

APPLIED ELASTICITY
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WEEK: 09
Lecture- 41

COURSE ON:
APPLIED ELASTICITY

Lecture 41
FIELD EQUATIONS IN POLAR COORDINATES

$T_{i,k}$

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Welcome back to the course on applied elasticity. The topic of today's lecture is the field equations in polar coordinates. So far, in all our previous lectures, whatever elastic deformation problems we have solved, all those were solved with the help of the rectangular Cartesian coordinate system, that is, the ϕ or stress function was a function of x , y , and z Cartesian coordinate system.

Now, instead of the Cartesian coordinate system, sometimes it becomes essential to use the polar coordinate system, that is, r , θ , z coordinates, because of the geometry of the problem. So, based on the geometry of the problem, we need to choose a proper coordinate system for ease of solution. Now, we will be looking into the field equations of elasticity. How do they look in the polar coordinate formulation?

So, first, we will start with a small elemental volume described in the polar coordinate like this. Earlier, we were taking a small element with length dx , dy , dz along x , y , and z directions.

3D Elasticity Problems in Polar Coordinates

The diagram shows a small element in a 3D coordinate system (x, y, z). The element is defined by radial distance r , angular span $d\theta$, and height dz . The element is projected onto the xy -plane and the z -axis. The stress components acting on the element are shown as vectors: normal stresses σ_{rr} , $\sigma_{\theta\theta}$, and σ_{zz} ; and shear stresses $\tau_{r\theta}$, $\tau_{\theta r}$, τ_{rz} , and τ_{zr} . The element is labeled with 'r-plane', ' θ -plane', and 'z-plane'. The equilibrium equations are given as:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + b_\theta = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + b_z = 0$$

Handwritten notes include: $\Sigma: \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0$

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Now, instead of that, we are going to take a small element, and the sides of that are aligned with r, θ, z axes. So, if you take this small element and project it on the xy plane and z -axis respectively, the lengths are described like this. So, these lengths this particular length of the elemental volume is along the r -direction, which is at a distance r from the centre. So, from the centre O till this point, the inner curved boundary, this length is r . Then, the length of the element along the radial direction is dr , the height of the element along the z -axis is dz , and the angular span of the element is $d\theta$, which, once projected in the xy -plane, that angle is shown here as $d\theta$, and it is lying at an angle θ from the x -axis. So, with this, we are going to write the stress components, and this length is dz .

Now, for this particular element, we will try to draw the stress components on different faces. So, we are trying to draw the Cauchy stress components for this particular element on the r -plane, θ -plane, and z -plane. So, if you do so, these are the components which will be coming on the r, θ , and z planes. Here, I have shown the components only on one r -plane, one θ -plane, and one z -plane.

On the opposite r, θ , and z planes, equal and opposite balancing shear stress components, normal and shear stress components, would also be there. Now, if you consider the r -plane, so this particular plane is the positive r -plane, where the unit normal to this plane is along the r -direction, the radially outward direction. So, this is having one normal component σ_{rr} , one normal stress, and two shear stresses: $\tau_{r\theta}$ and τ_{rz} .

So, $\tau_{r\theta}$ and τ_{rz} , these are the two shear components present on the positive r plane. If you consider the top face, the top one is the positive z plane. Now, on that, the normal stress, the normal stress component σ_{zz} is acting upward along the positive z axis. And two shear components, τ_{zr} along positive r and $\tau_{z\theta}$ along positive θ .

Note that we are considering theta to be positive in this direction. From the positive x axis, theta is increasing in the counterclockwise direction, positive as seen from the tau. Now, coming to this left-hand side plane, this plane is the theta plane. However, this is the negative theta plane because the unit normal for this particular plane is along the negative theta direction. So, if you look from the top.

So, the top view of this element in the $r - \theta$ plane would look like this. This is σ_{rr} . And these two are $\sigma_{\theta\theta}$, then we have $\tau_{r\theta}$ and $\tau_{\theta r}$ like this. So, similarly, on the opposite planes, on the negative r plane, negative z plane, and positive θ plane, equal and opposite stress components are also there.

Now, following a similar principle and procedure as we had done for the rectangular Cartesian coordinate system, for this elementary volume, by force balance along the r , θ , and z directions, we can derive the equilibrium equation. So, the equilibrium equation along the r direction for this particular problem, for this small element in $r - \theta - z$ coordinates, can be obtained as $\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r$. For the θ direction, the equation would be $\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + b_\theta$. And finally, for the z direction, the equilibrium equation would be $\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + b_z = 0$. Now, note that for the rectangular Cartesian coordinate system, we had three partial derivative terms of stress components. Along the x direction, it was like $\frac{\partial \sigma_{zz}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0$.

This was for the x direction in the rectangular Cartesian coordinate system. Here, we also have three terms: three partial derivative terms of different stress components involving r . So, stress components acting on the r plane must have one of the subscripts as r , preferably the first subscript. Thus, σ_{rr} , $\tau_{r\theta}$, and τ_{rz} are the three stress components acting on the r plane. The partial derivatives of these with respect to r , θ , and z are respectively present.

But there is one extra term before the body force, same is there for other directions as well, this is because If you consider the ah consider this 2D figure, this is let us say the r direction, this direction is r direction. So, along this we are having some unit vector e_r , now theta direction is perpendicular to this. So, it is taken e_θ in this particular direction. So, e_θ hat is let us say unit vector along theta direction.

You can see this sigma theta θ and tau r theta, they are having some component along e_θ , they are not not independent of that. So, ideally we are supposed to have $\sigma_{\theta\theta}$ oriented along e_θ and $\tau_{r\theta}$ should be parallel to e_θ , but that is not going to happen because of the small angle present. These are and this these small angle which is basically $d\theta$ by 2

from geometry that is giving rise to these additional terms in the r and θ equations. With the proper force balance if you write all this equation all this terms that force balance will give you these three terms where b_r , b_θ , and b_z are the are the body forces per unit volume along the r , θ , and z directions respectively. So these are the equilibrium equations for the polar coordinate in 3D along r , θ , and z directions involving the body forces now. Proceeding forward, we will try to reduce this 3D polar coordinate problem into 2D polar coordinate problem.

2D Elasticity Problems in Polar Coordinates

(b) **Plane Strain:** Infinitely long body along z direction

$$\epsilon_{zz} = \epsilon_{rz} = \epsilon_{\theta z} = 0, \quad \frac{\partial}{\partial z}(\) = 0$$

$$\Rightarrow \sigma_{rr} = \frac{E(\epsilon_{rr} + \nu\epsilon_{\theta\theta})}{(1-\nu^2)}, \quad \sigma_{\theta\theta} = \frac{E(\epsilon_{\theta\theta} + \nu\epsilon_{rr})}{(1-\nu^2)}, \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}), \quad \tau_{r\theta} = G\gamma_{r\theta} = 2G\epsilon_{r\theta}, \quad \tau_{rz} = \tau_{\theta z} = 0$$

Equilibrium equations:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{rz}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

$$\left. \begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned} \right\} \text{With zero body force}$$

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Problem with the assumption of either plane stress or plane strain. So, we had discussed in the rectangular Cartesian coordinate system that 3D problems can be reduced to 2D problems under certain assumptions of either plane stress or plane strain type problems. So, here also, the same thing can be done for the polar coordinate. So, we all know that plane stress problems are valid for bodies which have very small thickness along the axial direction.

So, along the z -direction, if the dimension or the thickness of the body is small, we can assume that to be a plane stress problem for which all the out-of-plane stress components—normal and shear stress components—out-of-plane components would go to 0. For this particular polar coordinate, σ_{zz} , τ_{rz} , $\tau_{\theta z}$. These are the three out-of-plane stress components acting on the z -plane. So, these should be 0 for the plane stress problem in polar coordinates. Now, σ_{zz} , τ_{rz} , $\tau_{\theta z}$ —these three being 0—we can write the strain components as this.

ϵ_{rr} would be $\frac{\sigma_{rr} - \nu\sigma_{\theta\theta}}{E}$. So, actually, there should be one more term, $\nu\sigma_{zz}$, but since σ_{zz} is 0, this term is going. The same is going to happen for $\epsilon_{\theta\theta}$ and σ_{zz} . So, $\epsilon_{rr} = \frac{\sigma_{rr} - \nu\sigma_{\theta\theta}}{E}$, $\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta} - \nu\sigma_{rr}}{E}$, ϵ_{zz} . The axial strain along the z -direction is $\frac{-\nu(\sigma_{rr} + \sigma_{\theta\theta})}{E}$.

Now, $\varepsilon_{r\theta} = \frac{\tau_{r\theta}}{2G}$, ε_{rz} , and $\varepsilon_{\theta z}$ are 0, as τ_{rz} and $\tau_{\theta z}$ are 0. So, note that even if we have σ_{zz} to be 0, the non-zero value of ε_{zz} is coming due to the presence of Poisson's ratio. Now, if you write the equilibrium equations, these are the three equilibrium equations. Now, we are going to impose these constraints— $\sigma_{zz} = 0$, $\tau_{rz} = 0$, $\tau_{\theta z} = 0$ —on these equilibrium equations. If you impose those, $\sigma_{zz} = 0$ means this particular term in the third equation will go to 0.

Then, forcing $\tau_{rz} = 0$, three more terms will go to 0. Then, forcing $\tau_{\theta z} = 0$, two more terms will go to 0. So, all the terms in the last equation are already 0. So, the last equilibrium equation along the z-direction is automatically satisfied for the plane stress polar coordinate problem.

And only two remaining equations are this: $\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$. This is along the r-direction, and along the theta-direction, the equation is $\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$. So, these are the two equilibrium equations for the plane stress polar coordinate problems with the absence of any body force, so b_r , b_θ , and b_z are taken to be 0.

2D Elasticity Problems in Polar Coordinates

2D equilibrium equations:

$$\left. \begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned} \right\} \text{Valid for both plane stress and plane strain problems with zero body forces}$$

which are automatically satisfied by the stress components defined in terms of a stress function $\phi(r, \theta)$ as,

$$\left. \begin{aligned} \sigma_{rr} &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\ \tau_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \end{aligned} \right\}$$

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Now, moving forward to the next case, which is the plane strain problem for the polar coordinate. This is valid for bodies where the length of the body along the z-axis is infinitely long. Or it is extremely large compared to other dimensions. For such cases, we can assume the problem to be of plane strain type. $\partial/\partial z$ of any quantity is 0, as the length of the body is very large along the z-direction. None of the quantities are going to vary along the z-direction. We are neglecting the variation of any quantity along z.

Hence, $\partial/\partial z$ is 0, and out-of-plane strain components— ε_{zz} (out-of-plane normal strain), ε_{rz} , and $\varepsilon_{\theta z}$ (out-of-plane shear strains)—are taken to be 0. Now, with ε_{zz} , ε_{rz} , and $\varepsilon_{\theta z}$ being 0, if we write the stress components, the σ_{rr} , $\sigma_{\theta\theta}$, and σ_{zz} —these three stress

components (normal stresses)—can be obtained as $\frac{E(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta})}{(1-\nu^2)}$; $\sigma_{\theta\theta} = \frac{E(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr})}{(1-\nu^2)}$. $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$.

This comes because $\varepsilon_{zz} = 0$, and then the two shear stresses, τ_{rz} and $\tau_{\theta z}$, would be 0, as the corresponding strains are 0. The only non-zero shear stress is $\tau_{r\theta}$, which will be equal to $2G\varepsilon_{r\theta}$. Or $G\gamma_{r\theta}$, where γ is the engineering shear strain, whereas $\varepsilon_{r\theta}$ is the tensorial shear strain. Now, writing the equilibrium equations and setting the 0 terms—here, we know that two of the tau components are 0 ($\tau_{rz}, \tau_{\theta z}$), and $\partial/\partial z$ of any quantity is 0. Since $\partial/\partial z$ of any quantity is 0, this particular term, $\frac{\partial\sigma_{zz}}{\partial z}$, should go to 0.

And then, enforcing τ_{rz} to be 0, we will get these 3 terms to 0. Then, forcing $\tau_{\theta z}$ to 0, 2 more terms will go to 0. Here, also, the last equation is directly satisfied, and the first two equations are the only remaining ones for the plane strain polar coordinate problems in the absence of any body force. Now, if you compare these equations with the plane stress problem equations, you will find that they are identical. So, for 2D elasticity problems in polar coordinates, the plane stress and plane strain approximations give us the same set of equilibrium equations, which are only two equations along the r and θ directions.

Thus, for any 2D polar coordinate problem, these are the two equilibrium equations which are valid for both plane stress and plane strain problems without any body force. If you recall, for the rectangular Cartesian coordinate also, coordinate system problems, the plane strain and plane stress equilibrium equations were the same.

They had only two equations with the same nonzero terms. Here also, in the polar coordinate, a similar analogy is valid. Now, if you want to add the effect of body forces, So, for that, the b_r , and b_θ —these two extra terms—are supposed to be added in both equations, respectively. The solution approach using the stress function would be slightly different for the case of polar coordinate problems with body force, which we would discuss in a separate lecture later. So, now, let us proceed with zero body forces.

These two equilibrium equations can be automatically satisfied if we choose the stress components in terms of a stress function $\phi(r, \theta)$ as $\sigma_{rr} = \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2}$. This is the normal stress in the r direction. Similarly, the normal stress in the theta direction, $\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$, and the shear stress $\tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\phi}{\partial\theta} \right)$. So, if you choose our stresses like this, in terms of this stress function $\phi(r, \theta)$, which is now a function of r and θ in polar coordinates, this is also called the Airy stress function in polar coordinates. So, in terms of ϕ , if you choose or define $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$ in this form and substitute it back into these equations, both equations will be automatically satisfied.

Thus, the solution of the polar coordinate problem would come down to the solution of the stress function ϕ , which would be the only unknown for such 2D elasticity problems.

Now, moving forward. Let us try to derive those stress component equations in terms of ϕ using another approach. This approach involves writing the equilibrium equations, then reducing some of the equations by either plane stress or plane strain assumptions, and finally, defining the stress components in terms of derivatives of the Airy stress function in polar form, meaning ϕ is a function of r and θ .

Now, we had already obtained the expression of stress components σ_{xx} , σ_{yy} , and τ_{xy} in terms of the derivative of the stress function in the rectangular Cartesian coordinate system. $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$, and $\tau_{xy} = \frac{\partial^2 \phi}{\partial x \partial y}$. Using those, and using the coordinate transformation from $x y$ to $r \theta$, we should also be able to get the same set of equations for σ_{rr} , $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ that we will try to verify.

2D Elasticity Problems in Polar Coordinates

$x = r \cos \theta, \quad y = r \sin \theta$
 $r = \sqrt{x^2 + y^2} \quad \frac{\partial r}{\partial x} = \frac{1}{r} \cdot \frac{\partial x}{\partial x} = \frac{x}{r} = \cos \theta$
 $\tan \theta = \frac{y}{x}$
 $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$
 $\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$

$\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2}$

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So, this figure is showing the simple transformation between rectangular Cartesian $x y$ coordinates to polar coordinates, that is $r \theta$, where $x = r \cos \theta$, and $y = r \sin \theta$.

Alternately, we can write $r = \sqrt{x^2 + y^2}$, and θ as \tan^{-1} of $\frac{y}{x}$. Now, if you try to recall, we had defined our $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$. Then, $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$.

So, all these partial derivatives mean $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of any variable. These we will try to write in terms of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$. So, basically, we will try to write these x as a function of r and θ , and it will have two terms: one is $\frac{\partial}{\partial r}$ term, and another is $\frac{\partial}{\partial \theta}$ term. So, for that, we are doing this exercise.

So, $\frac{\partial}{\partial x}$, this operator, partial derivative with respect to x can be written as $\frac{\partial r}{\partial x} \frac{\partial}{\partial r}$ and $\frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}$. So, how are you writing this? Because r is function of both x and y or x is also function

of both r and θ . Using that, this partial derivative $\frac{\partial}{\partial x}$ can be written in this particular form.

Now, these two terms $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ can be obtained. by using these two equation relation between r θ and x z . So, if you find out take the derivative of both these two equations with respect to x you will be getting $\frac{\partial r}{\partial x}$ as $\cos \theta$ and $\frac{\partial \theta}{\partial x}$ as $-\frac{\sin \theta}{r}$. So, replacing that here, we will get $\frac{\partial}{\partial x}$ operator is same as $\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$. So, partial derivative with respect to x operator is now changed to the partial derivative with respect to r and with respect to theta in this particular form.

So, I am just doing one of this. So, r being $\sqrt{x^2 + y^2}$ del r del x will be half of 1 by x square plus y square times 2 x . These two will get cancelled and you can write the square root of x square plus y square as r . So, $\frac{\partial r}{\partial x}$ will be x by r . Now, using this x by r can be written as $\cos \theta$.

So, $\frac{\partial r}{\partial x}$ is equals to $\cos \theta$ which is replaced here. Similarly, rest of the terms can also be obtained. So, in the same fashion, $\frac{\partial}{\partial y}$ can be written as $\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$. So, both of these two operators, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, partial derivatives with respect to x and y , are now converted into the combination or linear superposition of partial derivatives with respect to r and with respect to θ with the help of these equations. Now, with this, we will proceed with some stress transformation. So, the first is the transformation of σ_{xx} in polar coordinates. σ_{xx} was defined as $\frac{\partial^2 \phi}{\partial y^2}$. Now, this $\frac{\partial \phi}{\partial y}$, with the help of this newly defined $\frac{\partial}{\partial y}$ operator, we can write $\frac{\partial \phi}{\partial y}$ as $\sin \theta \frac{\partial \phi}{\partial r} + \frac{\cos \theta}{r} \frac{\partial \phi}{\partial \theta}$.

Now, $\frac{\partial^2 \phi}{\partial y^2}$ will be another partial derivative of this with respect to y . So, $\frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right)$, and once again using this definition of $\frac{\partial}{\partial y}$ here, you can expand this, and σ_{xx} will be obtained like this. So, you can see there are three terms: one term containing $\sin^2 \theta$, one containing $2 \sin \theta \cos \theta$, and another containing $\cos^2 \theta$. The σ_{xx} is written in terms of partial derivatives of ϕ with respect to r and θ . So, we started with this form where the stress component was defined in terms of rectangular Cartesian variables xy .

We ended up with this form of σ_{xx} , where it is defined with respect to partial derivatives of r and θ in the polar coordinate variables. Similarly, if you do the same thing for τ_{xy} , which was defined as $-\frac{\partial^2 \phi(x,y)}{\partial x \partial y}$. So, this $\frac{\partial \phi}{\partial y}$ equation on this, if you are taking $\frac{\partial}{\partial x}$ of that with a negative sign and use this form of $\frac{\partial}{\partial x}$ operator, τ_{xy} can be obtained as this with two terms: one with $-\sin \theta \cos \theta$, another with $\cos^2 \theta - \sin^2 \theta$ both multiplied by some partial derivatives of ϕ with respect to r and θ at different orders.

So, we are able to transform σ_{xx} and τ_{xy} , both the stress components acting on the x plane, into polar coordinates.

2D Elasticity Problems in Polar Coordinates

With $\theta \rightarrow 0$, $\sigma_{xx} \rightarrow \sigma_{rr}$, $\tau_{xy} \rightarrow \tau_{r\theta}$:

$$\sigma_{xx} = \sin^2\theta \frac{\partial^2\phi}{\partial r^2} + 2 \sin\theta \cos\theta \left(\frac{1}{r} \frac{\partial^2\phi}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial\phi}{\partial \theta} \right) + \cos^2\theta \left(\frac{1}{r^2} \frac{\partial^2\phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} \right)$$

$$\tau_{xy} = -\sin\theta \cos\theta \left(\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r} \frac{\partial\phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2\phi}{\partial \theta^2} \right) + (\cos^2\theta - \sin^2\theta) \left(\frac{1}{r^2} \frac{\partial\phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2\phi}{\partial r \partial \theta} \right)$$

$\sin\theta = 0, \cos\theta = 1$

$$\sigma_{rr} = \sigma_{xx}|_{\theta=0} = \frac{1}{r^2} \frac{\partial^2\phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial\phi}{\partial r}$$

$$\tau_{r\theta} = \tau_{xy}|_{\theta=0} = \frac{1}{r^2} \frac{\partial\phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2\phi}{\partial r \partial \theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\phi}{\partial \theta} \right)$$

When $\theta \rightarrow \frac{\pi}{2}$, $\sigma_{xx} \rightarrow \sigma_{\theta\theta}$:

$\sin\theta = 1, \cos\theta = 0$

$$\sigma_{\theta\theta} = \sigma_{xx}|_{\theta=\pi/2} = \frac{\partial^2\phi}{\partial r^2}$$

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Now, moving forward, we are considering this element with these stresses in the rectangular Cartesian coordinate system: σ_{xx} , σ_{yy} , and τ_{xy} . Now, coming to the $r \theta$ polar coordinate, the stress components are acting like this,

σ_{rr} , $\sigma_{\theta\theta}$ are normal stresses, and $\tau_{r\theta}$ is the shear stress. Now, if you carefully look, this angle is equal to θ , if we choose theta equal to 0 degrees, in that particular case, the σ_{rr} will coincide with the direction of x and thus, we can say for $\theta = 0$, σ_{rr} and σ_{xx} would be same. So, σ_{rr} can be written as σ_{xx} with θ replaced as 0. Similarly, with $\theta = 0$, $\tau_{r\theta}$ will be parallel to τ_{xy} . So, τ_{xy} at $\theta = 0$ is same as $\tau_{r\theta}$. Now, substituting $\theta = 0$ in the expression of σ_{xx} and τ_{xy} , $\theta = 0$ means $\sin\theta = 0, \cos\theta = 1$. So, with that $\sigma_{rr} = \sigma_{xx}|_{\theta=0}$ the $\sin\theta$ terms will go $\cos\theta$ would be equals to 1 we will only left with this term. So, $\sigma_{rr} = \frac{1}{r^2} \frac{\partial^2\phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial\phi}{\partial r}$. Similarly, $\tau_{r\theta}$ can be obtained as $\tau_{xy}|_{\theta=0}$ which would be having only these two terms and combining them in a single term it would be minus $-\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\phi}{\partial \theta} \right)$. Now, coming to $\theta \rightarrow \frac{\pi}{2}$ case. So, if this angle $\theta \rightarrow \frac{\pi}{2}$ then the element will be looking like this where σ_{rr} will go in this direction $\sigma_{\theta\theta}$ will be in this direction which is aligned along the x -axis. So, if $\theta \rightarrow \frac{\pi}{2}$, for that case, σ_{xx} at $\theta \rightarrow \frac{\pi}{2}$ is nothing but $\sigma_{\theta\theta}$. So, putting $\sin\theta = 1$ for $\frac{\pi}{2}$ and $\cos\theta = 0$, you can find out $\sigma_{\theta\theta} = \sigma_{xx}|_{\theta=\pi/2}$, which will be equal to $\frac{\partial^2\phi}{\partial r^2}$, which is this term in that case.

$\sin\theta$ will no longer be 0. The $\cos\theta$ would be 0 for that particular case. So, the last two terms would vanish, and $\sigma_{xx}|_{\theta=\pi/2}$ will be $\frac{\partial^2\phi}{\partial r^2}$, which is $\sigma_{\theta\theta}$ from the stress transformation.

2D Elasticity Problems in Polar Coordinates

Stress components in terms of Airy's stress function $\phi(r, \theta)$:

$$\begin{cases} \sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \\ \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \end{cases}$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

Biharmonic equation:

$$\begin{aligned} \nabla^4 \phi &= 0 \\ \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi &= 0 \\ \Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) &= 0 \end{aligned}$$

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So, if I write all those components in terms of any stress function ϕ , and note that now ϕ should be written or expressed as a function of r, θ , not as a function of x and y . With respect to that, these are the stress components for a 2D elasticity problem, and if you compare these with the previously obtained stress components from the equilibrium equation, they are identical. Now, these should also satisfy the bi-harmonic equation; the chosen stress function should satisfy the bi-harmonic equation, and using once again the $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \theta}$ operators.

Defined in terms of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, this $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are replaced here in the Laplacian operator. And with that, the biharmonic equation in the polar coordinate would be obtained like this. So, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, these operators are $\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$ in the polar coordinate. So, for the polar coordinate, the eddy stress function should satisfy this particular biharmonic equation.

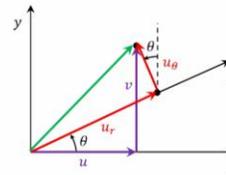
Strains in 2D Polar Coordinate Problems

Displacement components:

$$\begin{cases} u = u_r \cos \theta - u_\theta \sin \theta \\ v = u_r \sin \theta + u_\theta \cos \theta \end{cases}$$

Strain components:

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} \\ \Rightarrow \epsilon_{xx} &= \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) (u_r \cos \theta - u_\theta \sin \theta) \\ \Rightarrow \epsilon_{xx} &= \cos^2 \theta \frac{\partial u_r}{\partial r} + \sin \theta \cos \theta \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \sin^2 \theta \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \end{aligned}$$



$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

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Now, similar to the stress field, the strain fields can also be converted from the rectangular Cartesian to polar coordinates.

So, let us consider these displacements. So, this green line vector refers to the net displacement of a point, which has u and v components along the x and y directions. Along the r and θ directions, it has u_r and u_θ components. So, this is the r direction, along which it is u_r , and along θ , the displacement is u_θ . So, the total displacement vector is either u and v , meaning u along x and v along y , or u_r along r and u_θ along θ .

Now, the displacement components of these two, u and v in the xy coordinate, u_r and u_θ in the $r - \theta$ coordinate, can be related with this simple transformation with the help of an orthogonal tensor. And with respect to this, if I define the strain component normal strain ϵ_{xx} as $\frac{\partial u}{\partial x}$, using these expressions for u and the previously obtained expression for $\frac{\partial}{\partial x}$, the ϵ_{xx} can be expanded like this.

Strains in 2D Polar Coordinate Problems

$$\epsilon_{xx} = \cos^2 \theta \frac{\partial u_r}{\partial r} + \sin \theta \cos \theta \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \sin^2 \theta \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \quad \epsilon_{xy} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

With $\theta \rightarrow 0$, $\epsilon_{xx} \rightarrow \epsilon_{rr}$, $\epsilon_{xy} \rightarrow \epsilon_{r\theta}$:

$\sin \theta = 0, \cos \theta = 1$

$$\epsilon_{rr} = \epsilon_{xx} \Big|_{\theta=0} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{r\theta} = \epsilon_{xy} \Big|_{\theta=0} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

When $\theta \rightarrow \frac{\pi}{2}$, $\epsilon_{xx} \rightarrow \epsilon_{\theta\theta}$:

$\sin \theta = 1, \cos \theta = 0$

$$\epsilon_{\theta\theta} = \epsilon_{xx} \Big|_{\theta=\pi/2} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$$



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Similarly, the ϵ_{xy} , the shear strain, can be obtained like this. Now, following a similar approach as the stress transformation with θ equal to 0, ϵ_{xx} will be ϵ_{rr} , and ϵ_{xy} will be $\epsilon_{r\theta}$. So, substituting θ equal to 0 means $\sin \theta = 0, \cos \theta = 1$. ϵ_{rr} , the radial strain, would come out as $\frac{\partial u_r}{\partial r}$ and then $\epsilon_{r\theta}$, the in-plane shear strain, would come as $\frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$.

Similarly, with $\theta \rightarrow \frac{\pi}{2}$, ϵ_{xx} will go to $\epsilon_{\theta\theta}$ and that would come out to be $\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$, which can be obtained by substituting $\theta \rightarrow \frac{\pi}{2}$ in this particular equation.

Strains in 2D Polar Coordinate Problems

Strain components:

$$\begin{cases} \epsilon_{rr} = \frac{\partial u_r}{\partial r} \\ \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\ \epsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \end{cases} \Rightarrow \gamma_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \rightarrow \text{Engineering Shear Strain}$$

Strain compatibility relations:

$$\frac{\partial}{\partial r} \left(2r \frac{\partial \epsilon_{r\theta}}{\partial \theta} - r^2 \frac{\partial \epsilon_{\theta\theta}}{\partial r} \right) + r \frac{\partial \epsilon_{rr}}{\partial r} - \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} = 0$$

Only compatibility equation for 2D polar coordinate problems



Hence, for the 2D polar coordinate problem, all three strain components can be written like this. These are the three strain components.

The first two are normal strains, ϵ_{rr} and $\epsilon_{\theta\theta}$, in terms of displacement components u_r and u_θ , and the last one is the tensorial shear strain, $\epsilon_{r\theta}$. You can define $\gamma_{r\theta}$ as $2\epsilon_{r\theta}$, which is known as the engineering shear strain in polar coordinates. So, these are the three equations, known as the strain-displacement equations in polar coordinates, which differ from those in rectangular Cartesian coordinates. Similarly, you can convert the strain compatibility equation from rectangular Cartesian coordinates to polar coordinates.

For a 2D problem, we have only one strain-displacement compatibility equation. And that in polar coordinates would be something like this: $\frac{\partial}{\partial r} \left(2r \frac{\partial \epsilon_{r\theta}}{\partial \theta} - r^2 \frac{\partial \epsilon_{\theta\theta}}{\partial r} \right) + r \frac{\partial \epsilon_{rr}}{\partial r} - \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} = 0$. This is the only compatibility equation that any 2D polar coordinate problem should satisfy.

So, with this, we have converted all the field equations from the rectangular Cartesian system to the polar coordinate (r, θ) frame for the 2D elasticity problem.

Summary

- Field Equations in Polar Coordinates
- 2D Formulation in Polar Coordinates
- Stress, Strain, and Displacement Components in Polar Coordinates



So, in this lecture, we have talked about the field equations of elasticity in polar coordinates, then formulated the $2D$ problem in polar coordinates, either of plane stress or plane strain type, and expressed the strain, stress, and displacement components in polar coordinates as functions of r and θ . Thank you.