APPLIED ELASTICITY

Lecture42

Lecture 42 : General Solutions in Polar Coordinates

Welcome back to the course on applied elasticity. The topic of today's lecture is the general solution in polar coordinates. In the previous lecture, we discussed the field equations of elasticity in polar coordinates. Now, in this lecture, we will talk about the solution, the general solution of any elasticity problem, any $2D$ elasticity problem in polar coordinates. So, to have a quick recap,

We obtained the stress components for the $2D$ elasticity problem in terms of any stress function $\phi(r, \theta)$ as this. σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ were described in terms of ϕ, the stress function in polar coordinates, in this particular fashion, and that chosen phi must satisfy the biharmonic equation, which is explicitly written in this particular form for polar coordinate problems. Note that These expressions are valid for the $2D$ elasticity problem with either plane stress or plane strain assumption in polar coordinates.

This means we are neglecting the out-of-plane stress or out-of-plane strain components, either one. Now, coming to the general solution of that equation $\phi(r, \theta)$, how to choose this stress component, this stress function in polar coordinates in r θ coordinate, that solution is known as the general Michell solution. So, we will see how we can obtain that general Michell solution for the stress function in polar coordinates. For the rectangular Cartesian coordinate, we had we had used two types of solutions for ϕ, the stress function.

One was a polynomial form of solution, another was a Fourier form of solution. Here, we will see what type of solution, what form of solution is applicable for $\phi(r, \theta)$. Now, $\phi(r, \theta)$ must satisfy this biharmonic equation. $\nabla^4\phi = 0$, and the Laplacian operator in polar coordinates is written as $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. So, this particular part equals the Laplacian operator.

As it is biharmonic, we have the square of the Laplacian operator acting over ϕ, and that should equal 0. Now that we are choosing the general solution as $\phi(r, \theta) = R$, which is a function of the smaller radial variable times $e^{\beta\theta}$. So, we are assuming a separable solution, $\phi(r, \theta)$ is the product of two functions: one is a function of r, which is named as capital R, another is a function of θ, which is chosen to be an exponential function $e^{\beta\theta}$. Now, moving forward, we are going to substitute this form of ϕ in the biharmonic equation. So, ϕ is written as $R(r)e^{\beta\theta}$ in the biharmonic equation, and if you expand this, you will get this equation. $\frac{d^4R(r)}{dr^4}+\frac{2}{r}\frac{d^3R(r)}{dr^3}-\frac{(1-2\beta^2)}{r^2}\frac{d^2R(r)}{dr^2}+\frac{(1-2\beta^2)}{r^3}\frac{dR(r)}{dr}+\frac{\beta^2(4+\beta^2)}{r^4}R(r)=0$. Now, if you look at this equation, this is a differential equation, an ordinary differential equation of fourth order in the radial variable small r, but the constants.

So, if you look at the coefficients of these different-order derivatives, these are all functions of the variable small r. So, it would be convenient to solve this ODE if we are able to convert these coefficients of the different orders of differentiation into constants, and that can be done with a change in variable. We are defining a new variable k as our independent variable, where r and k are related using this. So, $r = e^k$.

Now, with this substitution of the independent variable from r (small r) to k, the above equation can be transformed into a fourth-order differential equation, a fourth-order ordinary differential equation with constant coefficients as this. So, earlier we were having small r-dependent coefficients. Now, it would be constant coefficients. You can see all these coefficients are becoming constant with this change in variable,

and note that earlier capital R was a function of small r. Now, capital R is a function of k because small r is now replaced with a new independent variable k. So, the equation is $\frac{d^4R(k)}{dk^4}-4\frac{d^3R(k)}{dk^3}+(4+2\beta^2)\frac{d^2R(k)}{dk^2}-4\beta^2\frac{dR(k)}{dk}+\beta^2(4+\beta^2)R(k)=0$. This is the fourth-order ODE with constant coefficients. And the solution of this can be easily obtained by assuming a general exponential solution of $R(k) = e^{\alpha k}$.

Substituting this $R(k) = e^{\alpha k}$ back into this equation, we would get a polynomial algebraic equation involving α and β like this. So, expanding this, it will have two factors. One is $\alpha^2 + \beta^2$. Another is $\alpha^2 - 4\alpha + 4 + \beta^2$.

The product of these two factors is 0. So, if α and β, these two exponential coefficients—one is for capital R, another is for θ, the θ function— if they are able to satisfy this particular equation, then the bi-harmonic condition will be satisfied by the chosen stress function. Now, to satisfy this particular equation, the roots of this equation are obtained, and these are $\alpha = \pm i\beta$ and $\alpha = 2 \pm i\beta$. So, this root is obtained from the first factor, and this root is obtained from the second factor.

Alternatively, instead of writing α in terms of β, you can write β in terms of α as $\beta = \pm i\alpha$ and $\beta = 2 \pm i(\alpha - 2)$. So, these are the two possible ways to represent the root, and you can clearly see there are four roots of α or four roots of β, as this was a fourth-order polynomial of α or a fourth-order polynomial of β. Now, in polar coordinate problems, we must have the periodic solution of θ, meaning in the polar coordinate, let us say this is the geometry.

We are starting from this line, the x-axis, which refers to $\theta = 0$. Now, as you start from here, after completing one complete revolution with $\theta = \pi$, the solutions must match with the $\theta = 0$ solution. Then, once again, with $\theta = 2\pi$, the solution should Sorry, this is $\theta = 2\pi$ after one complete revolution. So, at $\theta = \pi$, the solution should match with the $\theta = 0$ case.

Similarly, after two complete revolutions, the $\theta = 4\pi$ solution should match. So, this is called the periodic nature of the solution in polar coordinates for the variable θ. This must be ensured. And that can be done only if we choose beta to be some purely imaginary number in, where n is an integer. So, if you recall, the $\phi(r, \theta)$ in the previous slide was defined as some capital $R(r)e^{\beta\theta}$.

Now, $\beta = in$; then only this will be expressed in terms of $\sin\theta$ or cosine θ terms, $\sin n\theta$ or $\cos n\theta$, which will have this periodicity. So, thus, $\beta = in$, where n is an integer, and substituting that back, we will get $\phi(r, \theta) = R(r)e^{in\theta}$, where n will be an integer. If you enforce this n to be an integer in these roots, that will also ensure α is another integer number.

Now, moving forward, considering this exponential form of the solution of ϕ with $\beta = in$, $\phi(r, \theta)$ can be written as the combination of the cosine series and the sine series. where the first term is $\sum_{n=0}^{\infty} f_n(r)\, cos\, n\theta + \sum_{n=0}^{\infty} g_n(r)\, sin\, n\theta$. n can take values from 0, 1, 2, 3 till infinity. Now, this particular solution can be expanded explicitly and α and β should satisfy these conditions: $\alpha = \pm i\beta$ or $\alpha = 2 \pm i\beta$. Now, this total solution was initially proposed by Michael and later detailed by Little in 1973 and then we can represent the complete solution in the expanded form in this particular fashion. So, I will be explaining the different terms. If you look at the first row of terms, those are independent of $cos\,\theta$ or $sin\,\theta$. So, those are the terms which are not included in this $cos\,\theta$ or $sin\,\theta$. These are the terms with n equals to 0. If you put n equals to 0, then $\sin\theta$ terms will go to 0, and $cos\, n\theta$ will be 1 or minus 1. So, $f_n(r)$ with $n = 0$.

Basically, this is $f_0(r)$ term. And $g_0(r)$ with $n = 0$ will go to 0 because $\sin n\theta$ with n equals to 0 will vanish. Now, this n equals to 0 case with the cos term will result in these two big terms: one without θ, another with θ. Now, why two terms? Because of the repeated roots. If you put n equals to 0, β would be 0, and with β being 0, the two possible solutions of α are one is 0, another is 2, and both are repeated. So, you are having repeated solutions of α for n equals to 0, and due to that, we are getting two terms: one is constant, another term is another constant times θ, and both the constants are now functions of r. So, the first row of terms are the terms with n equals to 0 for repeated roots. Now, with n equals to 1, that is $cos\ \theta$, we will be getting this as $f_1(r)$ and this term as $g_n(r)$, $g_1(r)$ for n equals to 1.

For n greater than or equals to 2, all the terms are written in these two series. Where the summation is over n equals to 2 to infinity, $f_n(r)$ with n greater than or equals to 2 is this term which is multiplied with $\cos n\theta$ and the last row is the $\sin n\theta$ term $g_n(r)$ with n greater than or equals to 2. So, in total, these are the forms of $f_n(r)$ and $g_n(r)$ which satisfy this condition, this relation between alpha and beta. So, combining all these constraints, whatever we had obtained.

The $f_n(r)$ and $g_n(r)$ can be explicitly written like this, as proposed by Mikkel and later expanded by Little. So, this solution is called the general Mikkel solution for any $2D$ polar coordinate problem. With the help of this, most polar coordinate problems can be solved. Now, this involves many constants: a_0, a_1 till a_7, then a_{11}, a_{12}, b_{11}, b_{12}, a_{n1}, a_{n2}, b_{n1}, b_{n2}, and so on. These are required to be obtained by using the proper boundary conditions.

So, in most problems, we will have many of the constants to be 0, reducing it to a smaller form. Now, proceeding further to the axisymmetric type of problem in polar coordinates. We can call a problem axisymmetric in polar coordinates if it satisfies the following two conditions. The first is the geometry of the problem has one axis of symmetry.

So, if you look at this geometry here, the body is generated by revolving these particular curves. If you take this curve and revolve it around the z-axis, the entire body is generated, and the surface is created. So, a surface generated by revolving a curve will create an axisymmetric geometry with the z-axis being the axis of symmetry. So, a problem can be axisymmetric if the geometry of the body has one axis of symmetry, which is the z-axis for this particular case, and the second is the material properties, external loading, and boundary conditions—all of them—are also axisymmetric about that same z-axis. The same axis of symmetry. Only geometry being axisymmetric cannot guarantee an axisymmetric problem. Boundary conditions, loading, and material properties, material symmetry properties—all of them should also be symmetric. For example, with this axisymmetric geometry, if you have a boundary condition where only one quarter is fixed and the rest three quarters are free, the axisymmetry is immediately broken due to the non-axisymmetric boundary condition. Similarly, if a load is applied to only part of the body, let us say half of the body, from θ equals 0 to pi, and from π to 2π, the load is not there. In that case, the problem cannot be axisymmetric even if the geometry is axisymmetric.

So, geometry, loading, boundary conditions, and material properties—everything should be axisymmetric. Only then can we call the problem an axisymmetric problem. Now, for the axisymmetric problem that satisfies all these assumptions, all the field quantities are independent of θ. Thus, the partial derivative with respect to theta for any quantity, $\frac{\partial}{\partial\theta}$, equals 0 for any quantity. Now, to avoid non-axisymmetric deformation for the axisymmetric problems, u_θ the deformation component along the θ direction should be 0. If you allow u_θ to be non-zero that will immediately make the deformed profile to be non-axis symmetric which is not allowed.

u_r and u_z can be non-zero and functions of r and z only. They cannot be functions of θ. Hence, to avoid non-axisymmetric deformation, we should have our displacement field like this, with $u_\theta = 0$ and u_r, u_z being functions of r and z only. Now, with these assumptions, $u_\theta = 0$ and u_r, u_z being functions of r and z, two of the strain components, $\varepsilon_{r\theta}$ and $\varepsilon_{\theta z}$, would be 0 immediately.

The corresponding shear stress components, $\tau_{r\theta}$ and $\tau_{\theta z}$, would be 0, similar to the deformation. To ensure the axisymmetry of the body forces, b_r and b_z would be functions of r and z only, and the theta component of the body force, b_θ, should be forced to 0. Now, coming to the equilibrium equation in polar coordinates. These are the general equilibrium equations in polar coordinates. Now, we will try to reduce them to the equilibrium equations for axisymmetric problems in polar coordinates.

So, for axisymmetric problems, first we have $\frac{\partial}{\partial\theta} = 0$. The partial derivative of any quantity with respect to θ is 0. Three terms are set to 0. Then, b_θ is 0. The body force component along the theta direction is 0.

Then, two of the stress components, $\tau_{r\theta}$ and $\tau_{\theta z}$, are 0. Setting all of this to 0, the second equation in θ is automatically satisfied. So, for an axisymmetric problem, we are only left with these two equilibrium equations: one along the r direction and another one along the z direction. These two are called the equilibrium equations for any general $3D$ axisymmetric problem in polar coordinates. So, we need to solve these two equations for the axisymmetric problem.

For a general $3D$ problem in polar coordinates, we have these three total equations without any zero terms. With the assumption of axisymmetry, it would be reduced to these two equations. One equation in the theta direction is automatically satisfied. Now, this is for a $3D$ axisymmetric problem. We can further consider a $2D$ approximation of the axisymmetric problem.

From a $3D$ axisymmetric problem, by imposing either plane stress or plane strain, we can reduce it to a $2D$ axisymmetric problem. So, additionally imposing the plane stress constraints on the axisymmetric problem, we have σ_{zz}, τ_{rz}, and $\tau_{\theta z}$ to be 0. b_z will also go to 0 for the plane stress problem, and b_r will then only be a function of r. With this, out of these two equilibrium equations, imposing σ_{zz}, τ_{rz}, and b_z to be 0, These many terms will go to 0. The second equation in the z direction is automatically satisfied, and we are left with only one equation: $\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = 0$.

For the $2D$ axisymmetric plane stress problem. Similarly, for the axisymmetric plane strain problem, for the infinitely long axisymmetric body in the z direction, the out-of-plane strain components are 0. This would result in τ_{rz} and $\tau_{\theta z}$ being 0. Additionally, for the plane strain problem, we have a partial derivative of any quantity with respect to z, $\frac{\partial}{\partial z}() = 0$. So, forcing these assumptions on the equilibrium equation, τ_{rz} and $\tau_{\theta z}$, and $\frac{\partial}{\partial z}()$ means this one, which will also go to 0.

Body force $b_z = 0$, and $b_r = b_r(r)$, similar to the plane stress problem. So, with this, for the axisymmetric $2D$ plane strain problem, we also have the same equation as the axisymmetric plane strain case. So, for any $2D$ axisymmetric problem, either plane stress or plane strain, we are left with only one equilibrium equation, which is this. $\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = 0$. This is the only equation we need to solve for any $2D$ axisymmetric planar problems.

Now, coming to the strain-displacement equations. These are the three general strain-displacement equations. $\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$. $\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$. $\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$. These are without any approximation, just for a $2D$ polar problem. Now, with the assumption of axisymmetry, we additionally have $u_\theta = 0$ and $\frac{\partial}{\partial \theta}() = 0$.

So, if we impose those constraints on the general $2D$ strain-displacement equation of the polar coordinate, $u_\theta = 0$ means this term will go to 0, this term will go to 0, and this term will also go to 0 from u_θ. u_θ is 0, and as $\partial/\partial\theta$ of any quantity is 0, this is having a partial derivative of u_r with respect to θ. So, that term would also go to 0. We would have the strain-displacement equation for the axisymmetric problem like this: $\varepsilon_{rr} = \frac{du_r}{dr}$, and for this axisymmetric problem, u_r is a function of r only. Thus, the partial derivative $\partial u_r/\partial r$ can be written as the total derivative du_r/dr, as u_r is a function of r only. So, the normal strain along the radial direction $\varepsilon_{rr} = \frac{du_r}{dr}$. The circumferential strain $\varepsilon_{\theta\theta} = \frac{u_r}{r}$, and the shear strain $\varepsilon_{r\theta}$ (in-plane shear strain) equals 0 for the axisymmetric $2D$ planar problems. Now, coming to the displacement formulation, which is basically the Lamé-Navier equation that we had discussed for the rectangular Cartesian coordinate system. If I write that equation for the polar problem with the assumption of a $2D$ axisymmetric case, the Lamé-Navier equation in terms of the displacement field will be only one single equation of the only one displacement field u_r, and that would be like this. $\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$. So, that is a second-order ODE over u_r, and from this, u_r can be obtained. The general displacement field of the radial displacement component u_r can be written as $c_1 r + \frac{c_2}{r^2}$, where c_1 and c_2 are two constants which are required to be determined by using the displacement boundary conditions.

So, if you are having a stress problem, then we will be solving it in terms of stress components. If you are using displacement formulation for an axisymmetric $2D$ problem, then the Lame-Navier equation should be used, where this equation is solved. The general solution of that equation for the u_r variable is given like this, involving two constants which are solved with the help of displacement boundary conditions. Now, moving forward to the biharmonic equation, the chosen stress function ϕ must satisfy the biharmonic equation, which is given as $\nabla^4\phi(r,\theta) = 0$. Now, for axisymmetric problems, ϕ is a function of r only because it should be independent of θ, as all the field variables should be independent of θ. The stress function should also be independent of θ. So, ϕ is a function of only the radial field variable r. Thus, the biharmonic equation will be $\nabla^4\phi(r,\theta) = 0$, where ϕ is a function of r only. Now, in polar coordinates, this Laplacian operator The Laplacian operator in polar coordinates is written as $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$. Now, for axisymmetric problems, since we have del/del theta of any quantity to be 0, the last term would vanish, and thus the partial derivative with respect to r can be written as the total derivative with respect to r. Hence, the Laplacian operator will be $\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$ and that can be combined and written in this particular form: $\frac{1}{r}\frac{d}{dr}$ of r times d/dr. So, this is the Laplacian operator for $2D$ axisymmetric problems. Now, substituting this into the biharmonic equation, we would get this.

So, this is one Laplacian operator acting on the Laplacian of phi, and in total, this gives the biharmonic of ϕ. So, expanding this in this particular form, we will get $\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right\}\right] = 0$. Now, instead of expanding with the fourth-order derivative of r, I am writing this in this fashion because it would be helpful to solve for ϕ.

You can easily integrate this with respect to ϕ. So, with respect to r, for finding phi. So, let us say you can remove this $1/r$ from here, add an r on the right-hand side, then integrate it as d/dr is there on the left,

add one integration constant, then divide both sides by this r, once again integrate, and so on. If you do it chain-wise, all these d/dr, the four partial derivatives, can be opened one after another by doing four integrations and adding four integration constants. If you do so, you would get the general solution of the stress function ϕas a function of r only in this form.

$\phi(r)$ will be $A\ln r + Br^2 \ln r + Cr^2 + D$, where $\ln r$ means the log of r with base e, the natural logarithm of r. So, this is the stress function form for the $2D$ axis-symmetric problem involving four constants: A, B, C, and D. So, for the $2D$ axis-symmetric problem, it is not required to use the Michell solution. From the general Michell solution, by forcing the θ-dependent terms to zero, you can arrive at this particular form, but it is easier to start with the biharmonic equation for the $2D$ planar axis-symmetric problem and derive this form of ϕ. Now, for the stress components in the $2D$ planar axis-symmetric problem, σ_{rr} will have only one term, $\frac{1}{r}\frac{d\phi(r)}{dr}$, $\sigma_{\theta\theta} = \frac{d^2\phi(r)}{dr^2}$, and $\tau_{r\theta} = 0$ because ϕ is a function of r only. So, $d\phi/d\theta$ or $\partial\phi/\partial\theta$ equals zero. Thus, the stress equations in terms of stress components reduce to this form for the $2D$ planar axis-symmetric problem. And σ_{rr}, if you substitute this form of ϕ here, σ_{rr} would be obtained as $\frac{A}{r^2} + B(1 + 2\ln r) + 2C$, and $\sigma_{\theta\theta}$ would be $-\frac{A}{r^2}$ plus and $\tau_{r\theta} = 0$. So, these are the stress components as functions of r for any axisymmetric planar problem. Note that axisymmetric planar problems are subjected to only non-zero normal stresses σ_{rr} and $\sigma_{\theta\theta}$. The in-plane shear stress $\tau_{r\theta}$ is 0 for the planar axisymmetric problem. Now, there is one particular point to note: if you have an axisymmetric problem like this—let us say a solid disk or cylinder where the origin O is part of the domain— then the radial variable r varies from 0 to capital R. Let us say this is a disk or cylinder of radius R. So, the small r (the radial variable) varies from 0 to R. Now, at the origin—if this is part of the domain for the solid disk—if you look at the first term, $(1/r^2)$ or the $ln\, r$ term in both σ_{rr} and $\sigma_{\theta\theta}$, replacing $r = 0$ would make these terms approach infinity. if you replace r equals to 0, these two terms will shoot to infinity. So, the stress components would become indeterminate at the origin.

This has to be avoided. We cannot have infinite stress or singularity, stress singularity at the origin. To avoid that, we must force A to be 0 and B to be 0. To avoid the infinite stress value at the origin for the axisymmetric problem where the origin is part of the domain, we must have two of the constants A and B to be 0.

So, σ_{rr} and $\sigma_{\theta\theta}$ would be just the last term $2C$ left, and $\tau_{r\theta} = 0$. So, for such cases, axisymmetric problems involving the origin, the stress distribution σ_{rr} and $\sigma_{\theta\theta}$ must be a constant. We are having a constant stress field for this problem. Now, instead of a solid disk, if you have an annular disk where, let us say, r is varying between R_i to R_o, for such cases, if the origin is not part of the domain, then A and B can be nonzero.

So, we will take some example problems for this kind of solid and annular disk in the next lecture. So, in this lecture, we discussed the general Michell solution for polar coordinates, then formulated the axisymmetric problems in elasticity first for $2D$ and then reduced it to the $2D$ planar axisymmetric formulation and looked into the stress function and stress components for the $2D$ planar axisymmetric problems. Thank you.