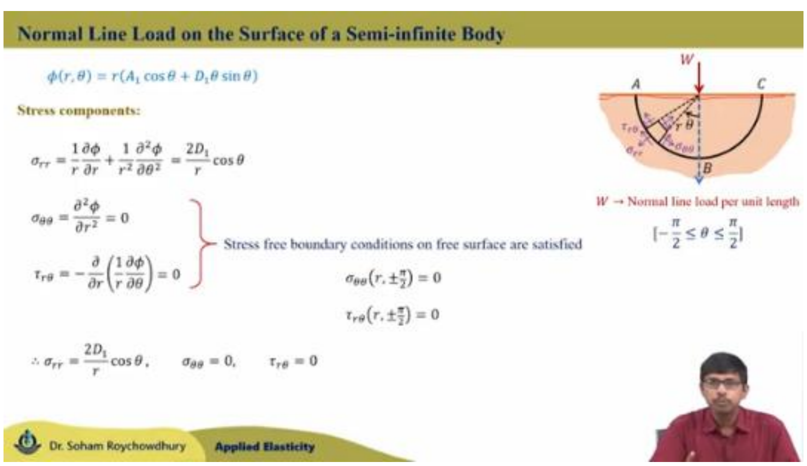
Now, we will be looking at all four terms individually and check if those terms are symmetric or not. Symmetric means θ being replaced with $-\theta$, there should not be any change in the stress function or stress component. Then, it is symmetric about $\theta = 0$ line.

Now, $\cos -\theta$ is same as $\cos\theta$. Hence, the first term is symmetric about $\theta = 0$. If I replace θ with $-\theta$, $\sin -\theta$ is $-\sin\theta$. Thus, this particular term is not symmetric and to avoid the asymmetric part, we must force the corresponding coefficient B_1 to 0.

Coming to third term, $\theta\cos\theta$. Separately $\cos\theta$ is symmetric about $\theta = 0$ but this θ term is breaking that symmetry. So, $\theta\cos\theta$ in total is one asymmetric term and to avoid asymmetric term, the coefficient of that, C_1, should be forced to 0.

Coming to the last term, $\theta\sin\theta$, they are individually asymmetric. But once they are taking a product, then overall function $\theta\sin\theta$ would be coming out as a symmetric function with respect to $\theta = 0$ line. This one is allowed to stay.

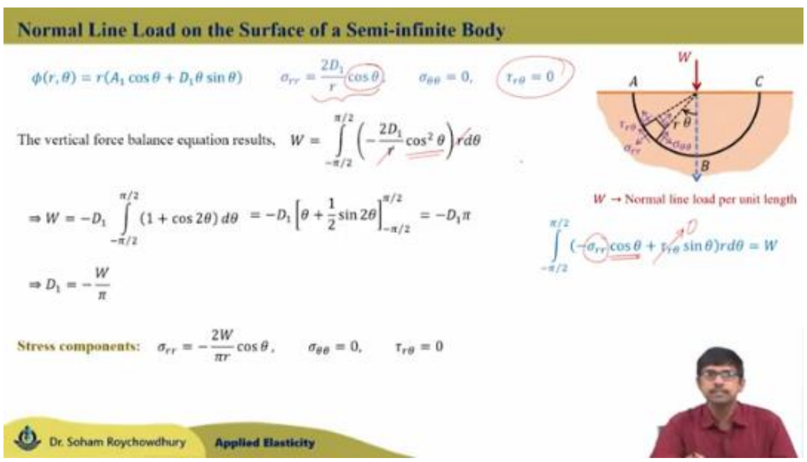
Thus, to ensure the symmetric solution with respect to $\theta = 0$ line, we must have B_1 and C_1, these two constants to be 0, and Θ would be $A_1\cos\theta + D_1\theta\sin\theta$ and total stress function ϕ will be $r(A_1\cos\theta + D_1\theta\sin\theta)$. These two - A_1 and D_1 - are unknown constants to be obtained using the boundary conditions.



Now, let us go to the stress components. We will try to find out σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ for the present problem with this chosen form of ϕ. If you substitute this ϕ in the expression of the stress components, σ_{rr} is obtained as $\frac{2D_1}{r}\cos\theta$, whereas $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are equal to 0.

With this, we will check the boundary conditions. As $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ both are 0 over the entire domain, both boundary conditions - that is, the free surface is free of any kind of surface traction - are satisfied. $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0 for $\theta = \pm\frac{\pi}{2}$ is automatically satisfied. Both $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0 for all θ values.

Obviously, at $\theta = \frac{\pi}{2}$ - that is, for the free surface - the traction components would vanish. Both normal and shear traction on the free surface are going to vanish. So, σ_{rr} is $\frac{2D_1}{r}\cos\theta$, and $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0. This is the stress field, where D_1 still remains as an unknown. We will solve for it by using the vertical force balance equation.

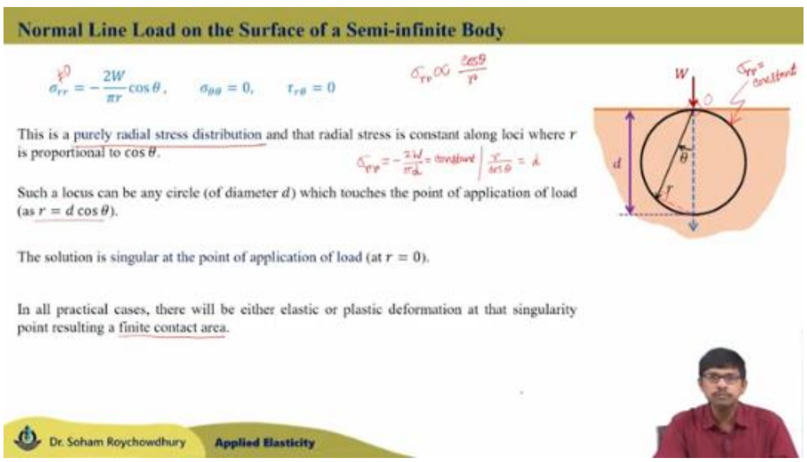


If you recall, by considering the vertical force balance of the semicircular region ABC, we have expressed W as $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}(-\sigma_{rr}\cos\theta + \tau_{r\theta}\sin\theta)rd\theta$. Here, $\tau_{r\theta}$ being 0, the second term would go to 0.

In the first term, σ_{rr} is replaced as $\frac{2D_1}{r}\cos\theta$, and replacing that here, W would be $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left(-\frac{2D_1}{r}\cos^2\theta\right)rd\theta$. One $\cos\theta$ term was already there in the integral, and another $\cos\theta$ is coming from σ_{rr}, thereby resulting in $\cos^2\theta$. Also, this $rd\theta$ term, that r and the $\frac{1}{r}$ present in σ_{rr}, would cancel each other, and thus it would be independent of r.

That is what was our objective for choosing the stress function. The aim was to ensure that W will be a constant. So, stresses must be proportional to $\frac{1}{r}$, and you can see the only non-zero stress, σ_{rr}, is proportional to $\frac{1}{r}$. Thus, this equation, the right-hand side of this integral, if you integrate, would result in $-\frac{D_1}{\pi}$. And thus, $-\frac{D_1}{\pi} = W$, or D_1 can be written as $-\frac{W}{\pi}$. The unknown constant D_1, is evaluated in terms of the applied line load intensity W as $-\frac{W}{\pi}$.

Substituting this back in the stress component σ_{rr}, the only non-zero stress component becomes $-\frac{2W}{\pi r}\cos\theta$. And the other two, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, are 0. This completes the present problem. The stress field, or stress components, are obtained like this for the semi-infinite elastic half-space subjected to line loading of intensity W.



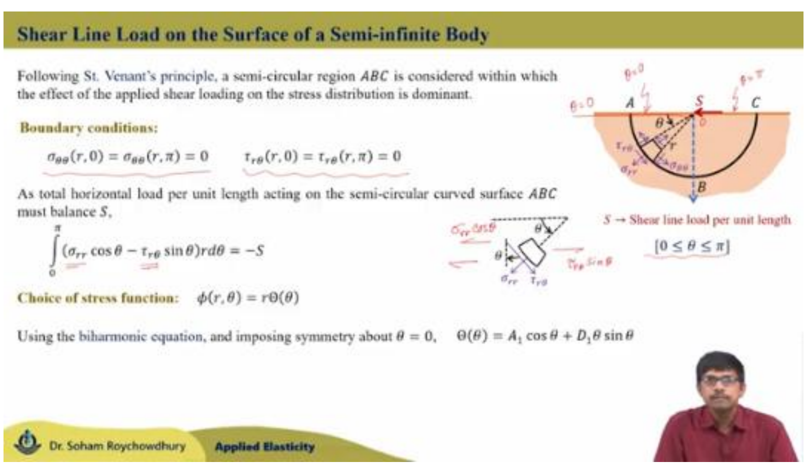
Now, if you carefully look at these stress fields. It is a purely radial stress distribution, as σ_{rr} is the only non-zero component, and rest of the terms are 0. So, it is a purely radial stress distribution, and this purely radial stress distribution, σ_{rr}, you can see, is proportional to $\frac{\cos\theta}{r}$. If we can find the locus where $\frac{\cos\theta}{r}$ is a constant, r is proportional to $\cos\theta$, for such cases, σ_{rr} would be constant. Such a locus can be obtained by drawing the circles like this, as shown in the figure.

Let us consider a circle of diameter d, which is touching the free surface at the point of application of the loading. This circle of diameter d is touching the point O, the point of application of load W at the free surface, and r is the distance of any point on the circle. Thus, d and r are related through this. If you think about this, this angle is 90°. Hence, we can write r as $d \cos\theta$ which results that $\frac{r}{\cos\theta} = d$, and substituting it back, σ_{rr} will be $-\frac{2W}{\pi d}$, which is a constant. So, for this particular circle, σ_{rr} is equal to constant. We will get constant radial stress circles, all touching point O, that is point of application of the load, for this purely radial stress distribution.

Exactly at point O, if you substitute $r = 0$ in σ_{rr}, σ_{rr} will tend to ∞. This means there exists a singularity of the solution at the point of application of the load, which is quite obvious because whenever we are subjecting any elastic body to a point load, at the point of application, the area being 0, the stress generated should be ∞, which is also observed from the solution of the present problem.

In real life, all practical scenarios, there will be some kind of elastic or plastic deformation at the point of application of the load, which will result in finite contact patch formation with some finite contact area. The singularity, or the possibility of having infinite stress, will be removed, and there will be some large but finite value of stress generated at the contact point.

This case, we will discuss at the end of this particular subject, in the last week of lecture, where we will talk about contact mechanics. Now, this particular solution is showing one singularity at the point of application of the load.



After considering the normal line load acting on the semi-infinite elastic half space, we are going to consider the semi-infinite elastic half space subjected to the shear line loading of intensity S acting per unit length. As shown in the figure, S is the shear line load intensity which is acting on the free surface of this elastic half space.

Here also, similar to the normal load case, considering St. Venant's principle, we assume that effect of the shear load S is confined within a semicircular boundary ABC. Within that only, the effect of this load or stress distribution due to S would be valid. And here, I am choosing the $\theta = 0$ line as this: along the direction of application of S, unlike the previous case where the vertical midline was chosen as $\theta = 0$. Here, we are changing this choice of $\theta = 0$ line because we want to match this datum of θ with the direction of applied line loading.

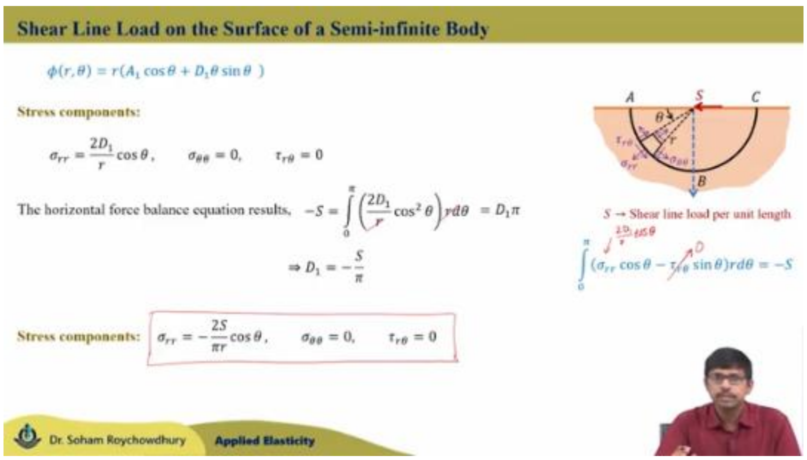
So, $\theta = 0$ is chosen along the direction of the application of shear load S. Here, θ would vary from 0 to π. Thus, the boundary is defined with $\theta = 0$ for the part O to A, and for the part O to C, the boundary or the free surface is defined by $\theta = \pi$. The boundary condition, which is the traction-free boundary condition for the entire free surface O to A ($\theta = 0$) and O to C ($\theta = \pi$), both $\sigma_{\theta\theta}$ (the normal stress) and $\tau_{r\theta}$ (the shear stress) should be 0. So, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are both 0 for $\theta = 0$ and $\theta = \pi$. This is our boundary condition.

Moving forward, instead of vertical load balance (which we have done for the normal load case), here we need to consider the horizontal load balance per unit length of the semicircular region ABC. The stress distribution on the curved part of ABC must balance the applied shear load S acting on the top free surface, and that equation can be obtained as follows: $\int_0^{\pi}(\sigma_{rr}\cos\theta - \tau_{r\theta}\sin\theta) r d\theta = -S$, from the stress distribution for this given figure, where the direction of theta is slightly different than the previous one. You should note that change in θ.

Here, if you consider the horizontal component, the horizontal component of $\tau_{r\theta}$ is $\tau_{r\theta}\sin\theta$ acting toward the right, and for σ_{rr}, it is acting toward the left ($\sigma_{rr}\cos\theta$). So, the net force acting toward the left is ($\sigma_{rr}\cos\theta - \tau_{r\theta}\sin\theta$) integrated from 0 to π times $r d\theta$. And since this is the net leftward force, it should be balanced by S, which is also acting in the leftward direction. They can balance only if there is a change in sign. Thus, $-S$ should equal this particular integral. This gives us the horizontal force balance.

Now, as we want to have the boundary condition satisfied, where both the free surface boundaries are 0. So, these stresses should be 0, which can be done only if σ_{rr} and $\tau_{r\theta}$ are proportional to $\frac{1}{r}$. This can be ensured if the stress function is proportional to r. Hence, similar to the previous normal loading case, the stress function ϕ is chosen as r times some unknown function $\Theta(\theta)$.

Using the bi-harmonic condition and imposing the symmetry about $\theta = 0$ line, we can get Θ as $A_1 \cos\theta + D_1\theta \sin\theta$. Substituting this back into the stress function, the stress function for this problem is achieved as $r(A_1 \cos\theta + D_1\theta \sin\theta)$.

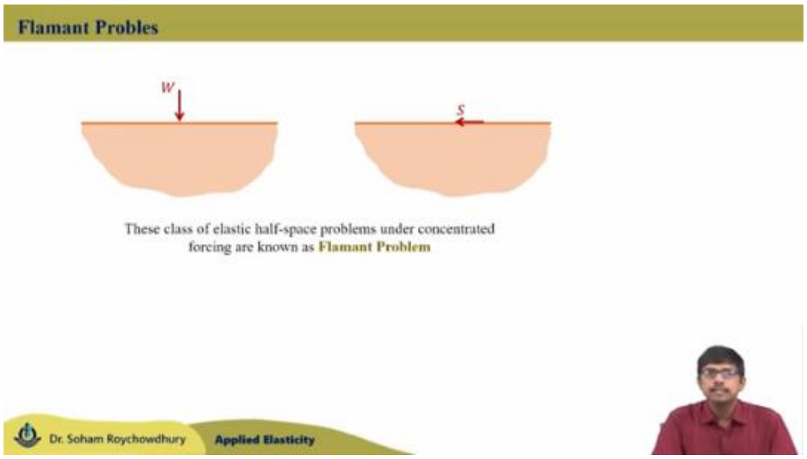


Moving forward with this form of ϕ, we can obtain the stress components as σ_{rr} is $\frac{2D_1}{r}\cos\theta$, and rest two, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$, both are equal to 0. Now, from the horizontal force balance equation, $\int_0^\pi(\sigma_{rr}\cos\theta - \tau_{r\theta}\theta\sin\theta) = -S$.

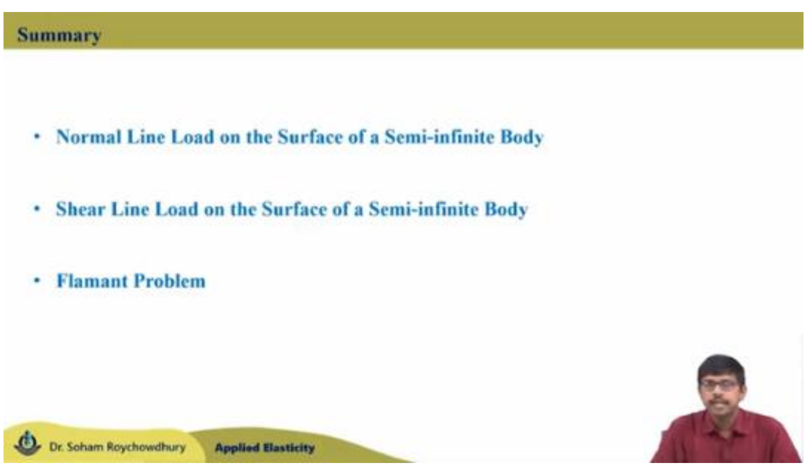
Substituting $\tau_{r\theta}$ to be 0 and also substituting this σ_{rr} as $\frac{2D_1}{r}\cos\theta$, we can simplify this equation as $-S = \int_0^\pi \left(\frac{2D_1}{r}\cos^2\theta\right) r d\theta$. This r will get canceled, and this integral will result in $D_1\pi$. Hence S and D_1 can be related, and D_1 can be written in terms of S as $-\frac{S}{\pi}$.

Substituting this D_1 back into the stress field, the stress field for this particular problem is obtained as: $\sigma_{rr} = -\frac{2S}{\pi r}\cos\theta$, and rest two, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are 0. This is the stress field generated for the present problem: an elastic half-space subjected to shear-type line loading S defined per unit length.

The solution of this and the previous problem are exactly the same: $\sigma_{\theta\theta}$, $\tau_{r\theta}$, are 0. Purely radial stress distribution is observed, and that stress distribution was $-\frac{2W}{\pi r}\cos\theta$ for the normal loading and $-\frac{2S}{\pi r}\cos\theta$ for the shear loading. So, the nature of the solution is exactly the same, only W is getting replaced with S.



Now, these two problems: one is the elastic half-space subjected to normal line loading, and another is the elastic half-space subjected to shear line loading. The first one is with W intensity, and the second one is with S intensity. These two in total are named as the Flamant problems. This class of elastic half-space problem, when it is subjected to the concentrated line loading either with intensity W or with intensity S, these are named as or termed as the Flamant problem in general.



In this lecture, we discussed two different elastic half-space problems: a semi-infinite elastic half-space subjected to normal line loading of intensity W and shear line loading of intensity S. Then, we defined this class of problems and named them as the Flamant problems.

Thank you.