**APPLIED ELASTICITY**

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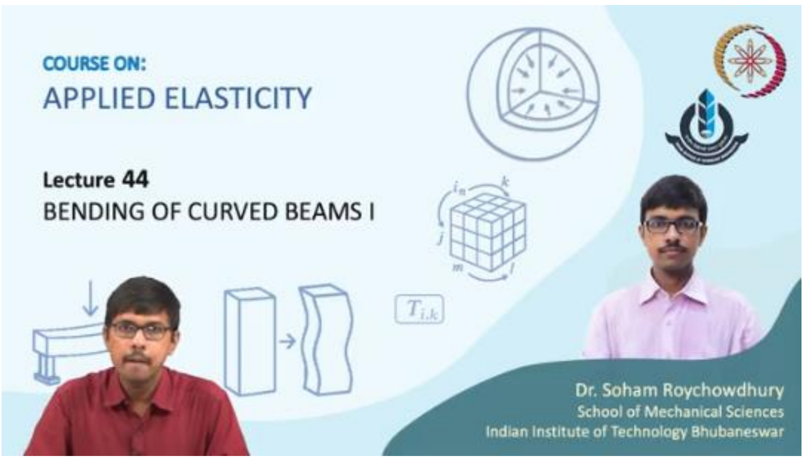
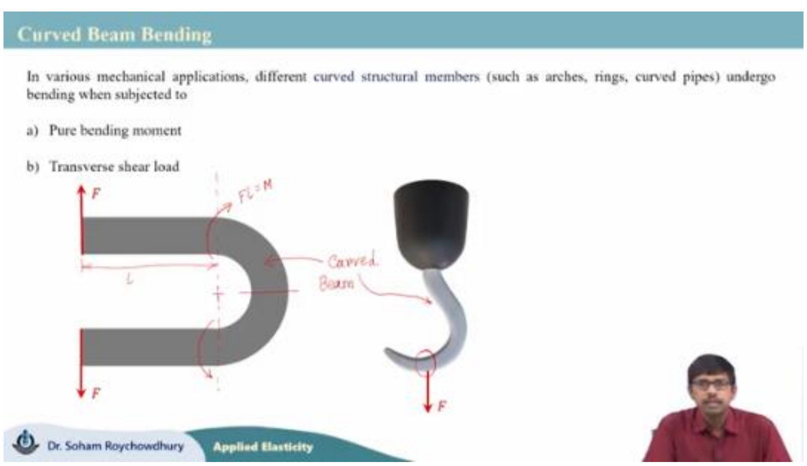
**WEEK: 09**
**Lecture- 44**



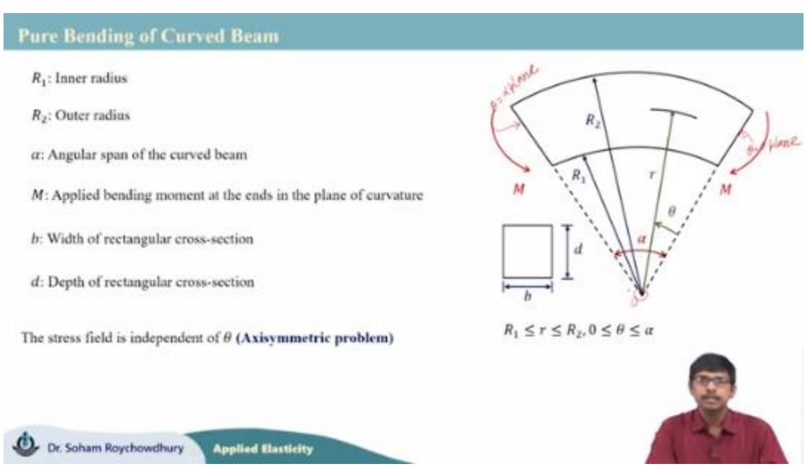

Welcome back to the course on applied elasticity. The topic of today's lecture is the bending of curved beams. In the previous lectures of this week, we discussed the axis-symmetric formulation and the bending of curved beams problem is one such class of problems that can be solved using the axis-symmetric formulation in polar coordinates.




So, curved beams are curved structural elements that can undergo bending when subjected to either bending moments or transverse shear loading. Now, in different mechanical applications, we can see arches, rings, bent rods, and curved pipes, all of which can be modeled as curved beam elements. In the undeformed configuration, there is a non-zero radius of curvature existing for the element. Now, if you consider a few examples, it would be like this. So, in the first case, we are considering a U-shaped bar.

which is subjected to end forces $F$ at both edges, which are equal and acting in opposite directions to each other. Now, these forces $F$ would cause bending of the curved portion. Here, the total domain can be divided into this region. There are two straight portions of the beam and one curved portion of the beam. Now, considering, let us say this length to be $L$, the curved portion of the beam would be subjected to a bending moment $F$ times $L$, similar here on both sides.

So, if you consider this particular section that would be subjected to a pure bending moment of $F$ times $L$, $M$ equals $F$ times $L$. Now, coming to the second one, which is very common for hooks, or hooks carrying weight in cranes or any other hook where a shear force $F$ is acting at the vertically downward section of the hook. So, this is the critical section of the hook where the downward force $F$ is acting due to the weight of the body which the hook is carrying. So, for these kinds of mechanical structures, these are modeled as curved beams because the geometry of this particular part or the geometry of the hook clearly shows curved members, and these are modeled as curved beams. Now, due to the application of either bending moment or transverse shear load, these curved members will undergo bending, and the theory discussed for the bending of straight beams would no longer be valid for discussing or solving the bending of this kind of curved beam structure.




So, here, a different solution approach is required. So, we will discuss that in this particular lecture and also in the next lecture. So, there are two possible types of loading which can act on curved beam structures: one is pure bending moment, and another is transverse shear load; both would cause bending. So, in today's lecture, we are going to talk about the bending of a curved beam when it is subjected to only a bending moment, that is called the pure bending problem of the curved beam structures. So, considering the pure bending of a curved beam structure, let us consider this particular curved beam element, which has a center somewhere here, let us say $O$, and The inner and outer radii are given as $R_1$ and $R_2$. $R_1$ is the inner radius of this curved beam structure.

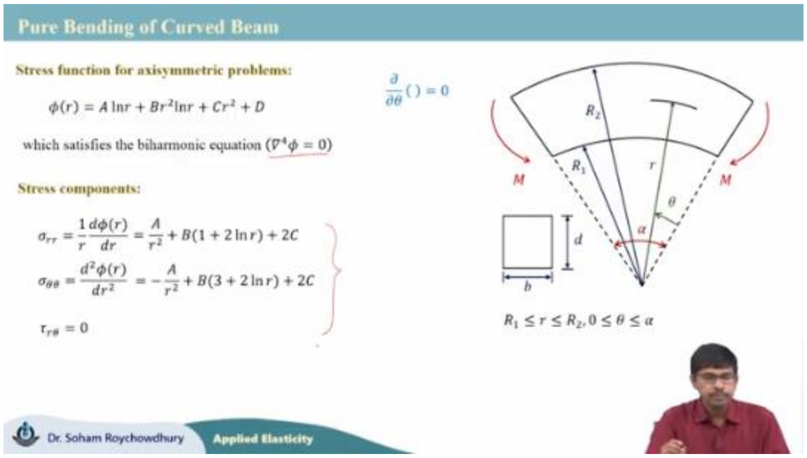
$R_2$ is the outer radius of this curved beam structure, and alpha defines the angular span of this curved beam structure. So, the curved beam is spanned over this angular range of alpha, and for solving the curved beam problems, normally we use the polar coordinate $(r, \theta)$ instead of the rectangular Cartesian coordinate, that is, the $xy$ coordinate. As we would be using the polar coordinate $(r, \theta)$, the range of $\theta$ is required to be defined, which is done with the help of this angular span of the curved beam.

So, $\theta$ varies between 0 to $\alpha$. This particular plane is, let us say, defined as the $\theta = 0$ plane, and this particular plane is defined as the $\theta = \alpha$ plane. Now, this curved beam is subjected to some external bending moment $M$, which is the applied pure bending moment, acting at both ends and on the plane of curvature.

So, this assumption is similar to the assumption of the bending of a straight beam in the plane of loading. The plane on which the bending moment or the transverse force is acting should coincide with the plane of curvature. We are considering a rectangular cross-section of the curved beam. So, the beam has curvature only in one plane; in another plane, the cross-section is a planar section with a rectangular geometry of width $b$ and depth $d$. So, these are the parameters we are going to consider.

Now, as the initial curved beam is part of a circle, a part of a circular beam, we can consider this problem to be an axisymmetric problem where the radial variable $r$ varies between ri to ro, and the angular variable theta varies between 0 to $\alpha$ over the total angular span of the beam. The problem being an axisymmetric problem, the stress field would be independent of theta.

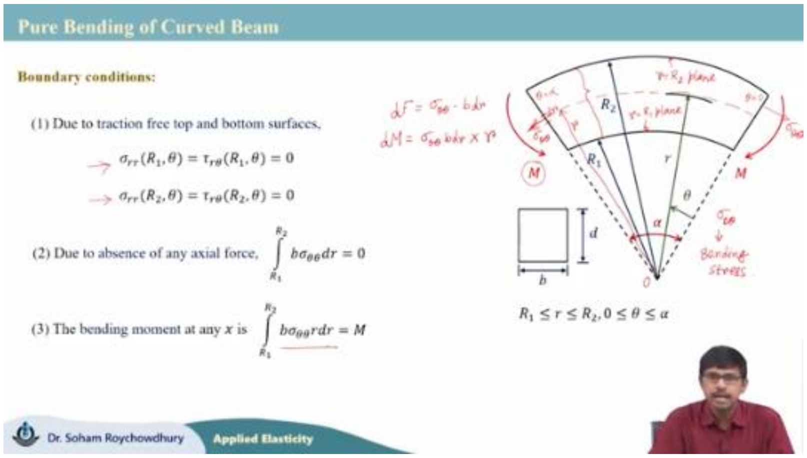
Here, the geometry, the loading—everything is symmetric about the outer plane axis passing through point $O$, that is, about the $z$-axis. Hence, the problem is modeled as or can be modeled as an axisymmetric problem, and for the axisymmetric problem, the generated stress—the bending stresses generated within the curved beam because of this pure bending—would be independent of the angular coordinate $\theta$.




Now, moving forward, we will be using the stress function of this particular form to solve this axisymmetric problem, where $\phi$, the stress function, is a function of $r$ only.

Independent of $\theta$ as the problem is axisymmetric, $\frac{\partial}{\partial\theta}$, the partial derivative of any quantity with respect to $\theta$ is equal to 0. To satisfy the biharmonic equation for such an axisymmetric problem, we have derived the stress function to be of this particular form where $\phi(r) = A \ln r + Br^2 \ln r + Cr^2 + D$, involving four unknown constants to be determined using the boundary conditions. $\phi$ is independent of $\theta$, and this form of $\phi$ directly satisfies the biharmonic equation.

Biharmonic of $\phi$ is 0 ensured. Now, with respect to this kind of stress function for the axisymmetric problems, the stress components $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ can be obtained as $\frac{1}{r}\frac{d\phi(r)}{dr}$, $\frac{d^2\phi(r)}{dr^2}$, and 0 respectively. $\tau_{r\theta}$, the shear stress, is 0; in-plane shear stress is 0 for the axisymmetric planar problems. Now, substituting $\phi$ here, the expression of $\sigma_{rr}$ would be $\frac{A}{r^2} + B(1 + 2\ln r) + 2C$ and $\sigma_{\theta\theta}$ would be $-\frac{A}{r^2} + B(3 + 2\ln r) + 2C$. So, these are the stresses generated for any axisymmetric problem and the present curved beam bending problem subjected to pure bending moment being an axisymmetric problem, these stress components must be valid for this problem as well, where $A, B, C$ we need to find out based on the boundary conditions of this curved beam bending.




Now, coming to the boundary conditions, which are extremely important for finding those unknown constants. So, considering the top and bottom surfaces, which are defined as $R$ planes. So, this top face is the $R$ equals to $R_2$ plane, and the bottom plane is defined as $R$ equals to $R_1$ plane. Now, both these top and bottom planes are free of any kind of surface traction.

Because no distributed normal or shear loads are acting on the top plane or on the bottom plane. Hence, at $r$ equals to $R_1$ and at $r$ equals to $R_2$, both the normal stress and shear stress must vanish. These being $r$ planes, the normal stress component present on these is $\sigma_{rr}$, and the shear stress component present is $\tau_{r\theta}$. Hence, for the inner boundary, for the inner plane, at $r$ equals to $R_1$, $\sigma_{rr}$ and $\tau_{r\theta}$ must be 0 at $r$ equals to $R_1$ for all values of theta.

Similarly, on the outer curved boundary, $\sigma_{rr}$ and $\tau_{r\theta}$ should be 0 when $r$ is $R_2$ for all values of $\theta$. So, these are the two boundary conditions coming due to the stress-free or traction-free top and bottom curved surfaces. Coming to the second boundary condition, here the element is just subjected to bending moment m. No axial force is acting. Axial force means we are not having any force acting like this.

If these forces are acting, then that would be called axial force. So, here the neutral fiber is a curved fiber, and we are calling that the curved axis of this curved beam. So, along that curved axis, we are not having any load as such loads are not existing in the present problem. We must have the net force along that curved axis to be 0. So, sigma theta is the stress component acting on these planes.

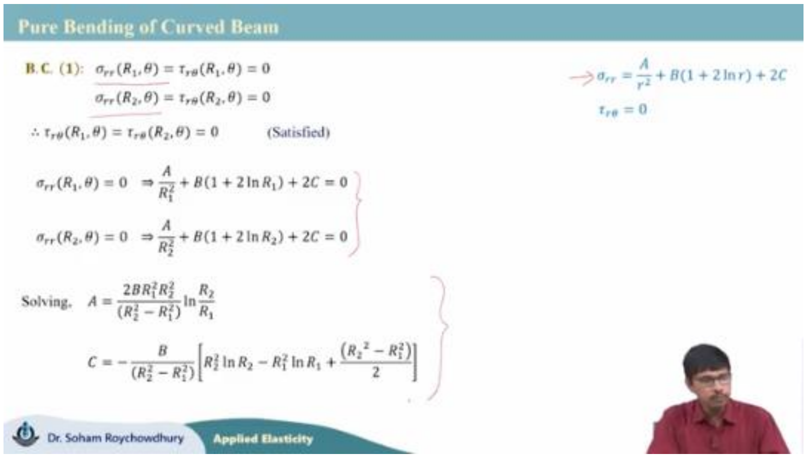
So, Considering these to be theta equals to alpha and theta equals to 0 plane, if you integrate the $\sigma_{\theta\theta}$ over the entire area from $R_i$ to $R_0$, then that would give us the net axial force acting on the curved beam.

So, considering that integrated axial force as integral $R_i$ $R_1$ to $R_2$ $b$ $\sigma_{\theta\theta}dr$, and that is set to 0, this gives us the second boundary condition. So, for the bending of a straight beam also, we were having a similar boundary condition where The axial stress was $\sigma_{xx}$. So, for the bending of a straight beam

$\sigma_{xx}$ was the bending stress for bending of the curved beam. $\sigma_{\theta\theta}$ is the bending stress which acts along the curved axis of the beam. So, this is the major stress generated due to bending in any curved beam structure. And the integral of that over any complete cross-section between $R_1$ to $R_2$ should be 0, as the net axial force acting on the element is 0. Coming to the third boundary condition, which is the definition of bending moment, the applied bending moment $M$.

As it is a problem of pure bending, that particular bending moment acting at any section is equal to a constant $M$. So, this should be the bending moment at any section of the beam. Now, how to write this bending moment in terms of the bending stress $\sigma_{\theta\theta}$. Consider the moment of $\sigma_{\theta\theta}$ with respect to point $O$ for any radius $r$. So, at any radius $r$, let us say a $\sigma_{\theta\theta}$ is acting. Thus, $\sigma_{\theta\theta}$ acting over a small area of $dr$ is having a force of $b\sigma_{\theta\theta}rdr$. So, df, the elemental force acting on the small element $dr$, is $\sigma_{\theta\theta}$ times the area, which is $bdr$. Now, moment $dm$ due to this $\sigma_{\theta\theta}$ is $b\sigma_{\theta\theta}dr$ times the distance of that element from the center point $O$, which is $r$.

This particular element is at a distance $r$ from the center. Thus, the moment created by $\sigma_{\theta\theta}$ is equal to $b\sigma_{\theta\theta}rdr$. If we integrate this over the total cross-section from $R_1$ to $R_2$, that would give us the bending moment $M$. So, m is the integral from $\int_{R_1}^{R_2} b\sigma_{\theta\theta}rdr = M$, and that is valid for any value of $x$ at any cross-section at any value of $\theta$; this should be valid.

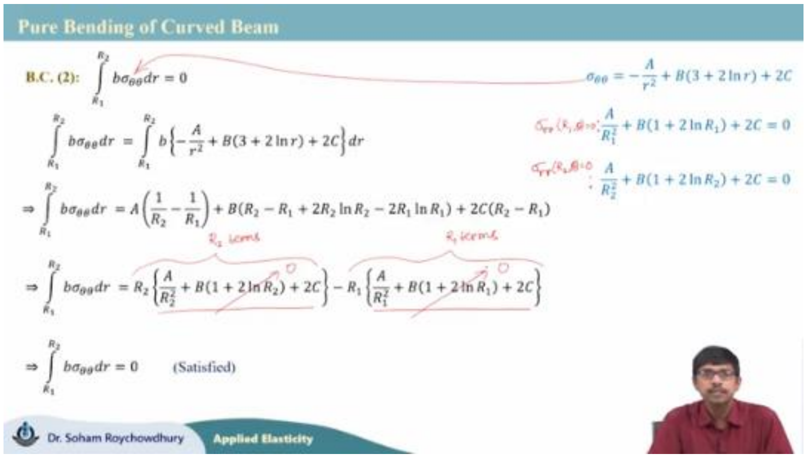



So, using the first boundary condition, $\sigma_{rr}$ and $\tau_{r\theta}$ are 0 at $r$ equals to $R_1$ and $r$ equals to $R_2$. For the axisymmetric problem, we know that $\sigma_{rr}$ and $\tau_{r\theta}$ are given by this equation, where $\sigma_{rr}$ is non-zero involving three unknown constants $A, B, C$, and $\tau_{r\theta}$ is 0. Since $\tau_{r\theta}$ is 0, this particular boundary condition on $\tau_{r\theta}$ at $R_1$ and $R_2$ equals to 0 is automatically satisfied. Then, coming to the $\sigma_{rr}$ condition, this one and this one, $\sigma_{rr}$ at $R_1$, $\theta$ is 0.

Substituting this form of $\sigma_{rr}$ here, we would get $\frac{A}{R_1^2} + B(1 + 2\ln R_1) + 2C = 0$ . Similarly, at the outer boundary, the normal stress $\sigma_{rr}$ at $R_2$, $\theta$ is 0, and that would result in this second equation. $\frac{A}{R_2^2} + B(1 + 2\ln R_2) + 2C = 0$. So, these are the two equations we got involving three unknowns $A, B, C$.

So, we must get one more equation to solve for these three unknowns, which would be obtained by using the other boundary conditions. Now, solving these two, we will get $A$ and $C$ in terms of $B$. So, as we have three unknowns and two equations, we are writing two equations with two unknowns by solving these two equations in terms of the third unknown.

So, by solving these two equations, we can express $A$ and $C$ in terms of $B$ in this particular form. Now, moving forward, We can write all the components in terms of a single unknown $B$ now because $A$ and $C$ are already written as functions of $B$.

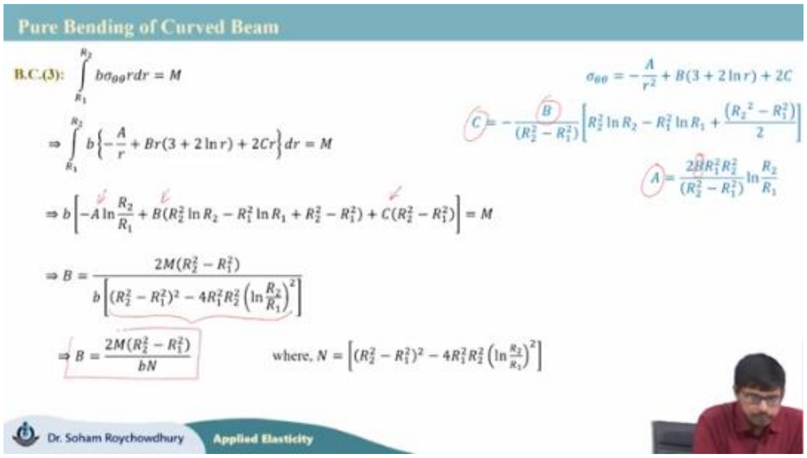



Now, coming to the second boundary condition, that is the $\int_{R_1}^{R_2} b\sigma_{\theta\theta} dr = 0$ , as no axial force is present over the curved beam. Here, if I replace the hoop stress expression $\sigma_{\theta\theta}$ for the axisymmetric problem, which was given as $-\frac{A}{r^2} + B(3 + 2\ln r) + 2C$, that $\sigma_{\theta\theta}$ is replaced here. And then, if you try to integrate it from $R_1$ to $R_2$ over $dr$, we would get this equation. So, the integral from $R_1$ to $R_2$ of $\sigma_{\theta\theta} dr$ equals this. Now, all these right-hand side terms can be carefully rearranged. Some terms contain $R_2$, and some terms contain $R_1$.

If I carefully rearrange those terms into 2 groups, one is the $R_2$ group, and the other is the $R_1$ group. So, the $\int_{R_1}^{R_2} b\sigma_{\theta\theta} dr$ can be written as a combination of 2 terms. The first term is $R_2\left\{\frac{A}{R_2^2} + B(1 + 2\ln R_2) + 2C\right\}$ minus The second term, which is dependent only on $R_1$, would be like this. So, I am just rewriting the previous equation in this particular fashion, and there is a reason for this rewriting.

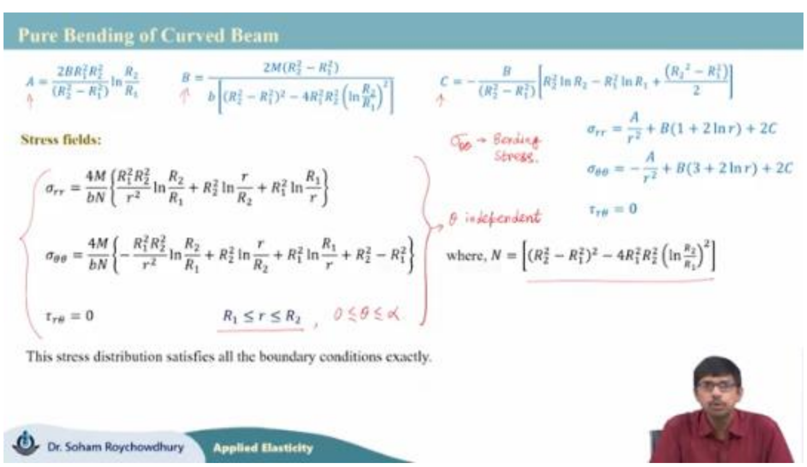
Why? Now, if you look at the terms present within the brackets for both of these two terms, these expressions are already shown to be 0. If you think about the condition $\sigma_{rr}$ at $R_1$, $\theta$ is 0, that gave us this equation. $\sigma_{rr}$ at $R_2$, $\theta$ equals to 0, that gave us this equation. Now, both terms within the brackets would go to 0.

From the previously obtained result of the stress-free curve boundaries, and thus the second boundary condition, the $\int_{R_1}^{R_2} b\sigma_{\theta\theta} dr = 0$ , is automatically satisfied.




Now, we are left with only one boundary condition: that at any $\theta$, $M$ is given by the $\int_{R_1}^{R_2} b\sigma_{\theta\theta} r dr$. Now, once again, I will substitute the expression of hoop stress $\sigma_{\theta\theta}$ in this integral, thus $M$ would be like this: $\int_{R_1}^{R_2} b\left\{-\frac{A}{r} + Br(3+2\ln r) + 2Cr\right\} dr = M$.

Now, if I integrate this and substitute the limits from $R_i$ $R_1$ to $R_2$, $M$ can be obtained like this. Now, here three unknowns are present. $A, B$, and $C$. Now, using the first boundary condition, we had already expressed two of the unknowns, $A$ and $C$, in terms of the third unknown, $B$. and those expressions were like this: $A$ and $C$ both were written as a function of $b$ and other given parameters $R_1$, $R_2$. So, substituting those here, we can express m just in terms of $b$. and from that, the remaining constant $b$ can be obtained as this: 2 $b$ equals to $\frac{2M(R_2^2 - R_1^2)}{bN}$. So, we got the last remaining constant $b$. Now, substituting these expressions of $B$ back in the $A$ and $C$ expressions, we can obtain the expressions of $A$ and $C$ in terms of $M$. So, here defining this numerator of the $B$ as a new quantity $N$, as this expression is getting longer, let us write the denominator of $B$ as this number $N$. So, $\left[(R_2^2 - R_1^2)^2 - 4R_1^2R_2^2\left(\ln\frac{R_2}{R_1}\right)^2\right]$ I am naming as $N$, and with that the expression of $B$ becomes $\frac{2M(R_2^2 - R_1^2)}{bN}$.



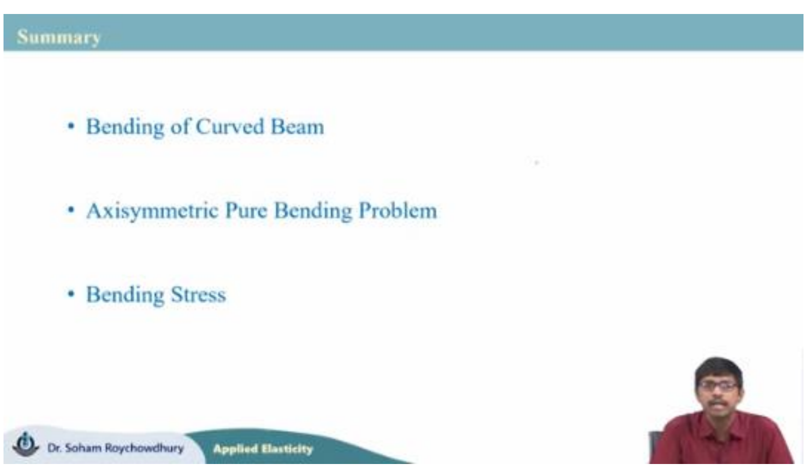
Now, these expressions of $B$ I will be substituting here and here so that we can write $A$ and $C$ in terms of bending moment $M$.

So, if I do so, the $A, B$, and $C$ expressions—all three—can be obtained in terms of the applied bending moment. And by putting all those back into the stress field equation, the stress components $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ for the axisymmetric problem, those would come out like this. So, this is the expression of $\sigma_{rr}$ where I have substituted $A, B$, and $C$—all three—in terms of the bending moment $M$.

So, $\sigma_{rr}$ would be $\frac{4M}{bN}\left\{\frac{R_1^2R_2^2}{r^2}\ln\frac{R_2}{R_1}+R_2^2\ln\frac{r}{R_2}+R_1^2\ln\frac{R_1}{r}\right\}$, where the denominator $N$ is given by this equation. Similarly, the hoop stress $\sigma_{\theta\theta}$ can be obtained like this: $\frac{4M}{bN}\left\{-\frac{R_1^2R_2^2}{r^2}\ln\frac{R_2}{R_1}+R_2^2\ln\frac{r}{R_2}+R_1^2\ln\frac{R_1}{r}+R_2^2-R_1^2\right\}$. And $\tau_{r\theta}=0$.

So, these give the complete stress distribution for the bending of a curved beam subjected to pure bending moment. Note that these stress distributions are valid within this range of the curved beam where the radius varies between $R_1$ to $R_2$ and $\theta$ varies between 0 to $\alpha$. Now, note that all these stress components are independent of theta. So, all the stress components are theta-independent, which is because these are the solutions of an axisymmetric problem.

So, for all values of $\theta$, the exact same stress would result where $\sigma_{rr}$ is the radial stress, $\sigma_{\theta\theta}$ is the hoop stress, which is the major stress generated due to bending. So, this is called the bending stress for the curved beam problem. So, these distributions, these stress distributions satisfy all the boundary conditions exactly and since they satisfy all the boundary conditions exactly, this solution is an exact solution. No approximation is used for solving this particular pure bending of the curved beam problem.



So, in this lecture, we discussed the bending of curved beam structures, and this was described as an axisymmetric pure bending problem. We derived the expressions for the bending stress components for this bending of a curved beam subjected to a pure bending moment. Thank you.