APPLIED ELASTICITY

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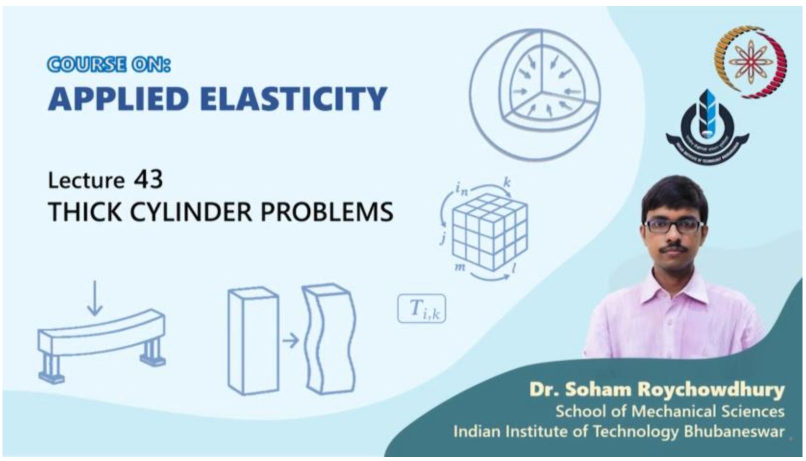
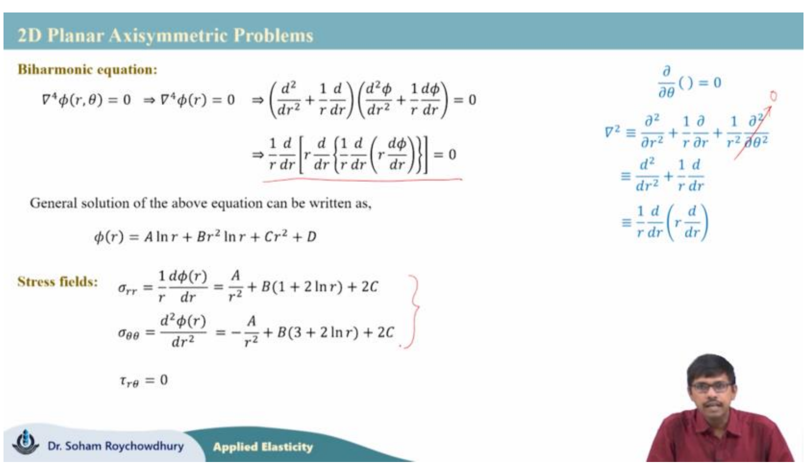
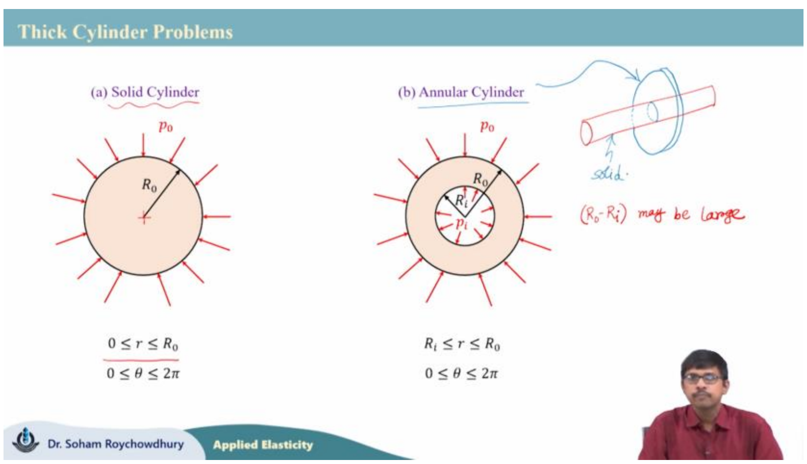
Lecture- 43



Welcome back to the course on applied elasticity. In today's lecture, we are going to talk about the solution to the thick cylinder problems. So, in the last lecture, we discussed the axisymmetric formulation, initially the three-dimensional axisymmetric formulation, and then with the approximation of either plane stress or plane strain, we arrived at the two-dimensional planar axisymmetric formulation.



So, for axisymmetric problems, the stress function ϕ is just a function of the radial coordinate r and is independent of the θ coordinate. Any quantity is independent of θ, any field variable is independent of θ for the axisymmetric problem, and thus $\frac{\partial}{\partial\theta}$, the partial derivative of any quantity with respect to θ, goes to 0. Now, with this assumption of axis symmetry, the Laplacian operator ∇^2 will just become $\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$. So, it would also be independent of θ because this $\frac{\partial}{\partial\theta}$ being 0, the last term of this Laplacian operator $\frac{\partial^2}{\partial\theta^2}$ would go to 0. And putting this back in the biharmonic equation, this was the form of the biharmonic equation for the $2D$ polar axisymmetric problems. Then the general solution of the stress function which satisfies this biharmonic equation was given by $A\ln r + Br^2\ln r + Cr^2 + D$ where A, B, C, D are the constants to be obtained using the boundary conditions. And now, with the help of this particular stress function for the planar axisymmetric problem, we can obtain the stress components like this. σ_{rr} and $\sigma_{\theta\theta}$, the two normal stresses, are given as these, whereas σ $\tau_{r\theta}$, the in-plane shear stress, is equal to 0 for the planar axisymmetric problem. A, B, C constants are required to be determined by using the displacement or traction boundary conditions for the problem, given for the problem. Now, this discussion we had in the last lecture.



In this particular lecture, we are going to use this set of equations derived for a planar axisymmetric problem for a specific application of thick cylinders. So, if you consider the thick cylinder problems, it is broadly categorized into two different geometries. The first case is a solid cylinder.

So, we are considering a solid cylinder of radius R_0, which may be subjected to external pressure p_0. Now here, for the solid cylinder, the radial coordinate r is varying between 0 to R_0. So, the range of small r, the radial variable, is 0 to R_0, whereas the angle, the circumferential variable θ, can vary between 0 to 2 π. So, this defines the domain of the problem. Now, instead of the solid cylinder, in many applications, we may come across the hollow cylinder or annular cylinder like this, with inner radius R_i and outer radius R_0.

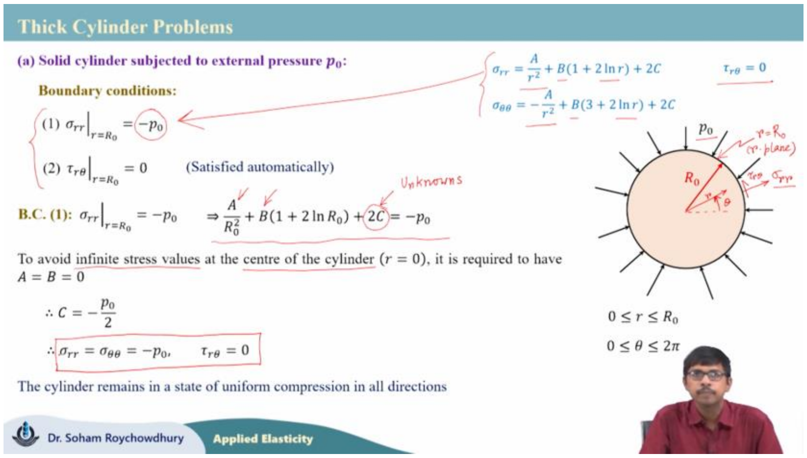
Now, this annular cylinder can be subjected to both internal pressure p_i as well as external pressure p_0. These problems for the annular thick cylinders The radial variable small r is varying between capital R_i to capital R_0, that is the range for r. The range for theta is the same, which is 0 to 2π. So, for the solid cylinder, the radial domain of the problem includes the origin, the center.

It is included within the solution zone for the case of a solid cylinder, whereas for the annular cylinder, the center is excluded. So, these For both of these solid and annular cylinders, we are assuming the thickness to be large, meaning For a solid cylinder, there is no thickness Which can be defined for the annular cylinder as R_0 minus R_i.

This value may be large So, that is why we are calling it a thick cylinder. If you have this difference to be small, that is a cylinder with a very thin wall, In that case, the pressure vessel theories are applicable, But for the thick cylinder, that theory is not going to be applicable.

And we are going to solve this problem as a $2D$ planar axisymmetric problem. Because you can clearly see here the geometry as well as the loading are axisymmetric, symmetric about the out-of-plane axis, let us say the z-axis. Now, these have wide applications in different mechanical engineering problems. So, let us consider a shaft that is carrying different gears or pulleys. So, at some particular section of the shaft,

At a particular section of the shaft, a pulley is attached or a gear is attached. For all such cases, this particular annular cylinder model should be used for the gear or pulley, which would be subjected to internal pressure coming at the contact surface between the shaft and the pulley, whereas the shaft should be modeled as a solid cylinder subjected to outer pressure. So, for such applications, which are quite common in different mechanical transmission drives, this model is useful.



Now, moving further to the solution technique. So, first, we will start our discussion on the solid cylinder subjected to external pressure p_0. R_0 is the radius of the solid cylinder. Here, the boundary conditions are defined like this. So, there exists only one boundary, which is the outer boundary.

This is the boundary which is defined by small r equals to R_0. So, this is basically a radial plane. So, the boundary here is the r plane on which two stresses are defined. On the r plane, we have σ_{rr} in the positive radial outward direction as the radial normal stress, and we also have $\tau_{r\theta}$ in this direction. So, we are assuming θ to be positive in this direction, and this is r.

So, $\tau_{r\theta}$ and σ_{rr} are the two possible boundary stresses acting on the boundary for these cylinders, for both solid and annular types of cylinders. Now, here the outer surface is subjected to only external pressure p_0, which is compressive in nature. Hence, the σ_{rr} direction and the direction of p_0 are different. Thus, the boundary condition σ_{rr} at r equals to R_0 should be equal to minus p. As P is compressive, the direction of p is opposing the σ_{rr} direction. σ_{rr} is radially outward positive.

p_0 is acting toward the center. So, this boundary condition, the first boundary condition, is $\sigma_{rr}|_{r=R_0} = -p_0$. R_0. Now, at the outer radius, there is no shear force acting on the system, so $\tau_{r\theta}|_{r=R_0} = 0$. These are the two boundary conditions for the solid cylinder subjected to external pressure. Now, if you recall the axisymmetric problem stress functions and the corresponding stress components, they were given like this.

σ_{rr} and $\sigma_{\theta\theta}$ were non-zero, and $\tau_{r\theta}$ was zero. This problem being an axisymmetric problem, we are going to use this form of stress solutions. So, since $\tau_{r\theta}$ is zero, the second boundary condition is automatically satisfied. Now, the first boundary condition we need to satisfy is for finding the unknown constants. So, substituting the σ_{rr} expression, this expression here, into the first boundary condition, we would get this.

$\frac{A}{R_0^2} + B(1 + 2\ln R_0) + 2C = -p_0$. Now, this is the only equation available to us to solve for all the unknowns, and we have three different unknown constants. So, only one equation is not sufficient for finding all three unknowns. Now, if you try to recall the discussion from our last lecture, we mentioned that we must ensure finite stress values within the entire domain of this axisymmetric problem, and that can be ensured at the origin, at the center, only if we force A and B to be zero. Otherwise, if you look at this term with r being zero, whatever the value of A, other than zero, this term will shoot to infinity. The same applies to the first term of $\sigma_{\theta\theta}$. So, if A is not equal to zero at the origin, both normal stresses would be infinite.

Similarly, due to the presence of this ln r term multiplied with B, If B is not equal to 0 at the origin, the stresses will shoot to infinity. So, to avoid the infinite stress values at the origin or the center of the cylinder, we must have the two constants A and B to be 0. And thus, from this equation, we can only obtain the single remaining unknown C as $-\frac{p_0}{2}$. Substituting that in the stress equation, σ_{rr} and $\sigma_{\theta\theta}$ both would be $-p_0$ with $\tau_{r\theta} = 0$.

So, this gives us the stress distribution of a thick solid cylinder subjected to external pressure. So, shear stress is not 0; the two non-zero stresses are normal stresses σ_{rr} and $\sigma_{\theta\theta}$, both of them equal to $-p_0$, which is a constant. Hence, the state of stress within this thick cylinder is basically a state of uniform compression along all the directions, which is free of any kind of shear stress. So, a thick cylinder which is solid and subjected to external pressure is going to have a state of uniform compression in all the directions.

Now, moving forward to the case of an annular cylinder. We are considering an annular cylinder with inner radius R_i and outer radius R_0. It is subjected to inner pressure p_i and outer pressure p_0. No shear stresses are present. The range of r for this case is R_i to R_0. Now, if we try to write the boundary conditions, here two boundaries exist.

One boundary is the outer boundary defined by r equals capital R_0. Another is the inner boundary defined by r equals capital R_i. Now, the boundary conditions can be written like this. There would be two boundaries and thus, each boundary results in 2 boundary conditions, so the total number of boundary conditions would be 4.

σ_{rr} and $\tau_{r\theta}$ are defined for both the inner boundary and outer boundary, resulting in 4 boundary conditions. Now, at the outer boundary, $\sigma_{rr}|_{r=R_0} = -p_0$, the negative of the external pressure. Why negative? The explanation is the same as the previous case of a solid cylinder.

For the outer boundary, σ_{rr} is outward, which is the positive r plane. For the inner boundary, σ_{rr} is inward because the inner boundary is the negative r plane; r is acting outward, positive. So, the outer boundary of the cylinder is defined as the positive r plane, and the inner boundary of the cylinder is defined as the negative r plane. The normal to the inner boundary is along the negative r direction, thus the σ_{rr} direction for the inner boundary is towards the center. As on both faces, outer as well as inner, the direction of σ_{rr} by the sign convention is opposing the direction of p . p_0 and p_i respectively. Both these boundary conditions: $\sigma_{rr}|_{r=R_0} = -p_0$, $\sigma_{rr}|_{r=R_i} = -p_i$. And shear stresses are 0 for both inner and outer boundaries.

So, these are the four boundary conditions available to us for solving the unknown constants.

Now, moving forward, the σ_{rr} and $\tau_{r\theta}$ equations for the axisymmetric problems are written here. Where σ_{rr} contains three constants A, B, C as a function of r, and $\tau_{r\theta}$ is 0. $\tau_{r\theta}$ being 0, the third boundary condition is satisfied automatically. So, the first two boundary conditions we need to solve for finding the unknown constants.

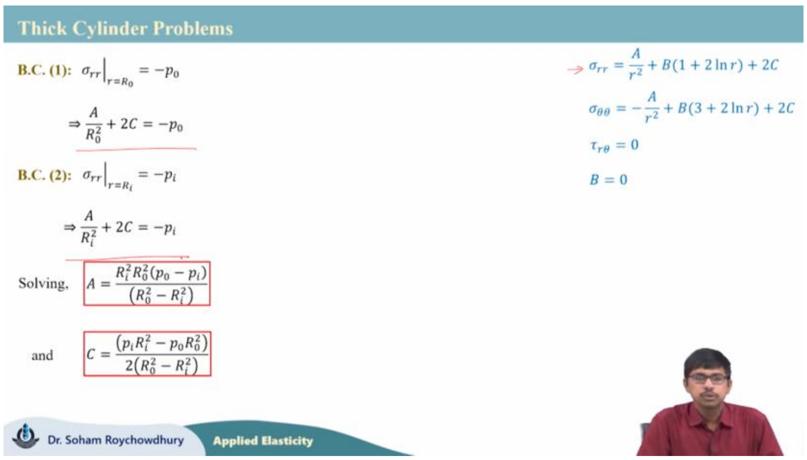
Now, If we are looking for the displacement conditions, from the restriction of ensuring a single-valued displacement along the theta direction at any particular point, we must have B equals to 0. So, with the help of this axisymmetric σ_{rr} and $\sigma_{\theta\theta}$,

If you write the expression of u_θ, which we derived in the last class, you can see this u_θ is proportional to b, which is this particular constant. There are other terms, but u theta is directly proportional to B. Now, for these axisymmetric problems, let us say we are starting from θ equals 0 on the x-axis. Now, after completing a full rotation, we should return to the x-axis at θ equal to 2π, and the result should be repetitive.

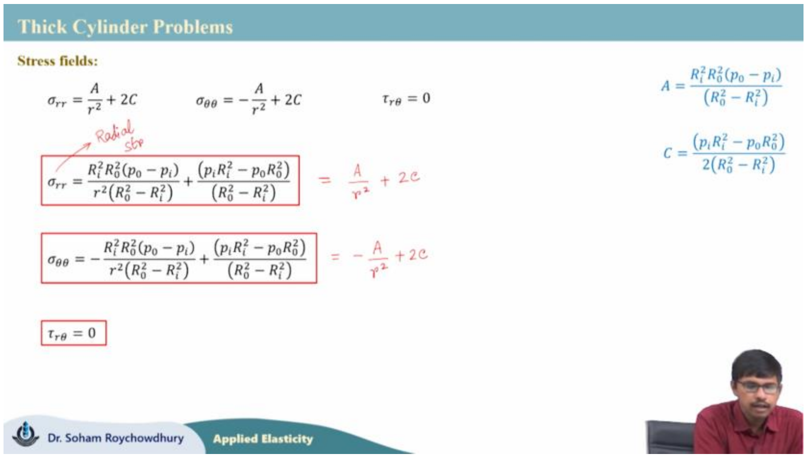
So, from 0 to 2π, whatever solution we obtained, then starting from 2π to 4π, the same solution should repeat. Now, if $u_\theta \propto B$, that would result in a zero value of u_θ or some specific value of u_θ at θ equals 0. So, let us say we obtained this much u_θ at θ equals 0. Now, after completing a full rotation, you should get the same value here.

So, this is proportional to $B\theta$; at θ equals 0, this would be 0. Now, after a full rotation, u_θ at θ equals 2π must equal this. Now, this is possible. This displacement single-value function is possible at any theta and θ plus 2π only if we enforce B equals 0. Otherwise, since it is directly proportional to $B\theta$, this will just have 2π as a scaling factor.

So, that should be avoided by forcing B equal to 0. So, this particular term should go to 0 for the axisymmetric problems like this to ensure the single-value displacement.



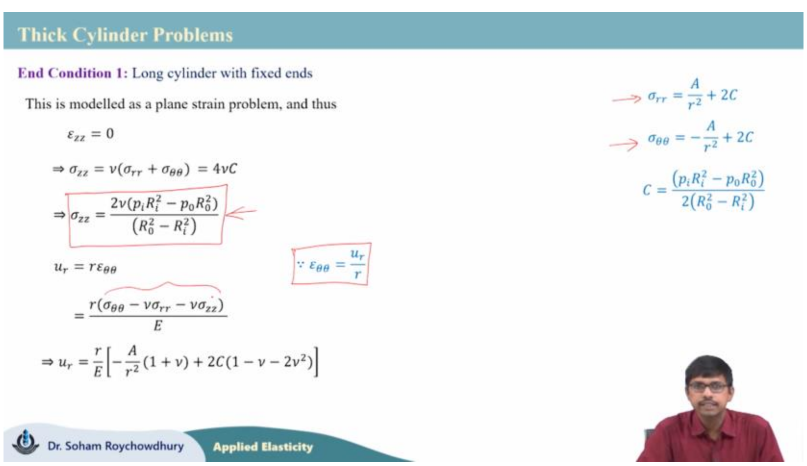
So, setting B equal to 0 and using boundary condition 1, $\sigma_{rr}|_{r=R_0} = -p_0$ would give us $\frac{A}{R_0^2} + 2C = -p_0$. Similarly, using the second boundary condition at the inner equator, $\sigma_{rr}|_{r=R_i} = -p_i$, substituting this expression of σ_{rr} there, we would get $\frac{A}{R_i^2} + 2C = -p_i$. Now, we have these two equations for solving two unknowns, A and C. If you solve for A and C from these two equations, you would get A as $\frac{R_i^2 R_0^2 (p_0 - p_i)}{(R_0^2 - R_i^2)}$. And the second constant, C, would be $\frac{(p_i R_i^2 - p_0 R_0^2)}{2(R_0^2 - R_i^2)}$. So now, we are able to get the complete solution: B is 0, and two non-zero constants are obtained in terms of applied pressure and the geometry, R_i and R_0, of the annular cylinder.



So, substituting those back—A, B, C expressions—in the stress components: $\sigma_{rr} = \frac{A}{r^2} + 2C$, $\sigma_{\theta\theta} = -\frac{A}{r^2} + 2C$, $\tau_{r\theta} = 0$. I had forced B to be 0. Now, with this, if you substitute A and C, the expressions of σ_{rr} and $\sigma_{\theta\theta}$, the two non-zero stress components for the annular cylinder are obtained like this.

Where you can see the first term is a constant multiplied by r square. And then the second term is another constant, with the second term I am saying $2C$. For σ_{rr}, whereas for $\sigma_{\theta\theta}$, the first term is the same as the first term of σ_{rr}, just the sign is changed, and the second term is exactly similar. $\tau_{r\theta}$ is 0. So, σ_{rr} and $\sigma_{\theta\theta}$ only differ in that there is a change in sign in one of the terms, which is proportional to 1 by r square. So, this is the formula for the normal stresses: radial stress and hoop stress. σ_{rr} is called radial stress.

And $\sigma_{\theta\theta}$ is called hoop stress. Both are normal stresses acting for this axisymmetric thick cylinder problem, which is annular in nature, and shear stress is 0.

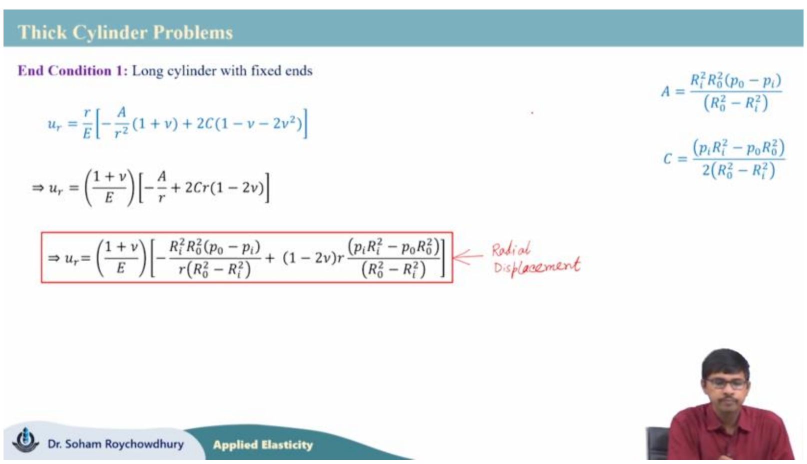


Now, moving to the end conditions: if you have a very long cylinder with fixed ends. So, as it is a long cylinder, we should approximate this as a plane strain problem, and thus the out-of-plane strain ε_{zz} should be 0. ε_{zz} being 0, we can write the out-of-plane stress σ_{zz} as $\nu(\sigma_{rr} + \sigma_{\theta\theta})$. Now, for the annular cylinder problem, we had already obtained these two expressions for σ_{rr} and $\sigma_{\theta\theta}$ as this.

$\frac{A}{r^2} + 2C$ is σ_{rr}, and $-\frac{A}{r^2} + 2C$ is $\sigma_{\theta\theta}$. Substituting those in this equation, σ_{zz} would come out to be $2\nu C$. Now, substituting the expression of C. This is the expression of σ_{zz}, so this much axial stress would be developed in a cylinder. Which is quite long or infinitely large theoretically, with the ends fixed.

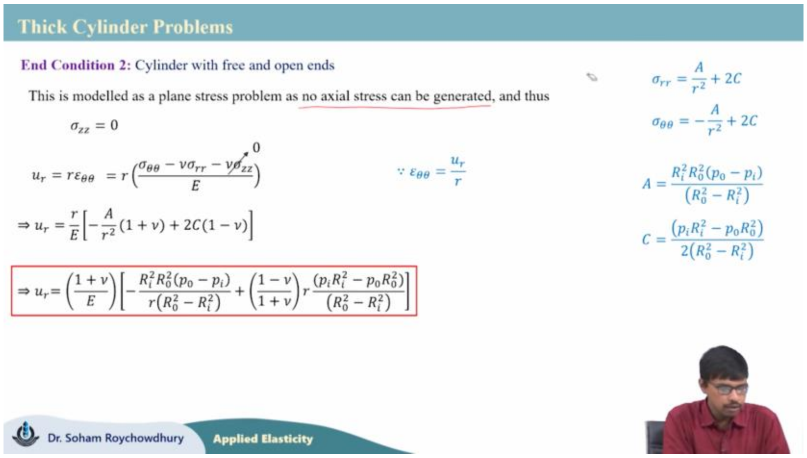
Now, if you are going to find the displacement—the radial displacement—how much the cylinder is going to expand, that can be measured with the help of u_r. Now, if you recall for the axisymmetric problems, the normal strain in the theta direction, $\varepsilon_{\theta\theta}$, was defined as u_r by r. So from there, we can easily write u_r as r times $\varepsilon_{\theta\theta}$, where $\varepsilon_{\theta\theta}$ is nothing but $\frac{(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz})}{E}$.

Now, we know the expressions of $\sigma_{\theta\theta}$ and σ_{rr}, and we also got the expression of σ_{zz}. Substituting all those here, we can write the u_r expression like this, involving two constants A and C, which are already determined, and two material properties: E and ν, Young's modulus and Poisson's ratio.



Now, substituting the explicit expressions for A and C in the u_r and simplifying it further, we can get the radial deformation u_r for this particular case that is, a long cylinder with fixed ends, as this. So, this would be the expression for the radial displacement. So, at any particular value of r, if you substitute in this equation, you would get how much a particular radial point is going to move out. So, the change of radius—inner radius or outer radius—of the thick annular cylinder can be obtained by using this equation.

The cylinder is quite large, and the ends are fixed. This is the first possible type of end condition.

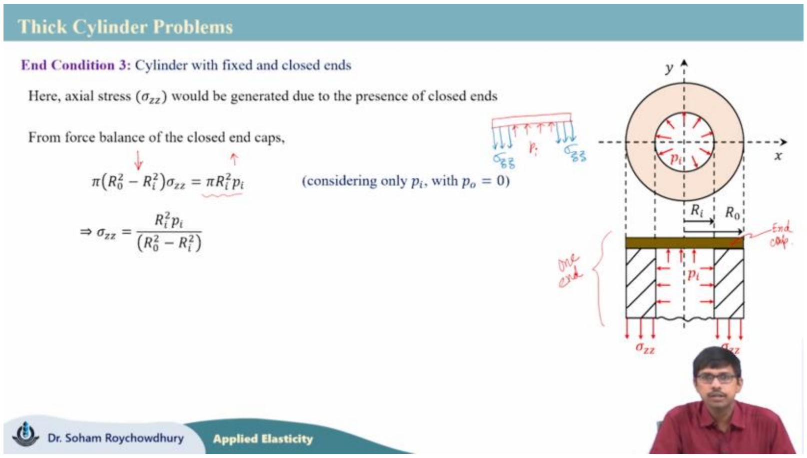


Now, let us move to the second one, where the cylinders have free and open ends, so it does not have a very large length. It may or may not have but the cylinder ends are free to expand. We are not forcing any fixed condition at the two end caps.

And for such cases, as the cylinder ends are free, no axial stress is allowed to be generated if you have a cylinder that is free. So, it is just kept like this. So, we are not forcing any one end like this. If this is forced to be fixed, then this condition is not applicable.

For this particular case, we are not having any fixed boundary condition at any end of the cylinder. So, we are just having a cylinder with free ends, and hence axial stress should be 0; σ_{zz} should be 0. For such cases, u_r can be written as $r\varepsilon_{\theta\theta}$, writing $\varepsilon_{\theta\theta}$ in terms of the stress components and then setting σ_{zz} to be 0, we can get u_r as this.

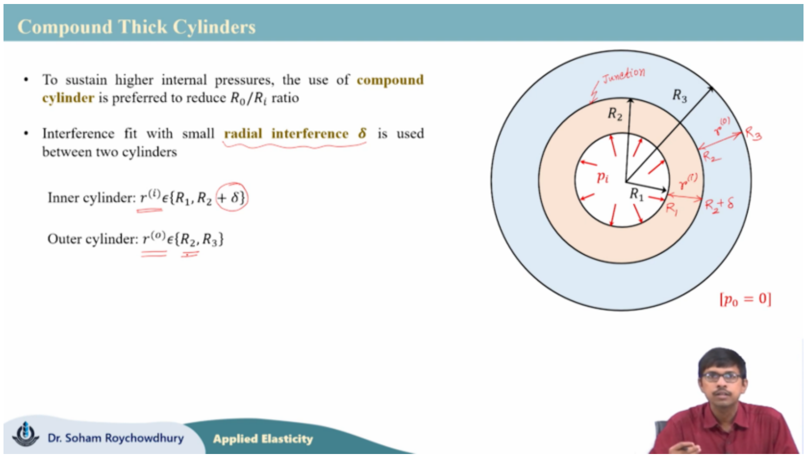
So, Putting A and C values in this expression of u_r, the radial displacement for the cylinder with open and free ends can be written like this. So, this expression is different from the previous end condition. So, the radial displacement equation depends on the end conditions.



Now, coming to the third type of end condition, which is this. The cylinder is fixed, and the ends are closed. So, for the previous case, the cylinder had free and open ends. Now, at both ends, the cylinder is closed. Now, due to the presence of the closed end caps, there will be axial stress generated for such cylinders. So, let us consider the annular cylinder as shown in the figure, which has the closed end cap.

So, this figure shows one end of the cylinder where this is the end cap that closes the cylinder. R_i and R_0 are the inner and outer radius. Let us consider the cylinder is subjected to only inner pressure p_i. p_0 may also be there. So, let us consider only p_i; p_0 is 0. Now, it would have some non-zero value of σ_{zz} as the end cap is present with the cylinder.

And from the force balance of the end cap, if you consider the vertical force balance of the end cap, that would give us this equation. So, if you consider the end cap like this in this particular region, the central circular region is subjected to p_i acting in the upward direction. So, $\pi R_i^2 p_i$—this is the net upward force—whereas, for the annular zone where the material is between R_i and R_0, In this annular region, it is subjected to stress σ_{zz}. So, σ_{zz} multiplied by the annular area, $\pi(R_0^2 - R_i^2)$, that is the net downward force, and these two should be balanced for the equilibrium of the end cap. From that, you can get the expression of axial stress σ_{zz} as $\frac{R_i^2 p_i}{(R_0^2 - R_i^2)}$. Note that this equation is obtained with p_0 equal to 0. Now, substituting this σ_{zz} into the expression of u_r, you can obtain the radial displacement for this third type of end condition—that is, a cylinder with a fixed and closed end with the help of end caps.



Now, moving to the problem of a compound thick cylinder, where instead of one cylinder, multiple thick cylinders are stacked in a concentric manner. Now, why is it required? Thick cylinders are often used for the storage of high-pressure fluids. Now, for such cases, if the fluid pressure is large, to withstand the generated stresses, we may need to increase the size—the R_i and R_0 values, the radius values of the cylinder.

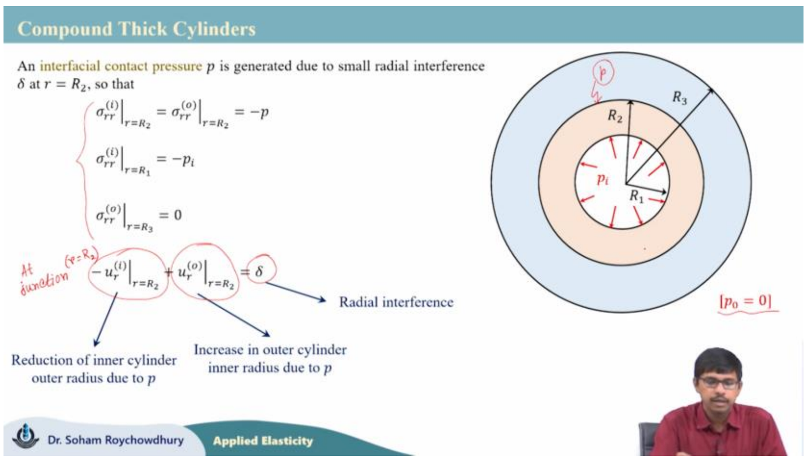
the thickness of the cylinder—that is, the R_i by R_0 ratio is required to be increased to sustain the higher internal pressure and avoid failure. From the space constraints, that may not be feasible for certain cases. So, hence the use of compound cylinder is preferred where there will be multiple layers in the cylinder which which will be fitted with a radial interference small radial interference of δ like this.

So, we are considering a compound cylinder consisting of two thick cylinders which are concentric. So, this is one cylinder the inner one and there is another outer cylinder both are press fitted one within another with a small radial interference delta and R_1, R_2, R_3 are let us say three radius. R_1 is the inner radius of the compound cylinder, R_3 is the outer radius of the compound cylinder and R_2 is the radius of the junction. So, where the two cylinders are press fitted.

So, this is the junction boundary of two cylinders where they are interacting with each other with a radial interference of delta. Now, p_0 is taken to be 0, let us say it is subjected only to p_i, the compound cylinder is used as a fluid storage device subjected to internal pressure p_i coming from that high pressure fluid. Now, here for the inner cylinder radial variable is written as r with superscript small i, for the outer cylinder the radial variable is written as r with superscript o.

So, the superscript small i refers to the inner cylinder quantity, and superscript small o refers to the outer cylinder quantity. Now, for the outer cylinder, the radial variable is varying between R_2 to R_3, that is for R_0. For the inner cylinder, this is varying between R_1 to R_2. But with a small radial interference. So, the range of the inner cylinder radial coordinate is from $R_1, R_2 + \delta$, which is the radial interference. So, the inner radius of the outer cylinder and the outer radius of the inner cylinder are not exactly the same.

So, the inner cylinder's outer radius is slightly greater. Greater by the small amount δ as compared to the inner radius of the outer cylinder. And that is why press fitting is required, to ensure that these two cylinders are not going to slip one out from the other. To ensure that the radial interference is added in press fitting.



Now, coming to the solution procedure, due to the presence of the press fitting, the radial interference δ.

When we are internally pressurizing the compound cylinder, a non-zero contact pressure p would be generated at the junction at the contact boundary R_2. And hence, the boundary condition can be written like this: stress boundary condition at R_2, at R_2. Both the inner radius and outer radius are going to experience this contact pressure p. So, the radial stress for the inner and outer cylinder at r equals to R_2 would be equal to minus p. For the inner cylinder, at the inner boundary, we have $\sigma_{rr}^{(i)}\big|_{r=R_1} = -p_i$.

For the outer cylinder at the outer boundary, as p_0 is 0, this would be equal to 0. $\sigma_{rr}^{(o)}\big|_{r=R_3} = 0$. So, these are the four stress boundary conditions on the normal stresses.

The shear stress boundary conditions are all 0. $\tau_{r\theta}$ is 0 for both inner and outer cylinders at R_1, R_2, and R_3.

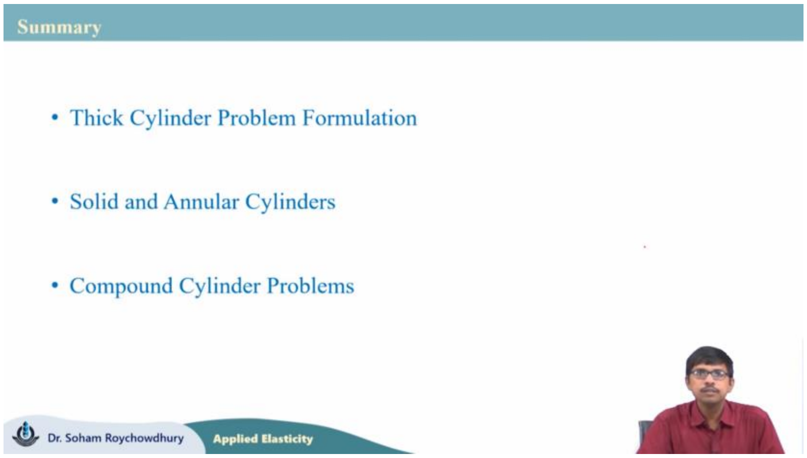
Now, moving to the displacement boundary condition. Through which the interference is defined. So, at r equals to R_2 at the junction, this is defined at the interface where two cylinders are interacting with each other, that is at r equals to R_2 . $-u_r^{(i)}\Big|_{r=R_2} + u_r^{(o)}\Big|_{r=R_2} = \delta$. Where the first term refers to the reduction of the outer radius of the inner cylinder due to the generated contact pressure p.

The second term is equal to the increase in the inner radius of the outer cylinder due to the generated contact pressure p. And the summation of this should be equal to the given radial interference delta. Now, why is this equation coming? Because as p is present, as p is resulted due to the presence of δ, due to radial interference, some interfacial contact pressure is generated between two cylinders at R_2, r equals to R_2. That would be causing the inner boundary of the outer cylinder to expand by how much?

By this second term, the inner boundary of the outer cylinder will expand by this much due to the presence of contact pressure p. Similarly, the effect of contact pressure on the outer boundary of the inner cylinder would be a decrease in the outer radius of the inner cylinder by this much. So, why the minus sign? Due to p, the outer radius of the inner cylinder is going to shrink. So, the summation of these two should balance the initially given interference δ.

So, the contact pressure, interfacial contact pressure, is actually unknown. How much pressure will be generated depends on the radial interference given to the problem δ, and using this equation, we can obtain p for a given δ. Then, using a similar procedure as described for a single thick cylinder, the compound cylinder problems can be solved by finding the constants A, B, C.

Now, for this case, there would be different constants A_1, B_1, C_1 for the inner cylinder, and A_2, B_2, C_2 for the outer cylinder. But these are the basic governing equations or boundary conditions using which the compound cylinder problems can be solved.



So, in this lecture, we discussed the formulation of the thick cylinder problem and then the solution procedures for the solid cylinder, annular cylinder, and compound cylinder. Thank you.