

# APPLIED ELASTICITY

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WEEK: 09

Lecture- 45

COURSE ON:  
APPLIED ELASTICITY

Lecture 45  
BENDING OF CURVED BEAMS II

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Welcome back to the course of applied elasticity. In this lecture, we are going to continue our discussion on the bending of curved beams, which we had started in the previous lecture.

Curved Beam Bending

In various mechanical applications, different curved structural members (such as arches, rings, curved pipes) undergo bending when subjected to

a) Pure bending moment

b) Transverse shear load

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So, the different curved beam elements or the curved structural elements, such as the rings, bent pipes, curved members, curved rods, or arches, undergo bending when they are subjected to either pure bending moment or to transverse shear loading.

So, some such applications were shown like this, where if you have this U-shaped bar subjected to the end shear force  $F$  in the equal in the two opposite directions of equal magnitude, that would be causing the bending of the curved part of the structure. So, this part is curved for this U-shaped structure, which will be undergoing bending due to the application of this force  $F$ . Similarly, if you consider the second figure, which is the hook problem.

So, these hooks are common weight-carrying elements. So, when the hooks are carrying some weight due to the weight of that element or the quantity that it is carrying, that would apply a vertically downward force  $F$  at this particular section of the hook, and this hook is also modeled as a curved beam or curved beam element. Now, for such problems, along with the bending moment, a transverse shear force is also acting on that particular section. So, if you consider this particular section or if you consider any other section of the hook, let us say here.

So, here  $F$  is creating some bending moment, along with that,  $F$  itself is acting as a vertical shear force. So, the effect of this transverse force, along with the bending, both would be present for this kind of hook problem. The curved members, curved structural members, may undergo bending when they are subjected to only bending moment. They may undergo bending when they are subjected to transverse shear load or even a combination of bending moment and shear load. A transverse shear load will cause a bending moment, along with that, the direct load will be creating some non-zero value of direct transverse shear stress as well.

So, in today's lecture, we are going to talk about the bending of curved beams under transverse shear loading. In the previous lecture, we discussed the first case: the pure bending of a curved beam. In this lecture, our discussion will focus on the bending of curved structures when they are subjected to transverse shear loading.

## Bending of Curved Beam with End Shear Load

$R_1$ : Inner radius

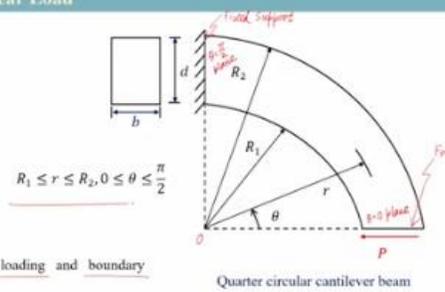
$R_2$ : Outer radius

$P$ : End shear force

$b$ : Width of rectangular cross-section

$d$ : Depth of rectangular cross-section

This is a **non-axisymmetric problem** as the loading and boundary conditions are not axisymmetric.



So, let us consider this particular geometry, where a quarter-circular cantilever beam is analyzed. Quarter-circular means it spans one quarter of a complete circle.

So, theta varies between 0 to  $\pi/2$ , or 0 to 90 degrees. We are considering this quarter-circular curved beam, which is fixed at one end. At this particular end, we have the fixed support. The other end, this end, is free. Thus, we have this quarter-circular cantilever curved beam structure.

Now,  $R_i$  or  $R_1$  and  $R_2$  are the inner and outer radii of this cantilever curved beam, respectively.  $R_i$  is the inner radius measured from center  $O$ , and  $R_2$  is the outer radius measured from center  $O$ . Now, theta and  $r$  are the polar coordinates we will use to solve the problem. Now, theta starts from the free edge. This free edge refers to the  $\theta = 0$  plane, and the cantilever edge refers to the  $\theta = \pi/2$  plane.

So, the  $\theta$  value is 0 for the free edge, and the  $\theta$  value is pi by 2 for the cantilever or the fixed edge. Now,  $P$  is the end shear force. So, as we are going to consider the curved beam bending with the transverse shear loading at the free end, we are subjecting the beam to a transverse shear load of magnitude  $P$ . At  $\theta$  equals 0, this end force  $P$  is acting toward the

center of the beam, that is, toward point  $O$ . The direction can be reversed; in that case, the solution would be almost similar, with  $P$  just having a minus sign. Now, coming to the cross-section, we are considering the beam cross-section to be uniform, and that is given by this rectangular geometry with width  $b$  and depth  $d$ . So,  $b$  and  $d$  are the two dimensions of the rectangular cross-section, and those are not varying over the curved length of the beam.

Now, here if you carefully look at the problem, the given geometry, loading, and boundary condition. And if you compare this with the previous curved beam bending problem when it was subjected to pure bending. The previous case was a case of an axisymmetric problem because the geometry, the loading, and the boundary condition all have one axis of symmetry. Now, coming to the present problem, as the cantilever beam is part of a circle for one quarter,

the geometry is axisymmetric about an out-of-plane axis passing through point  $O$ . However, the loading  $P$  and boundary condition defined at  $\theta$  equals to  $\frac{\pi}{2}$  plane are both non-axisymmetric. They are breaking the nature of axisymmetry of the problem. So, here in general, we cannot consider the problem to be an axisymmetric problem because of the non-axisymmetric nature of the external loading  $P$ . and the non-axisymmetric nature of the boundary condition at  $\theta$  equals to  $\frac{\pi}{2}$  plane.

So, this problem is required to be solved as a non-axisymmetric problem in the polar coordinate. The problem can be axisymmetric only if loading, boundary condition, body force, as well as geometry, also material properties, everything is axisymmetric about the same axis of symmetry. That is violated here. Thus, this curved beam bending problem when subjected to transverse end shear load should be solved as a non-axisymmetric bending problem.

Here, for this problem, the range of  $\theta$  is defined from 0 to  $\frac{\pi}{2}$ , and the range of the radial coordinate  $r$  is defined from  $R_1$  to  $R_2$ . So, these define the domain of the present non-axisymmetric problem. Now, moving forward, as it is a case of a non-axisymmetric problem, the axisymmetric solution technique is not going to be applicable for this. We have to solve it with the help of a general stress function, which would be dependent on  $\theta$ . The solution, or the bending stresses, would be dependent on  $\theta$ , and we need to solve it with the help of the required boundary conditions.

## Bending of Curved Beam with End Shear Load

### Boundary conditions:

(1) Due to traction free top and bottom surfaces,

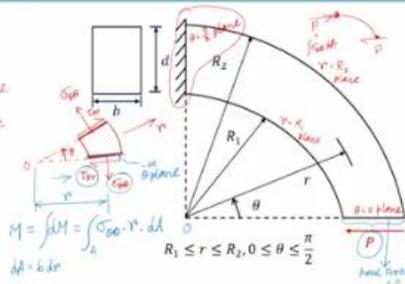
$$\sigma_{rr}(R_1, \theta) = \tau_{r\theta}(R_1, \theta) = 0 \quad \because r = R_1 \text{ plane}$$

$$\sigma_{rr}(R_2, \theta) = \tau_{r\theta}(R_2, \theta) = 0 \quad \because r = R_2 \text{ plane}$$

(2) Shear force at free end ( $\theta = 0$ ),  $\int_{R_1}^{R_2} b\tau_{\theta r} dr = P$

(3) Axial force at free end ( $\theta = 0$ ),  $\int_{R_1}^{R_2} b\sigma_{\theta\theta} dr = 0$

(4) Bending moment at free end ( $\theta = 0$ ),  $\int_{R_1}^{R_2} b\sigma_{\theta\theta} r dr = 0$



So, let us look into the boundary conditions for the present problem. So, the first boundary condition of this problem is the same as the previous problem's first boundary condition, that is, the top and bottom curved surfaces are free of any kind of surface traction. So, the inner curved face is defined by  $r$  equals to capital  $R_1$  plane, and the outer surface is defined as small  $r$  equals to capital  $R_2$  plane. Both of these two curved surfaces, top and bottom curved surfaces, are free of any external normal traction or any external shear traction.

Thus,  $\sigma_{rr}$  and  $\tau_{r\theta}$  both should be 0 for  $R$  equals to  $R_1$  and  $R$  equals to  $R_2$  surface for any value of  $\theta$ . So,  $\sigma_{rr}(R_1, \theta) = \tau_{r\theta}(R_1, \theta) = 0$ . This is for  $r$  equals to capital  $R_1$  plane, and the second one is for  $r$  equals to capital  $R_2$  plane:  $\sigma_{rr}$  and  $\tau_{r\theta}$  0 at  $(R_2, \theta)$ . Now, coming to the next boundary condition, which is the definition of shear force  $P$  at the free end, the transverse shear force is acting at the free end, which is the end load  $P$ .

Now, that should give rise to shear stress distribution at the free edge. The free edge was defined by theta equals to 0 plane. The fixed edge was defined as theta equals to  $\frac{\pi}{2}$  plane for the present problem. So, at the free end defined by  $\theta$  equals to 0, the resultant shear stress distribution should be equal to the applied load  $P$ .

Now, if you consider one small element in the polar coordinate like this, the center is somewhere here. Now, here this plane being the  $\theta$  plane, theta starts in this direction measured counter-clockwise positive. So, on the theta plane, we will be having this  $\sigma_{\theta\theta}$  direction;  $r$  direction is radially outward positive,  $\theta$  is counter-clockwise positive. Now, following the sign convention on the positive  $\theta$  plane.

That is, on this particular face, we are having  $\tau_{r\theta}$  or  $\tau_{\theta r}$  to be positive in this direction. So, on the positive  $\theta$  plane,  $\sigma_{\theta\theta}$  and  $\tau_{\theta r}$  would be positive along the positive  $\theta$  and

positive  $r$  directions, whereas, on the negative theta plane, this is the negative  $\theta$  plane where the direction of  $\tau_{\theta r}$  would be along the negative  $r$  direction, the direction of  $\sigma_{\theta\theta}$  would be along the negative  $\theta$  direction.

So, the direction of the given  $P$  and direction of  $\tau_{r\theta}$  at  $\theta$  equals to at negative  $\theta$  plane, this is negative  $\theta$  plane that is at  $\theta$  equals to 0, the direction of  $\tau_{\theta r}$  is towards left, towards centre, the direction of the given  $P$  is also towards centre, they are matching, thus they would be having same sign. So, this  $\tau_{\theta r}$  variation, when it is integrated over the entire cross section from  $R_1$  to  $R_2$ ,

So,  $\int_{R_1}^{R_2} b\tau_{\theta r} dr$  these should be equals to the applied external load  $P$  as  $P$  is giving rise to these non-zero  $\tau_{r\theta}$ . This is the second boundary condition shear force at the free end  $\theta$  equals to 0. Now, coming to the axial force at the free end here at the free end the axial force in this direction is equals to 0. No axial force is acting at the free end for the bar.

So, as the axial force is equals to 0, the integration of this normal stress  $\sigma_{\theta\theta}$  over the entire area at the free end should be 0. So,  $\sigma_{\theta\theta}$  is the normal stress responsible for axial force. As there is no axial force at the free end,  $\int_{R_1}^{R_2} b\sigma_{\theta\theta} dr = 0$  and now coming to the bending moment. Bending moment is defined as the moment of  $\sigma_{\theta\theta}$ .

So, about the center, if from the center to this point the distance is  $r$ . So, the moment created,  $dm$ , due to this  $\sigma_{\theta\theta}$  is  $\sigma_{\theta\theta}r$  times the small elemental area of this face,  $da$ . So, this is the elemental moment created about the center point  $O$  due to the  $\sigma_{\theta\theta}$ . If you want to find out the total bending moment about the cross-section, that should be the integral of this  $dm$ .

So, if you integrate this  $dm$ , we would be getting the total bending moment over the entire area. So, this expression  $b\sigma_{\theta\theta}rdr$ . So, for our problem of a rectangular cross-section,  $da$  is  $bdr$ . So, putting it there,  $\int_{R_1}^{R_2} b\sigma_{\theta\theta}rdr$ . This gives us the expression of the total bending moment.

This is the bending moment  $M$ . Now, as the free end for this cantilever beam should be subjected to 0 bending moment. For any cantilever beam, at the free end, the bending moment should be 0. So, thus at  $\theta$  equals to 0, which refers to the free end, at  $\theta$  equal to 0 or free end, we must have the bending moment to be 0. This gives us the fourth boundary condition. So, these are the boundary conditions for the present problem which we need to solve.

## Bending of Curved Beam with End Shear Load

Choice of stress function:

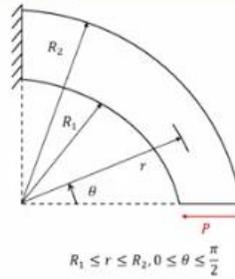
$$\phi(r, \theta) = R(r) \sin \theta \quad [\text{motivated by the boundary conditions}]$$

Biharmonic equation:

$$\begin{aligned} \nabla^4 \phi &= 0 \\ \Rightarrow \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right) &= 0 \\ \Rightarrow R(r) &= Ar^3 + \frac{B}{r} + Cr + Dr \ln r \end{aligned}$$

$$\therefore \phi(r, \theta) = \left( Ar^3 + \frac{B}{r} + Cr + Dr \ln r \right) \sin \theta$$

$$\begin{aligned} \nabla^2 \phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ &= \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right) \sin \theta \end{aligned}$$



Now, if you look at this at the  $\theta$  equals 0 plane or free end, axial force and bending moment are 0, and shear force is at its maximum. Now, if you go for  $\theta$  equals  $\frac{\pi}{2}$ , the fixed there, the shear force in the vertical direction would be 0. The force along the horizontal direction, which is basically axial force, would be non-zero because that should be balancing  $P$ . So, if you draw the free body diagram of this bar which is subjected to an end shear force  $P$  here at the free end, on the fixed end you should have a  $P$  like this, which will be basically the integral of  $\sigma_{\theta\theta} da$ , because that is acting along the axial direction at the  $\theta$  equals  $\frac{\pi}{2}$  plane. Hence, the axial force varies from 0 to a non-zero value as  $\theta$  goes from 0 to  $\frac{\pi}{2}$ , whereas the shear force  $P$  varies from a non-zero value to 0 as you go from 0 to  $\frac{\pi}{2}$ . From the free edge to the fixed edge.

Now, these kinds of stress distributions, these kinds of boundary conditions or stress distributions, can be obtained if we choose the stress function of this form. which is dependent on sine  $\theta$ . So, the  $\theta$ -dependent part of the stress function is chosen to be sine  $\theta$  multiplied by some arbitrary function of small  $r$  which is taken to be capital  $R$ . So, we are taking a separable solution for the stress function  $\phi(r, \theta)$ , which has the radial component as capital  $R, \theta$ .

Function of smaller times sin  $\theta$ . Now, why sin  $\theta$ ? Sin  $\theta$  would be 0 at  $\theta$  equals 0, and it would be 1 at  $\theta$  equals  $\frac{\pi}{2}$ . Some of the stress components would have a sin  $\theta$  term, while others would have a cos  $\theta$  term. The given boundary conditions motivate us to choose the stress function like this.

From that, we can get the shear force to be non-zero at  $\theta$  equals 0, but zero at  $\theta$  equals  $\frac{\pi}{2}$  or at the fixed state, same for the axial stress that we will verify. Now, this stress function must satisfy the biharmonic equation  $\nabla^4 \phi = 0$ . Substituting this stress function  $\phi(r, \theta) = R(r) \sin \theta$  in the biharmonic equation. So, this is the biharmonic equation.

This is the Laplacian operator of  $\phi$ . If Laplacian  $\phi$  is expanded with  $\phi$  being substituted as  $R(r) \sin \theta$ , Laplacian of  $\phi$  can be obtained as  $\sin \theta \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right)$ . Now, this is the Laplacian operator of  $\phi$ . The biharmonic operator would be double of this.

So, the same operator will be acting twice. So, this is one Laplacian, and this is another Laplacian of  $R$ . So, in total, this should be 0, then only the chosen form of  $\phi$  would satisfy the biharmonic equation. And from this equation, if we integrate it and solve for capital  $R$ , because sine theta is getting canceled, as the right-hand side is 0, all the terms of this Laplacian operator contain sine  $\theta$  and thus it would be canceled.

So, the equation would only be an ordinary differential equation of capital  $R$ , which can be solved as  $Ar^3 + \frac{B}{r} + Cr + Dr \ln r$ . This is the solution of capital  $R$ , the  $R$ -dependent part of the stress function, which satisfies the biharmonic equation. So, substituting it back in  $\phi$ , the stress function would be  $Ar^3 + \frac{B}{r} + Cr + Dr \ln r$ , the entire thing multiplied with  $\sin \theta$ .

**Bending of Curved Beam with End Shear Load**

$$\phi(r, \theta) = \left( Ar^3 + \frac{B}{r} + Cr + Dr \ln r \right) \sin \theta$$

Stress components:

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \left( 6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = -\left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta$$

$\left\{ \begin{array}{l} M, \theta=0: \sigma_{rr} = \sigma_{\theta\theta} = 0, \tau_{r\theta} \neq 0 \\ M, \theta=\frac{\pi}{2}: \sigma_{rr} \neq 0, \sigma_{\theta\theta} \neq 0, \tau_{r\theta} = 0 \end{array} \right.$



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Now, with this particular stress function, we can obtain the stress components using the definition of the stress component for the 2D polar problem in terms of the stress function  $\phi$ ,  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$ , and  $\tau_{r\theta}$  can be obtained respectively as  $\sigma_{rr} = \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$ ,  $\sigma_{\theta\theta} = \left( 6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$ , and  $\tau_{r\theta} = -\left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta$ . Note that the variation of  $\sigma_{rr}$  and  $\tau$  and  $\sigma_{\theta\theta}$  are similar.

Both of them have a  $\sin \theta$  type variation. Whereas,  $\tau_{r\theta}$  has a  $\cos \theta$  variation. Hence, at  $\theta = 0$ ,  $\sigma_{rr}$  and  $\sigma_{\theta\theta}$  will be 0. However,  $\tau_{r\theta}$  would be non-zero. On the other hand, at  $\theta$  equals  $\frac{\pi}{2}$ ,  $\sigma_{rr}$  is non-zero, and  $\sigma_{\theta\theta}$  is non-zero.

But,  $\tau_{r\theta}$  would be 0. Now, as we have chosen a  $\sin \theta$  form or  $\sin \theta$  term in the stress function, that results in this set of stress components, which has these particular properties as  $\theta$  varies from 0 to  $\frac{\pi}{2}$ , from the free edge to the cantilever's fixed edge.

This will help us satisfy the boundary conditions. Note one more thing: the coefficient present with  $\sin \theta$  in the  $\sigma_{rr}$  term is the same as the coefficient present with  $\cos \theta$ , but with a negative sign, in the  $\tau_{r\theta}$  term.

**Bending of Curved Beam with End Shear Load**

**B.C. (I):**  $\sigma_{rr}(R_1, \theta) = \tau_{r\theta}(R_1, \theta) = 0$   
 $\sigma_{rr}(R_2, \theta) = \tau_{r\theta}(R_2, \theta) = 0$

$$\sigma_{rr} = \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$$

$$\tau_{r\theta} = - \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta$$

$$\sigma_{rr}(R_1, \theta) = \tau_{r\theta}(R_1, \theta) = 0 \Rightarrow 2AR_1 - \frac{2B}{R_1^3} + \frac{D}{R_1} = 0$$

$$\sigma_{rr}(R_2, \theta) = \tau_{r\theta}(R_2, \theta) = 0 \Rightarrow 2AR_2 - \frac{2B}{R_2^3} + \frac{D}{R_2} = 0$$


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Now, we will move forward and try to satisfy the boundary conditions. So, looking at the first boundary condition, which is traction-free top and bottom curved faces,  $\sigma_{rr}$  and  $\tau_{r\theta}$  are 0 at  $r$  equals  $R_1$ , and they are also 0 at  $r$  equals  $R_2$ .

Now,  $\sigma_{rr}$  and  $\tau_{r\theta}$  expressions were obtained like this. Substituting smaller as  $R_1$  and then smaller as  $R_2$ , we can write these four boundary conditions as this. Now, as I had told, the coefficient of  $\sin \theta$  within  $\sigma_{rr}$  is the same as the coefficient of  $\tau_{r\theta}$  with a negative sign. Then, at  $r$  equals to  $R_1$ , For any value of  $\theta$ , if we are equating  $\sigma_{rr}$  to be 0 and  $\tau_{r\theta}$  to be 0, both of them would result in the same equation because, for both cases, either  $\sin \theta$  or cosine theta would get canceled. So,  $\sigma_{rr}(R_1, \theta) = 0$  would result in this. That is,  $2AR_1 - \frac{2B}{R_1^3} + \frac{D}{R_1} = 0$ . Also, this part,  $\tau_{r\theta}(R_1, \theta) = 0$ , will result in the same equation once again.

So, both boundary conditions at  $R_1$  for the inner boundary or inner curve phase being traction-free, we are getting a single equation. We are not getting two different equations from these two conditions. Similarly, at the outer boundary, that is, the outer curve phase  $r$  equals to  $R_2$ , both  $\sigma_{rr}$  condition and  $\tau_{r\theta}$  condition would give us the same equation, which is  $2AR_2 - \frac{2B}{R_2^3} + \frac{D}{R_2} = 0$ . So, from boundary condition 1, we have these two equations available, which we will use to solve for  $A$ ,  $B$ , and  $D$ , the three unknowns.

### Bending of Curved Beam with End Shear Load

B.C. (3):  $\int_{R_1}^{R_2} b \sigma_{\theta\theta}|_{\theta=0} dr = 0$  (Satisfied)

B.C. (4):  $\int_{R_1}^{R_2} b \sigma_{\theta\theta}|_{\theta=0} r dr = 0$  (Satisfied)

B.C. (2):  $\int_{R_1}^{R_2} b \tau_{\theta r}|_{\theta=0} dr = P$

$$\Rightarrow - \int_{R_1}^{R_2} b \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) dr = P$$

$$\Rightarrow -A(R_2^2 - R_1^2) + B \left( \frac{R_2^2 - R_1^2}{R_1^2 R_2^2} \right) - D \ln \frac{R_2}{R_1} = \frac{P}{b}$$

$$\sigma_{\theta\theta} = \left( 6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$$

$$\sigma_{\theta\theta}|_{\theta=0} = 0$$

$$\tau_{r\theta} = - \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta$$



Now, moving forward to the third boundary condition, which is the  $\int_{R_1}^{R_2} b \sigma_{\theta\theta}|_{\theta=0} dr = 0$ . So, what was the physical significance of this boundary condition?

At  $\theta$  equals 0, that is, at the free edge, the total axial force is 0. The integral of sigma theta theta over the entire area of the free edge is 0. Also, at the free edge, the bending moment is 0, which was expressed in the fourth boundary condition. So, the  $\int_{R_1}^{R_2} b \sigma_{\theta\theta}|_{\theta=0} r dr = 0$ . If we are integrating it from  $R_1$  to  $R_2$ . So, this third condition was axial force 0, the fourth condition was bending moment 0, and both were defined with respect to  $\sigma_{\theta\theta}|_{\theta=0}$ . Now, if you look at the  $\sigma_{\theta\theta}$  expression, that was for this chosen stress function, the obtained expression of  $\sigma_{\theta\theta}$  was  $\left( 6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta$ . Now, at  $\theta$  equals 0, due to the presence of the  $\sin \theta$  term,  $\sigma_{\theta\theta}$  would be the integrand in both boundary condition 3 and 4 would vanish; they will go to 0, and thus both the boundary conditions are automatically satisfied.

No need to impose any extra constraint for satisfying the third and fourth boundary conditions, that is, zero axial force. and zero bending moment at the free end of the curved beam at theta equals zero. Now, moving to the remaining last boundary condition, that is, boundary condition number 2, which was the shear force definition at the free end. So, the integral of the shear stress at the free end theta equals zero should be equal to shear force  $P$ .  $\int_{R_1}^{R_2} b \tau_{\theta r}|_{\theta=0} dr = P$ . Now, theta  $\tau_{r\theta}$  or  $\tau_{\theta r}$  expression was obtained as  $-\left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta$ . Now, this term would be non-zero at  $\theta$  equals zero, these  $\cos \theta$  would be equal to one. Thus,  $\tau_{r\theta}$  would be  $-\left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right)$ . If I substitute that, this integral would be like this:  $-\int_{R_1}^{R_2} b \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) dr = P$ .

Now, if I integrate this and then substitute the boundary condition  $R_1$  to  $R_2$ , this is the equation we would be getting. So, using boundary condition 1, two expressions we got involving three unknown constants  $A$ ,  $B$ , and  $D$ . From boundary condition two, we are

getting the third equation, the third expression, involving once again those three boundary those three unknown constants capital  $A$ , capital  $B$ , and capital  $D$  and remaining two third and fourth boundary conditions are satisfied automatically now these three equations involving three unknowns first two are obtained from boundary condition one that last one the third one was obtained using boundary condition 2 and in these equations in these three equations we have three unknowns capital  $A$  capital  $B$  and capital  $D$ .

**Bending of Curved Beam with End Shear Load**

$$A = \frac{P}{2bN}, \quad B = -\frac{PR_1^2R_2^2}{2bN}, \quad D = -\frac{P}{bN}(R_1^2 + R_2^2)$$

$$N = R_1^2 - R_2^2 + (R_1^2 + R_2^2) \ln \frac{R_2}{R_1}$$

$R_1 \leq r \leq R_2$   
 $0 \leq \theta \leq \pi/2$

$$\left\{ \begin{aligned} \sigma_{rr} &= \left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta \\ \sigma_{\theta\theta} &= \left( 6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta \\ \tau_{r\theta} &= -\left( 2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta \end{aligned} \right.$$

**Stress fields:**

$$\left\{ \begin{aligned} \sigma_{rr} &= \frac{P}{bN} \left( r + \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \sin \theta \\ \sigma_{\theta\theta} &= \frac{P}{bN} \left( 3r - \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \sin \theta \\ \tau_{r\theta} &= -\frac{P}{bN} \left( r + \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \cos \theta \end{aligned} \right.$$

Following St. Venant's principle, this stress distribution is valid for regions remotely located from the end support.



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So, simultaneously solving these three equations we would be getting the capital  $A$  unknown as  $\frac{P}{2bN}$ ,  $B = -\frac{PR_1^2R_2^2}{2bN}$  and  $D = -\frac{P}{bN}(R_1^2 + R_2^2)$ , where capital  $N$ , the denominator expression is equals to  $R_1^2 - R_2^2 + (R_1^2 + R_2^2) \ln \frac{R_2}{R_1}$ . So, with this, we are able to obtain all three unknown constants in the stress function or in the stress field. Now, with respect to this  $A$ ,  $B$  and  $D$ , if we substitute those in the stress components for the present problem.

The stress fields can be explicitly written like this  $\sigma_{rr} = \frac{P}{bN} \left( r + \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \sin \theta$ .  $\sigma_{\theta\theta} = \frac{P}{bN} \left( 3r - \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \sin \theta$  and  $\tau_{r\theta} = -\frac{P}{bN} \left( r + \frac{R_1^2R_2^2}{r^3} - \frac{R_1^2 + R_2^2}{r} \right) \cos \theta$ . So, these gives us the complete stress distribution.

The expression for all the stress components for the present curved beam bending problem subjected to the end shear load. This solution is valid for small  $r$  varying between  $R_1$  to  $R_2$  and  $\theta$  varying between 0 to  $\pi/2$  for the quarter circular arch, and the capital  $N$  expression is given here. Note that all three stress components are non-zero for the axisymmetric curved beam bending problem, which was pure bending of a curved beam; the  $\tau_{r\theta}$  was 0 here. As it is subjected to transverse shear load, the beam must feel some kind of non-zero transverse shear stress.

And here also, we are getting that non-zero transverse shear stress. The bending stress  $\sigma_{\theta\theta}$  and  $\sigma_{rr}$ , the radial stress, are functions of  $\sin \theta$ , whereas the transverse shear stress is a function of  $\cos \theta$ . These obtained stress distributions are valid for the regions away from the end support, following the Saint-Venant's principle. So only near that end support, where  $\theta$  equals  $\pi/2$ , there may be certain distortion. These solutions may not be valid, but at all the far-field points away from that cantilever end support, these obtained stress distributions are valid.

#### Summary

- Bending of Curved Beam Subjected to End Shear Load
- Non-axisymmetric Bending Problem
- Bending Stress



So, in this lecture, we discussed the curved beam bending problem when it is subjected to end shear load. And this problem is a non-axisymmetric bending problem in nature. We obtained the bending stress distribution for this particular bending problem. Thank you.