

APPLIED ELASTICITY

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Week 10

Lecture 46: Semi-infinite Domain Problems I

COURSE ON:
APPLIED ELASTICITY

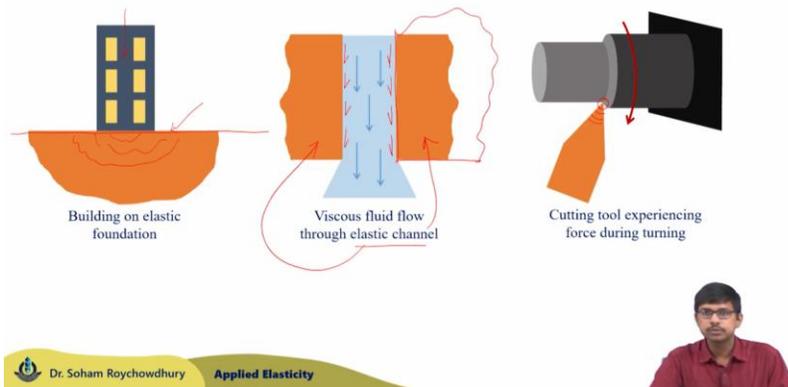
Lecture 46
SEMI-INFINITE DOMAIN
PROBLEMS I

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The slide features a yellow background with several diagrams and logos. On the left, there is a diagram of a beam on two supports with a downward arrow indicating a load. Next to it is a diagram of a rectangular block being deformed into a wavy shape. In the center, there is a 3D grid of a cube with axes labeled i , j , and m , and a stress tensor symbol $T_{i,k}$. Above the grid is a circular diagram with arrows pointing outwards. On the right, there is a portrait of Dr. Soham Roychowdhury, a man with glasses and a mustache, wearing a light pink shirt. The IIT Bhubaneswar logo is visible in the top right corner.

Welcome back to the course on Applied Elasticity. In today's lecture, we are going to start our discussion on a new topic: semi-infinite domain problems. We are going to talk about different types of semi-infinite domain problems in the next couple of lectures. What do we mean by semi-infinite domain problems? Let us first consider a few examples.

Semi-infinite Domain Problems



The first example is like this: a building standing on an elastic foundation. This is a common soil-structure interaction problem. Whenever we are constructing buildings or ships, this is mostly related to structural or civil engineering applications, where we need to determine the stress distribution generated at the base of the building and, accordingly, design the foundation of the building. Because of the weight due to this vertical building, there will be a certain stress field generated at the elastic foundation at the base of the building. Depending on the nature of the elastic foundation, which is basically the properties of the soil where the building is constructed, we need to design the type of foundation required for the building.

Here, the building can be modeled as a vertical load acting on this elastic foundation, and this elastic foundation is basically infinitely large below the free surface. This is the ground level; below that, this particular foundation is infinitely large. That is why we call it a semi-infinite domain problem. The boundary is only defined on one side through this surface line; below that, the body or the elastic foundation stretches to infinity. That is why, this is one example of a semi-infinite domain problem where the loading comes from the vertical weight of the building. This is a typical soil-structure interaction problem.

Considering another case where a viscous fluid is flowing through an elastic channel. This elastic channel is this part and this part, through which a fluid is flowing in the vertically downward direction. Due to the flow of the fluid, there will be shear stresses acting on the vertical edges of both sides of the elastic channel. If fluid is viscous,

viscosity is high or it is of corrosive nature, there will be high value of shear stresses acting on the vertical elastic channels, which may lead to wear of the surface of that particular channel. Thus, we need to understand the stresses generated in the elastic channel. These elastic channels are defined by this vertical boundary and horizontal boundary, but in these directions this is infinitely large. So, once again this is a semi-infinite domain problem.

If you consider the third example where this cutting tool is modeled as a plain elastic wedge. The cutting tool is trying to turn the workpiece in, let us say, lathe. In the turning operation being performed in lathe, this cutting tool is subjected to high value of cutting force at the contact point. So, near the cutting edge, at the point of contact between tool and work-piece, some large amount of force will be acting on the cutting tool which will try to result a stress distribution near the tip of the cutting tool. Here also, this cutting tool can be modeled as a plane elastic wedge which falls under the category of semi-infinite domain problem. To avoid the failure of the tip of the cutting tool, assessing the stress generated near the tip during this operation is important, and based on that, proper cutting parameters are required to be chosen.

These are the classes of few typical examples of the semi-infinite domain problem, where the semi-infinite elastic space, or semi-infinite elastic domain is subjected to various types of loading. In this particular week, in the next few lectures, we are going to take different such semi-infinite domain problems and discuss the solutions of those.

Quarter Plane with Uniform Face Shear Loading

Viscous fluid flow through elastic channel

$S \rightarrow$ Shear stress generated due to fluid flow

$[x \geq 0, y \geq 0] \rightarrow$ defines the domain of the problem

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In this particular lecture, we are going to take this quarter-plane problem subjected to uniform face shear loading, which is basically the modeling of viscous fluid flow through an elastic channel. As I had already told, as the viscous fluid is flowing through this elastic channel, this can be modeled as a problem like this.

So, this part of the elastic channel is considered here, where the horizontal axis x and vertical axis y mark the two boundaries of this channel. However, on this side, this is infinitely extended. That is why this is called a quarter-plane. If you consider one quarter of a complete elastic plane, that defines this particular semi-infinite domain, where it is bounded by $x = 0$ and $y = 0$ axes on two sides, while the other two sides are unbounded. Thus, it is a semi-infinite domain problem.

$x \geq 0$ and $y \geq 0$ defines the domain of this quarter-plane. For $x < 0$ and $y < 0$, the quarter-plane does not exist. Beyond $x \geq 0$ and $y \geq 0$, that domain is defined as the elastic domain.

Coming to the type of loading, which is taken to be a face shear load. S is the shear stress generated, which is acting on the vertical wall, that is, along the y -axis or $x = 0$ axis of this particular quarter-plane. And this shear stress is typically a result of the flow of viscous fluid through the elastic channel. As this fluid flows, it will give rise to some kind of shear stress on the elastic channel, on the vertical edge of the elastic channel, and this is the problem we are going to solve.

Our objective is to see what kind of stresses will be generated within this elastic quarter-plane, when this end shear force of stress S is acting on the vertical edge. A right-angle wedge or the quarter-plane is considered, which is subjected to the shear force stress S along the vertical edge, coming due to viscous fluid flow. For solving this particular problem, we need to write the boundary conditions.

Quarter Plane with Uniform Face Shear Loading

Boundary conditions:

$$\sigma_{yy}(x, 0) = \tau_{yx}(x, 0) = 0 \quad : y = 0 \text{ edge}$$

$$\sigma_{xx}(0, y) = 0, \quad \tau_{xy}(0, y) = S \quad : x = 0 \text{ edge}$$

In polar coordinates, $[0 \leq \theta \leq \frac{\pi}{2}]$

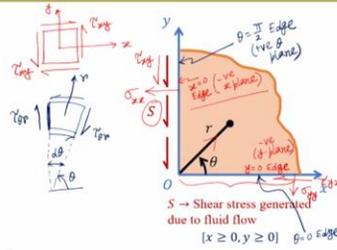
$$\sigma_{\theta\theta}(r, 0) = \tau_{\theta r}(r, 0) = 0 \quad : \theta = 0 \text{ edge}$$

$$\sigma_{\theta\theta}(r, \frac{\pi}{2}) = 0, \quad \tau_{\theta r}(r, \frac{\pi}{2}) = -S \quad : \theta = \frac{\pi}{2} \text{ edge}$$

The general solution to satisfy the biharmonic equation ($\nabla^4 \phi = 0$) is chosen as:

$$\phi(r, \theta) = R(r)\theta(\theta)$$

where, $R(r) = r^2$ so that the stress fields become independent of r , which is required to satisfy the given boundary conditions.



Coming to the surface traction boundary conditions. There are two finite boundaries prescribed or defined for this problem. One is $x = 0$, another is $y = 0$.

If I consider $y = 0$ edge, which is this, the x -axis, the bottom edge and another edge is this which is $x = 0$ edge. Considering the $y = 0$ edge, in this particular edge, this is free of any kind of surface traction. No normal traction, no shear traction is acting on this surface. So, $y = 0$ edge should be free of σ_{yy} and τ_{yx} .

Note that this is a negative y plane, and this one is negative x plane because the unit outward normal to this particular plane is along negative y direction; unit outward normal for this plane, $y = 0$ edge, is along negative y direction. Thus, we call it to be negative y plane. Similarly for the vertical plane, on which shear is acting, the unit outward normal is along the negative x direction thus, we call that to be negative x plane.

The stress boundary condition for the $y = 0$ edge can be defined on normal stress σ_{yy} and shear stress τ_{yx} as both of them are 0 if y is 0 for all values of x . So, $\sigma_{yy}(x, 0) = \tau_{yx}(x, 0) = 0$. These are the traction free boundary conditions on $y = 0$ edge.

Coming to the vertical edge, $x = 0$ edge, the normal traction is 0. Thus, the normal shear stress should be 0. Now, for this particular plane, this being the x -plane, the normal stress is defined as σ_{xx} , and the shear stress for this plane is defined as τ_{xy} . For the bottom plane, it was σ_{yy} as normal stress and τ_{yx} as the shear stress. Coming to the $x = 0$ edge,

the normal stress σ_{xx} is 0 for all values of y . So, $\sigma_{xx}(0, y)$ is 0, but $\tau_{xy}(0, y)$ should be equal to S , because S is the downward shear stress acting on the vertical edge, $x = 0$.

Why is this S , even if S is downward, and not $-S$? That is because if you consider the convention, this is the x -axis, this is the y -axis, and the shear stress convention is like this. τ_{xy} is positive along the positive y -axis for the positive x -plane, that is on the right-hand side vertical face. τ_{xy} is positive on the left-hand side face if it is acting downward along the negative y -direction on the negative x -plane. The present $x = 0$ edge being the negative x -plane, τ_{xy} is positive here along the downward direction, which is in the same direction as the applied loading S . Hence, τ_{xy} and S should have the same sign: $\tau_{xy}(0, y) = S$.

If the vertical shear stress on the vertical wall is upward, in that case, the boundary condition should have a negative sign: $\tau_{xy}(0, y)$ would be $-S$ in that case, but here as the shear is downward, $\tau_{xy}(0, y)$ will be $+S$.

These are the four boundary conditions we are having on two edges of this quarter plane defined due to the surface traction boundary conditions. These boundary conditions, we will convert into polar coordinates because it is convenient to solve the semi-infinite domain problem in terms of polar coordinates.

So, at point O , we are defining the origin of the polar coordinate with r being the radially outward coordinate and θ being the circumferential or angular coordinate measured counter-clockwise positive from the positive x -axis. The range of θ is defined between 0 to $\frac{\pi}{2}$. θ varying between 0 to $\frac{\pi}{2}$, we can convert these boundary conditions into the polar coordinate form as this. Here, the $y = 0$ edge is nothing but $\theta = 0$ edge. This edge equal to this can also be written as $\theta = 0$ edge. Whereas, this is defined by $\theta = \frac{\pi}{2}$ edge.

So, for $\theta = 0$ edge, we are having normal stress $\sigma_{\theta\theta}$ and shear stress $\tau_{\theta r}$ both to be 0. Whereas, for the $\theta = \frac{\pi}{2}$ edge, normal stress $\sigma_{\theta\theta}$ is 0, but $\tau_{\theta r}$ is $-S$.

Why $-S$? Let us draw the sign convention of this element in the polar coordinate. You have to be extremely careful regarding the sign convention while writing the boundary

condition. This is positive r direction, this is positive θ direction, with counter clockwise positive, and this angle is $d\theta$. Now, this particular plane is positive θ plane. For the positive θ plane, the direction of $\tau_{\theta r}$ is positive along the positive r direction. For the negative θ plane, $\tau_{\theta r}$ would be positive along the negative r direction.

Similarly, for this. Hence, at $\theta = \frac{\pi}{2}$ edge, which is the positive θ plane, $\tau_{\theta r}$ should be upward which is opposing the direction of given S . Thus, this boundary condition: $\tau_{\theta r}$ at $(r, \frac{\pi}{2})$ should be equal to $-S$ in polar coordinate. So, for every coordinate you have to draw the element to avoid the confusion following the sign convention and then write the boundary conditions correctly.

Moving forward, the general solution of this problem is chosen as a separated solution of R which is function of only r and Θ which is function of only θ . For the present problem this R , the radial variable-dependent function is chosen to be r^2 so that the stress field becomes independent of r , which is required to satisfy the boundary conditions.

Now, we will be looking at the boundary conditions in the polar coordinate. All the boundary conditions are either 0 or constant. They are independent of r . This can be satisfied only if we have the stress components independent of r , which is possible if the R part of the stress function is equal to r^2 . Only then, all the stress components σ_{rr} , $\sigma_{\theta\theta}$, $\tau_{r\theta}$ can be independent of r . They will be constant, either 0 or a non-zero constant.

Hence, we are choosing R to be a function of r as r^2 . And Θ is the unknown which we need to find out. How to obtain this Θ ? By solving the biharmonic equation, and for that, we will be using the general Michell solution discussed for the polar coordinate planar problems.

Quarter Plane with Uniform Face Shear Loading

The general Michell solution for polar coordinate elasticity formulation is given as,

$$\phi(r, \theta) = r^2 \Theta(\theta)$$

$$\phi(r, \theta) = R(r)e^{in\theta}, (n = 0, 1, 2, 3, \dots) = \sum_{n=0}^{\infty} f_n(r) \cos n\theta + \sum_{n=0}^{\infty} g_n(r) \sin n\theta$$

which can be expanded in the following form as,

$$\begin{aligned} \phi(r, \theta) = & (a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r) + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r) \theta \\ & + (a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r \theta + a_{16} r \theta \log r) \cos \theta + (b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r \theta + b_{16} r \theta \log r) \sin \theta \\ & + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta \end{aligned}$$

For a solution of the form $\phi(r, \theta) = r^2 \Theta(\theta)$, using the general Michell solution, the $\Theta(\theta)$ function is chosen as,

$$\phi(r, \theta) = r^2 (a_2 + a_6 \theta + a_{21} \cos 2\theta + b_{21} \sin 2\theta) \quad \text{considering } n \leq 2$$

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If you recall the Michell solution, that ϕ was given as $R(r)e^{in\theta}$, with n varying from 0, 1, 2, 3, and so on. This $e^{in\theta}$ can be expanded as the summation of the cosine series: $\cos(n\theta)$ term multiplied with $f_n(r)$ and the $\sin(n\theta)$ term multiplied with $g_n(r)$, summed over n equals 0 to ∞ .

In the expanded form, $\phi(r, \theta)$ was written like this. The first row of terms are the terms which are out of $\sin \theta$ and $\cos \theta$ series with $n = 0$. Then, the second row terms refer to $n = 1$, and the last row is the terms for $n \geq 2$, the series solution for over $\sin(n\theta)$ and $\cos(n\theta)$. This we had discussed in the previous week's lecture.

Here, as we have chosen our $\phi(r, \theta)$ as $r^2 \Theta(\theta)$, from the general solution, we will only choose the terms which contain the r^2 term. If you carefully look, here we have one term, and here also we have one term. No such term is present here in the $\cos \theta$ and $\sin \theta$ coefficients. However, here from $\cos(n\theta)$ and $\sin(n\theta)$, we can get such terms if I choose such terms by restricting n to 2.

We are taking terms up to $n = 2$. So, with $n = 2$, this r^n will be r^2 . Here also, this r^n will be r^2 . Thus, we would have four terms. So, with $n = 2$, we have only four terms available for the general Michell solution, which have coefficients of r^2 . Thus, ϕ would be $r^2(a_2 + a_6 \theta + a_{21} \cos 2\theta + b_{21} \sin 2\theta)$.

Here, the series is truncated for $n = 2$. Post that, $n = 3$ onwards, we have dropped the terms from the series. With this, we will try to obtain the solution. If it is achievable to satisfy all the boundary conditions and to get the stress field, then it is fine. If it's not, we

are not able to get it, then higher-order n values are required to be taken in the series solution.

For this present problem, it will be sufficient to consider n value up to 2. Thus, the stress function $\phi(r, \theta)$ will have a form like this, which is obtained from the general Michell solution.

Quarter Plane with Uniform Face Shear Loading

$$\phi(r, \theta) = r^2(a_2 + a_6\theta + a_{21}\cos 2\theta + b_{21}\sin 2\theta)$$

Stress components:

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 2a_2 + 2a_6\theta - 2a_{21}\cos 2\theta - 2b_{21}\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 2a_2 + 2a_6\theta + 2a_{21}\cos 2\theta + 2b_{21}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = -a_6 - 2b_{21}\cos 2\theta + 2a_{21}\sin 2\theta$$

$S \rightarrow$ Shear stress generated due to fluid flow
[$x \geq 0, y \geq 0$]

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Moving forward, with the help of this ϕ , using the standard equations of the stress components, we can obtain three non-zero stress components: σ_{rr} , $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ as this, involving these constants a_2 , a_6 , a_{21} , b_{21} , and some θ , $\sin 2\theta$, $\cos 2\theta$ terms. We will try to solve for these constants by using the boundary conditions.

Quarter Plane with Uniform Face Shear Loading

$$\sigma_{rr} = 2a_2 + 2a_6\theta - 2a_{21}\cos 2\theta - 2b_{21}\sin 2\theta$$

$$\sigma_{\theta\theta} = 2a_2 + 2a_6\theta + 2a_{21}\cos 2\theta + 2b_{21}\sin 2\theta$$

$$\tau_{r\theta} = -a_6 - 2b_{21}\cos 2\theta + 2a_{21}\sin 2\theta$$

Imposing the boundary conditions,

$$\sigma_{\theta\theta}(r, 0) = 0 \Rightarrow 2a_2 + 2a_{21} = 0$$

$$\tau_{\theta r}(r, 0) = 0 \Rightarrow -a_6 - 2b_{21} = 0$$

$$\sigma_{\theta\theta}(r, \frac{\pi}{2}) = 0 \Rightarrow 2a_2 + a_6\pi - 2a_{21} = 0$$

$$\tau_{\theta r}(r, \frac{\pi}{2}) = -S \Rightarrow -a_6 + 2b_{21} = -S$$

Solving,

$$a_2 = -\frac{S\pi}{8}$$

$$a_6 = \frac{S}{2}$$

$$a_{21} = \frac{S\pi}{8}$$

$$b_{21} = -\frac{S}{4}$$

$$\therefore \phi(r, \theta) = S \left(\frac{r^2\theta}{2} - \frac{\pi r^2}{8} + \frac{\pi r^2 \cos 2\theta}{8} - \frac{r^2 \sin 2\theta}{4} \right)$$

$S \rightarrow$ Shear stress generated due to fluid flow
[$x \geq 0, y \geq 0$]

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Imposing the boundary conditions in the polar coordinate. We were having four boundary conditions: two for $\theta = 0$ edge, and two for $\theta = \frac{\pi}{2}$ edge. $\sigma_{\theta\theta}$, $\tau_{\theta r}$, both are 0 for $\theta = 0$ or

this horizontal edge. $\sigma_{\theta\theta}$ is 0 for the vertical edge with $\theta = \frac{\pi}{2}$, but $\tau_{r\theta}$ at $\theta = \frac{\pi}{2}$ or for vertical edge is equal to $-S$. These are the boundary conditions for the vertical edge.

Substituting these expressions of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ in these four boundary conditions, we would be getting this set of equations: $2a_2 + 2a_{21} = 0$; $-a_6 - 2b_{21} = 0$; $2a_2 + a_6\pi - 2a_{21} = 0$; $-a_6 + 2b_{21} = -S$. These are the four equations we get by using the obtained stress components in the four boundary conditions. We can solve these four equations for our four unknowns because in the stress function we have four unknowns: a_2, a_6, a_{21}, b_{21} , and here we are having four equations involving those four unknowns.

Solving these four algebraic equations simultaneously, we will get the solution of a_2 as $-\frac{S\pi}{8}$, a_6 as $\frac{S}{2}$, a_{21} as $\frac{S\pi}{8}$, b_{21} as $-\frac{S}{4}$. Substituting all of these constants back in the stress function ϕ , the stress function for this present problem will be obtained as: $S \left(\frac{r^2\theta}{2} - \frac{\pi r^2}{8} + \frac{\pi r^2 \cos 2\theta}{8} - \frac{r^2 \sin 2\theta}{4} \right)$.

Quarter Plane with Uniform Face Shear Loading

$$\phi(r, \theta) = S \left(\frac{r^2\theta}{2} - \frac{\pi r^2}{8} + \frac{\pi r^2 \cos 2\theta}{8} - \frac{r^2 \sin 2\theta}{4} \right)$$

Stress fields:

$$\begin{cases} \sigma_{rr} = \frac{S}{2} \left(2\theta - \frac{\pi}{2} - \frac{\pi}{2} \cos 2\theta + \sin 2\theta \right) \\ \sigma_{\theta\theta} = \frac{S}{2} \left(2\theta - \frac{\pi}{2} + \frac{\pi}{2} \cos 2\theta - \sin 2\theta \right) \leftarrow \\ \tau_{r\theta} = \frac{S}{2} \left(-1 + \cos 2\theta + \frac{\pi}{2} \sin 2\theta \right) \leftarrow \end{cases}$$

At point O ($r = 0, \theta = 0$ or $r = 0, \theta = \frac{\pi}{2}$), $\sigma_{\theta\theta} = \tau_{r\theta} = 0$
(for both the edges)

Thus, no singularity is present in the obtained solution.

This is the stress function for the present problem and putting these constants back in the stress components, we can also obtain the complete stress field as this. $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$ for the present problem as: $\sigma_{rr} = \frac{S}{2} \left(2\theta - \frac{\pi}{2} - \frac{\pi}{2} \cos 2\theta + \sin 2\theta \right)$, then $\sigma_{\theta\theta}$, the normal stress along the θ direction would be $\frac{S}{2} \left(2\theta - \frac{\pi}{2} + \frac{\pi}{2} \cos 2\theta - \sin 2\theta \right)$, and the in-plane shear stress $\tau_{r\theta}$ would be $\frac{S}{2} \left(-1 + \cos 2\theta + \frac{\pi}{2} \sin 2\theta \right)$.

This is the stress field obtained for the present problem, which satisfies all the boundary conditions exactly, and thus, the solution, whatever we had assumed from the general Michell solution, that is, restricting the n value up to 2, is admissible for the present problem. The obtained solution by the restriction of n up to 2 is going to satisfy all our boundary conditions, and thus, these stress fields are valid for the present problem.

Now, if you look at the origin, that is point O . Point O is the intersection of two edges. One is the horizontal edge, and another is the vertical edge. The horizontal edge is given by $\theta = 0$. The vertical edge is given by $\theta = \frac{\pi}{2}$. If you substitute $\theta = 0$ with $r = 0$. So, considering the horizontal edge, the coordinate of point O is $r = 0, \theta = 0$. $r = 0, \theta = 0$ is the coordinate of point O if you are considering the x -axis or the horizontal edge, not the vertical edge. Considering the vertical edge or the y -axis, the coordinate of point O is $r = 0$ and $\theta = \frac{\pi}{2}$.

If you substitute both cases at point O , we will get $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ to be 0. As both these two are θ planes, we can only define $\sigma_{\theta\theta}$ and $\tau_{r\theta}$. With $\theta = 0$, in the $\sigma_{\theta\theta}$, the 2θ term, the $\sin 2\theta$ term will vanish. $\cos 2\theta$ at $\theta = 0$ is 1, so this $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ will cancel each other, thus $\sigma_{\theta\theta}$ would be 0.

Similarly, in $\tau_{r\theta}$, this $\cos 2\theta$ would be equal to +1 for $\theta = 0$, and the $\sin 2\theta$ term would go to 0. So, this -1 and the $+1$ will cancel each other and result in $\tau_{r\theta}$ being 0. Similarly, substituting θ to be $\frac{\pi}{2}$ in both these equations, you can verify that both $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ would come out to be 0.

So, there is no discontinuity in the solution at the junction point O . Considering both horizontal and vertical edges, $\theta = 0$ edge or $\theta = \frac{\pi}{2}$ edge, we can end up with the same solution of $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ to be 0 at the junction point of these two edges, that is at point O .

And note that at point O , both the stresses are 0. There is no singularity present in the obtained solution at the junction. It is common to have singularity at the junction point for many structural applications. Where the semi-infinite domains are subjected to some point loads, at the point of loading or where the edges meet, at those junction points,

there is a possibility of having singularity, which is not there for the present problem if described with this r and θ coordinate.

Quarter Plane with Uniform Face Shear Loading

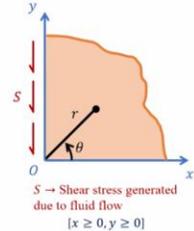
$$\phi(r, \theta) = S \left(\frac{r^2 \theta}{2} - \frac{\pi r^2}{8} + \frac{\pi r^2 \cos 2\theta}{8} - \frac{r^2 \sin 2\theta}{4} \right)$$

Stress function in terms of Cartesian variables:

Using, $r = \sqrt{x^2 + y^2}$, $\tan \theta = \frac{y}{x}$

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{x^2 - y^2}{x^2 + y^2} \Rightarrow r^2 \cos 2\theta = (x^2 - y^2)$$

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2xy}{x^2 + y^2} \Rightarrow r^2 \sin 2\theta = 2xy$$

$$\therefore \phi(x, y) = S \left[\frac{(x^2 + y^2)}{2} \tan^{-1} \left(\frac{y}{x} \right) - \frac{\pi(x^2 + y^2)}{8} + \frac{\pi(x^2 - y^2)}{8} - \frac{xy}{2} \right]$$


$S \rightarrow$ Shear stress generated due to fluid flow
[$x \geq 0, y \geq 0$]



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Now, let us convert this problem into x, y coordinates from this r, θ or polar coordinates, let us convert it into the rectangular Cartesian coordinate system, and the stress function ϕ , we will be writing in terms of Cartesian variables x and y by using this transformation. So, θ and r are defined like this: $\tan \theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. With the help of this, $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$, and $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$.

Using these trigonometric identities and substituting $\tan \theta$ as $\frac{y}{x}$ here, we can write $\cos 2\theta$ as $\frac{x^2 - y^2}{x^2 + y^2}$, and $\sin 2\theta$ as $\frac{2xy}{x^2 + y^2}$. Note that $x^2 + y^2$ is nothing but r^2 in the denominator of both terms, and from this, we can write $r^2 \cos 2\theta = x^2 - y^2$, and this $r^2 \cos 2\theta$ term is already present here in the stress function. Similarly, from the next equation, $r^2 \sin 2\theta$ can be written as $2xy$, which is also present in the last term of the stress function.

This r and θ we already know in terms of this; this r can also be written in terms of x and y . $r^2 \cos 2\theta$ is written as $x^2 - y^2$, and $r^2 \sin 2\theta$ is written as $2xy$. With these substitutions, the stress function ϕ in Cartesian variables, in terms of x and y , can be written like this.

Quarter Plane with Uniform Face Shear Loading

$$\phi(x, y) = S \left[\left(\frac{x^2 + y^2}{2} \right) \tan^{-1} \left(\frac{y}{x} \right) - \frac{\pi(x^2 + y^2)}{8} + \frac{\pi(x^2 - y^2)}{8} - \frac{xy}{2} \right]$$

The shear stress in terms of Cartesian variables becomes $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{Sy^2}{x^2 + y^2}$

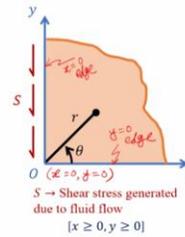
At $x = 0$ edge, $\tau_{xy} = S$
 At $y = 0$ edge, $\tau_{xy} = 0$ } Predicts the edge boundary conditions correctly

However, at point O ($x = 0, y = 0$), τ_{xy} becomes indeterminate.

Thus, the same stress function, when expressed in Cartesian coordinate system, has a singularity at tip.

This is known as **Timoshenko's paradox**.

This occurs due to the presence of $r^2 \theta$ term, which makes stresses bounded function of θ only, but not so in x - y frame of reference.



Now, we will try to get the shear stress component in the Cartesian coordinate by using ϕ as a function of x and y . The shear stress in the Cartesian coordinate is given by τ_{xy} as $-\frac{\partial^2 \phi}{\partial x \partial y}$. If you substitute this form of ϕ here as a function of x and y , you will get τ_{xy} as $\frac{Sy^2}{x^2 + y^2}$. This is the shear stress distribution in terms of the Cartesian variables x and y .

Now, let us see if this satisfies the boundary conditions or not. Along the $x = 0$ edge, this vertical edge is given by the $x = 0$ edge. Along that if you put $x = 0$, you will get τ_{xy} to be S and if you check with the boundary condition which we had initially written in terms of x, y coordinates, this was the boundary condition for $x = 0$ edge. So, getting satisfied.

Coming to the horizontal edge, which is $y = 0$ edge, for that, putting $y = 0$, τ_{xy} would be 0 because y^2 is there in the numerator. So, τ_{xy} is 0 in the $y = 0$ edge, that boundary condition is also satisfied. τ_{xy} when it is described in terms of Cartesian variable x, y , that is also predicting the edge boundary conditions properly.

Coming to the point O , *i.e.*, solution of the junction point at point O in the Cartesian coordinates $x = 0$ and $y = 0$. The position vector of point O is given by or denoted by $(0, 0)$. At $(0, 0)$, due to presence of this $x^2 + y^2$ in the denominator, this τ_{xy} would be indeterminate. Thus, there exists a singularity in the stress function at the tip, at the origin O , when it is expressed in the Cartesian coordinate system and this is surprising.

The same stress function defined in polar coordinate was not giving any singularity at O . Just being converted into Cartesian coordinate by using $x = r \cos \theta, y = r \sin \theta$ relation,

and then using the Cartesian coordinate shear stress variable τ_{xy} , the same stress function is resulting a singularity at the same point O . The solution is becoming dependent on the type of the coordinate system being used. In polar coordinates, there is no singularity at point O , but in the Cartesian coordinate stress function, there is a singularity existing at the tip, which is normally unexpected.

This particular phenomena is named as Timoshenko's paradox, which is observed for the quarter plane problem subjected to shear face loading, and this occurs due to the presence of $r^2\theta$ term in the stress function, which makes the stresses bounded in the r, θ plane, but once converted into x, y , that would result a $\tan^{-1}\left(\frac{y}{x}\right)$ term which would become unbounded at $(0,0)$. This is the reason for Timoshenko's paradox observed in such quarter-plane problems.

Summary

- Semi-infinite Domain Problems
- Quarter Plane with Uniform Face Shear Loading
 - Stress Distribution
 - Timoshenko's Paradox



In the present lecture, we discussed the introduction to various types of semi-infinite domain problems. Then, we solved one particular semi-infinite domain problem: an elastic quarter-plane subjected to uniform shear loading on the vertical edge. We obtained the corresponding stress distribution, and also discussed the phenomenon of Timoshenko's paradox.

Thank you.