APPLIED ELASTICITY

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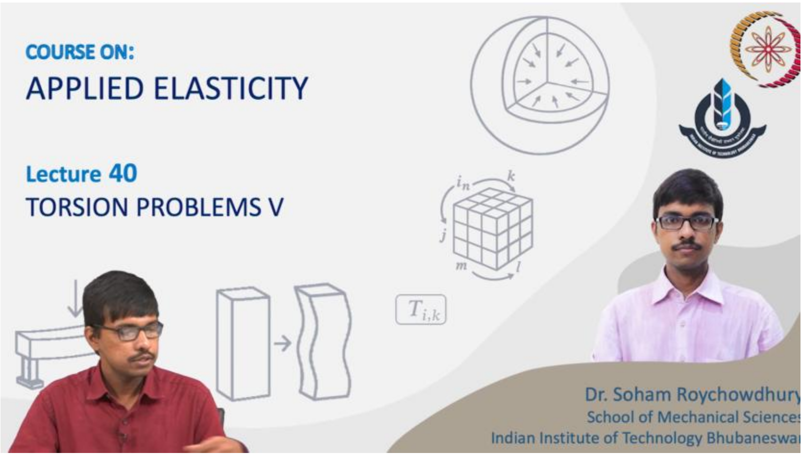
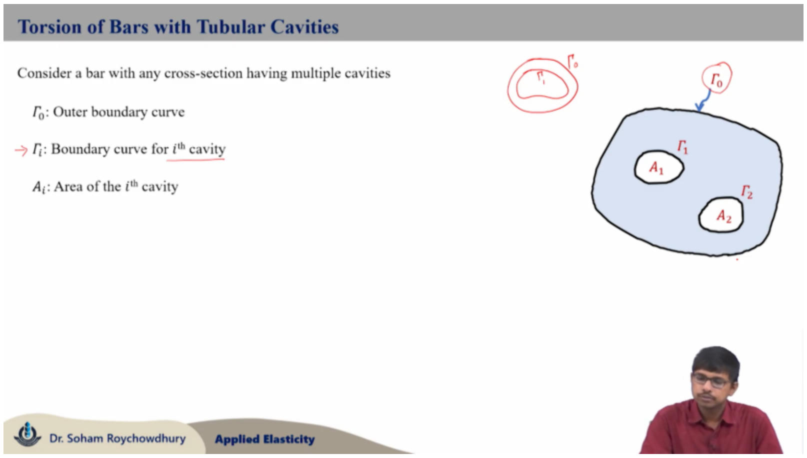
Week 8

Lecture 40: Torsion Problems V



Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the torsion problems. In the previous lecture, we were discussing about the torsion problems of various non-circular cross-sectional bars with elliptic cross-section, then triangular and then rectangular cross-section with the help of the St. Venant's theory of torsion using the Prandtl stress function approach.

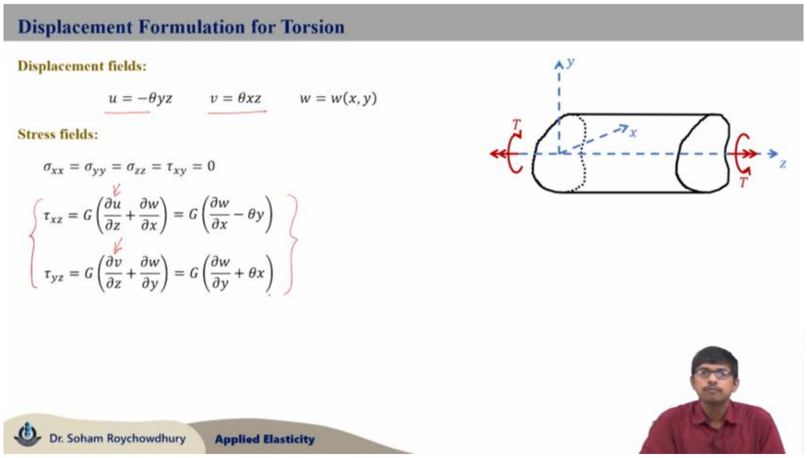


In this particular lecture, we are going to talk about the torsion of the hollow tubes, *i.e.*, torsion of the prismatic bars with tubular cavities. There may be a single cavity, there may be multiple cavities. If you are having a single cavity, then that is simply a hollow tube with constant or variable thickness. Instead of a single cavity, we may have multiple tube like cavities present within the cross section. It is shown in this particular figure.

Here, this is the cross section which we are considering, which is defined by this outer boundary curve Γ_0. This refers to the equation of the outer boundary of the bar and in between we are having two cavities here, the boundary curves of these two cavities are defined as Γ_1 and Γ_2. Γ_i is the boundary curve equation for the ith cavity.

There may be a single cavity. If you have a single cavity, then let us say this is outer boundary, this is inner boundary, referred by Γ_0 or Γ_1. This is simply a hollow tube.

Now, instead of one, we may have two cavities as shown in this figure, or you may have three or more. Depending on the number of cavities, the i will vary from 1 to some finite number n, where n is the number of tubular cavities present within the tube prismatic bar, and this $i = 0$ refers to the outer boundary. Γ_0 refers to the outer boundary curve. The area enclosed within the ith cavity is defined by A_i. For the first cavity, the enclosed area is A_1; for the second cavity, the enclosed area equals A_2, and so on.



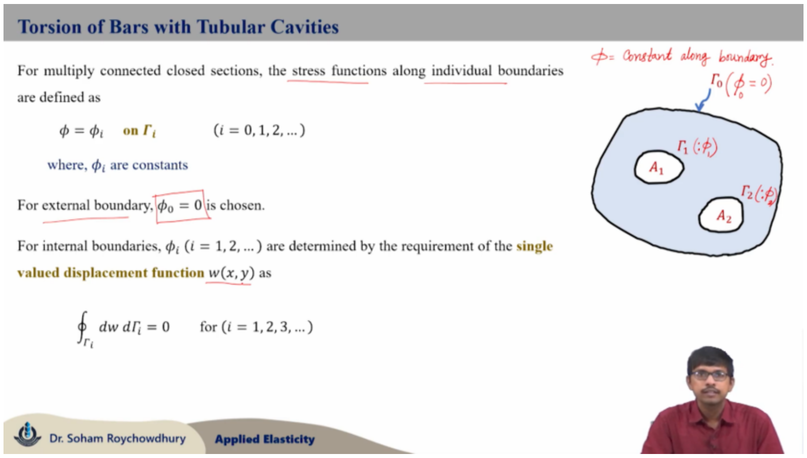
Moving forward to the displacement formulation of the torsion, which we had discussed. I will just have a quick recap, and then, using this formulation, we will try to solve the torsion problem of the bars with tubular cavities.

For the displacement formulation, we started with the assumption of the displacement fields: u, v, and w, as $u = -\theta yz$, $v = \theta xz$, and $w = w(x, y)$, some unknown function of x and y. The out-of-plane displacement for the non-circular shafts, in general, is taken to be non-zero, but it is a function of in-plane variables x and y only, independent of the axial variable z.

With these particular displacement fields, we can obtain the strain fields and then the stress fields as σ_{xx}, σ_{yy}, σ_{zz}, and τ_{xy}. These four stress components would be 0. Only two non-zero stress components will be there: τ_{xz} and τ_{yz}. τ_{xz} can be defined with the help of ε_{xz}, that is $2G\varepsilon_{xz}$, and τ_{yz} can be defined as $2G\varepsilon_{yz}$.

Substituting the expressions of ε_{xz} and ε_{yz}, these two strain components, in terms of displacement components, with the help of the strain-displacement relation, we can get these two equations for τ_{xz} and τ_{yz}.

Using $u = -\theta yz$ and $v = \theta xz$, these two terms can be simplified, and τ_{xz} is obtained as $G\left(\frac{\partial w}{\partial x} - \theta y\right)$, and τ_{yz} is $G\left(\frac{\partial w}{\partial y} + \theta x\right)$. We will be using these two expressions of nonzero stress components in terms of the out-of-plane displacement, w, for finding the solution of the torsion problem of a prismatic bar with tubular cavities.



Moving forward, once again going back to that figure of the cross-section of a prismatic bar. With two tubular cavities, it may have n number of cavities as well. Γ_0 refers to the outer boundary curve, Γ_1 and Γ_2 refer to the boundary curves of the inner cavities with corresponding areas being A_i.

For multiply-connected closed sections, the stress functions are required to be defined along the individual boundaries. Here, this particular cross-section is having multiple boundaries. It is a closed cross-section defined by multiple boundaries. For all the boundaries - outer boundary and each of the inner boundaries - we need to define the individual cross-sections for individual stress functions for individual boundaries. So, for the ith boundary Γ_i, we define $\phi = \phi_i$, where ϕ_i will be a constant.

If you recall the boundary condition of the St. Venant's theory of torsion, ϕ should be constant along the boundary. This was the condition, the boundary condition we had obtained: ϕ equals constant along the boundary. This is required to be satisfied for ensuring the traction-free boundary surfaces.

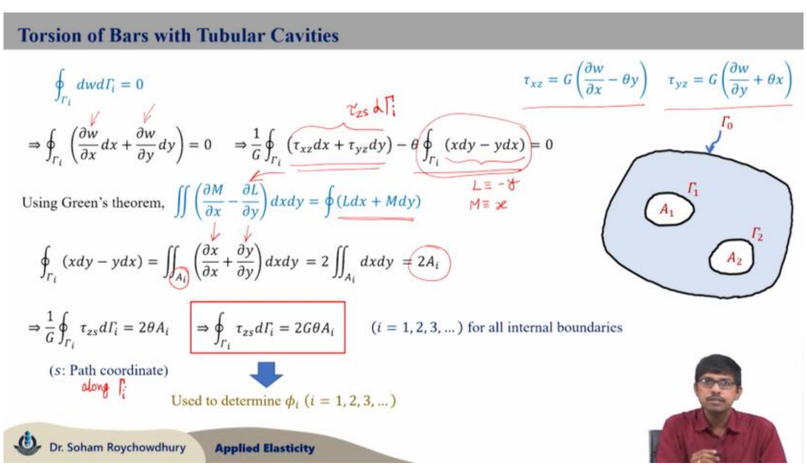
Now, to ensure $\phi = 0$ or ϕ equals constant along the boundary, if we are having a single boundary, that constant is taken to be 0. Here, we have multiple boundaries: one outer boundary and multiple inner boundaries. So, all the constants cannot be 0. We are only choosing the outer boundary constant to be 0. With respect to the outer boundary for Γ_0, the corresponding ϕ_0 is chosen to be 0. Whereas, for ϕ_1, ϕ_2, which corresponds to Γ_1

and Γ_2, inner boundaries, we will have some non-zero but constant values of the stress function.

For finding those ϕ_i, where i varies from 1, 2, 3, and so on ($i \neq 0$; the $i = 0$ case is the external boundary for which ϕ_0 is chosen to be 0)... For the inner boundaries, ϕ_i are constant but non-zero, and those are obtained by ensuring the requirement of a single-valued displacement function w.

So, $w(x, y)$ is our out-of-plane displacement function, and it must be single-valued over the boundary. That is the requirement; we cannot have different values of w resulting from different ϕ_i. We must have a single displacement function, and to ensure that, this condition must be satisfied for all the inner boundaries: $\oint_{\Gamma_i} dw d\Gamma_i$ should be equal to 0, for $i = 1, 2, 3, ...$ and so on.

$i = 0$ is not considered - this is not for the outer boundary. For inner boundaries, ϕ_1, ϕ_2, ϕ_3 , and so on can be obtained by using this requirement of the single-valued displacement function w over the inner boundary. The integral of dw over any inner boundary Γ_i is equal to 0, where i can take values of 1, 2, 3, and so on.



Now, we will try to simplify this equation. $\oint_{\Gamma_i} dw d\Gamma_i = 0$ can be written as this: $\oint_{\Gamma_i} \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy\right) = 0$. Any boundary curve $d\Gamma$ is a function of dx and dy, and w is also a function of x and y. Thus, $dwd\Gamma_i$ is written as $\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy\right)$. This is a closed curve integral, the integral over Γ_i.

Substituting this $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of the shear strain components τ_{xz} and τ_{yz}. If you recall the displacement formulation, we had obtained τ_{xz} as $G\left(\frac{\partial w}{\partial x} - \theta y\right)$ and τ_{yz} as $G\left(\frac{\partial w}{\partial y} + \theta x\right)$. From here, we can write $\frac{\partial w}{\partial x}$ as $\frac{\tau_{xz}}{G} + \theta y$ and $\frac{\partial w}{\partial y}$ as $\frac{\tau_{yz}}{G} - \theta x$.

Replacing these $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of τ_{xz} and τ_{yz}, we can rewrite this equation as the combination of two boundary integrals. The first term is $\frac{1}{G}\oint_{\Gamma_i}\left(\tau_{xz}dx + \tau_{yz}dy\right)$. The second term is $-\theta\oint_{\Gamma_i}(xdy - ydx)$. The total should go to 0, and both boundary integrals are over Γ_i, the ith inner boundary curve.

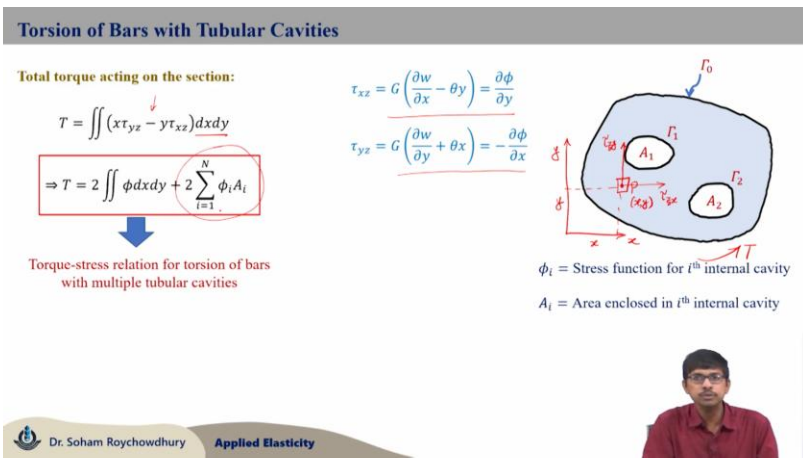
Using Green's theorem, which converts an area integral to a closed curve boundary integral, $\iint\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dxdy = \oint(Ldx + Mdy)$. Here, we will try to simplify the second term using Green's theorem.

If you compare $(Ldx + Mdy)$ with $(xdy - ydx)$, what would be our L and what would be our M? Here, the coefficient of dy is M, which is x, and the coefficient of dx is L, which is $-y$. So, L is equivalent to $-y$ and M is equivalent to x for this problem. Substituting them back here on the left-hand side, $\frac{\partial M}{\partial x}$ will be $\frac{\partial x}{\partial x}$, and $-\frac{\partial L}{\partial y}$ will be $\frac{\partial}{\partial y}(-y)$. The minus signs will cancel each other. It will be $\frac{\partial y}{\partial y}$, and this integral is now over A_i, the area of the ith inner boundary. $\frac{\partial x}{\partial x}$ is 1, and $\frac{\partial y}{\partial y}$ is also 1. This is nothing but $2\iint_{A_i} dxdy$ which is nothing but total A_i, the area enclosed within the ith inner boundary.

So, this second term, this entire term, is simply becoming $2A_i$. Hence, the first term, this one divided by G, minus $2A_i\theta$ equals to 0, I have taken $2A_i\theta$ on the right hand side, and rewriting this equation, this particular term as $\tau_{zs}d\Gamma_i$, where s is the path coordinate.

$\left(\tau_{xz}dx + \tau_{yz}dy\right)$ can be written as $\tau_{sz}ds$, or ds is nothing but $d\Gamma_i$, where s is the path coordinate along Γ_i and then $\frac{1}{G}$ is also there. From this we can get this boundary curve integral $\oint_{\Gamma_i} \tau_{zs} d\Gamma_i = 2G\theta A_i$.

If you know Γ_i and A_i, with the help of this equation, we can obtain the corresponding ϕ_i. This equation is used to determine the stress function at any inner boundary defined by Γ_i, where i varies from 1, 2, 3 and so on. So, outer boundary stress function is 0, and for every inner boundary, we will be getting one equation like this, using which the corresponding stress function for that respective inner boundary can be obtained.



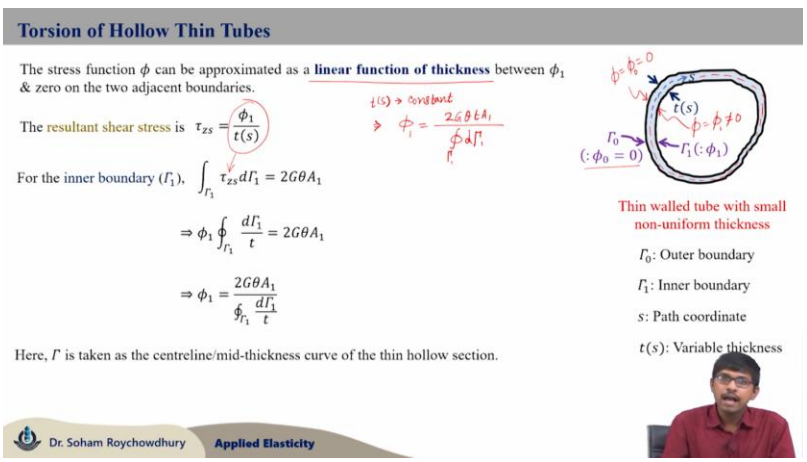
Moving forward, τ_{xz} and τ_{yz} can be written either in terms of w or in terms of ϕ. In terms of w, τ_{xz} is $G\left(\frac{\partial w}{\partial x} - \theta y\right)$, and in terms of ϕ, τ_{xz} is $\frac{\partial \phi}{\partial y}$. This is the relation between the two forms of τ_{xz}. Similarly, for τ_{yz}, we have $G\left(\frac{\partial w}{\partial y} + \theta x\right) = -\frac{\partial \phi}{\partial x}$.

Now, if you try to get the torque acting on the system. T is equal to $\iint\left(x\tau_{yz} - y\tau_{xz}\right)dxdy$. If you try to recall this, this equation was obtained in one of the previous lectures.

Consider any point P, and we are taking a small element of lengths dx, dy around P. This is τ_{xz}, and this is τ_{yz}. The coordinate of point P is given by x, y. This distance is x, and this distance is y. So, the moment created in the counterclockwise direction along the direction of P due to τ_{yz} is $x\tau_{yz}$. And the moment created by τ_{xz} is $y\tau_{xz}$, but that is in the opposite direction. That is why this minus sign is there. If you integrate this moment over the total area, then you will get T. Note that this area is the actual area excluding the hollow portions. This is the actual area of the tube cross-section with tubular cavities.

Here, if you replace this expression of τ_{xz}, τ_{yz} in terms of ϕ, and then integrate this, and expand, you will end up with $T = 2\iint \phi dxdy + 2\sum_{i=1}^{N} \phi_i A_i$, where i varies from 1 to N. This N is the number of tubular cavities.

For the bars without any cavity, this second term was not there. For the solid bar, T was $2\iint \phi dxdy$. Now, for the bars with tubular cavities, this extra term, which is $2\sum_{i=1}^{N} \phi_i A_i$, is being added. ϕ_i is the stress function for the ith internal cavity, A_i is the area enclosed within the ith internal cavity and N is the number of internal cavities. This gives us the torque-stress relation for the torsion of bars with multiple tubular cavities.



Moving to the torsion of hollow thin tubes. Hollow thin tubes mean the thickness of the tube is small, and we have a single cavity. We consider a bar like this: a thin-walled tube with small non-uniform thickness. t is the thickness, which is a function of the path coordinate s and is not uniform. As s changes, the thickness of the bar also changes.

We define the outer boundary with this Γ_0, which corresponds to the stress function $\phi_0 = 0$. As we had already seen in the previous discussion, for the outer boundary Γ_0, the stress function is chosen to be 0. Coming to the inner boundary, that is defined by Γ_1, and the corresponding stress function is ϕ_1, which must be non-zero; this cannot be 0. s is chosen as the path coordinate; along this thin-walled tube, s is going.

As the tube is thin, this thickness is small, and we are assuming that this path coordinate s is parallel to both the outer boundary and the inner boundary at any particular s. And $t(s)$ is the variable thickness of the bar, which is a function of s. As you are going along the path coordinate; the thickness $t(s)$ is continuously changing.

The stress function ϕ can be approximated as a linear function of thickness. On the outer boundary here, $\phi = \phi_0$, which is 0. And on the inner boundary, $\phi = \phi_1$, which is non-zero. In between, within the body, we can assume that ϕ_1 is varying linearly along the thickness t_1, between ϕ_1 and ϕ_0 ,or ϕ_1 and 0, which are two adjacent boundaries.

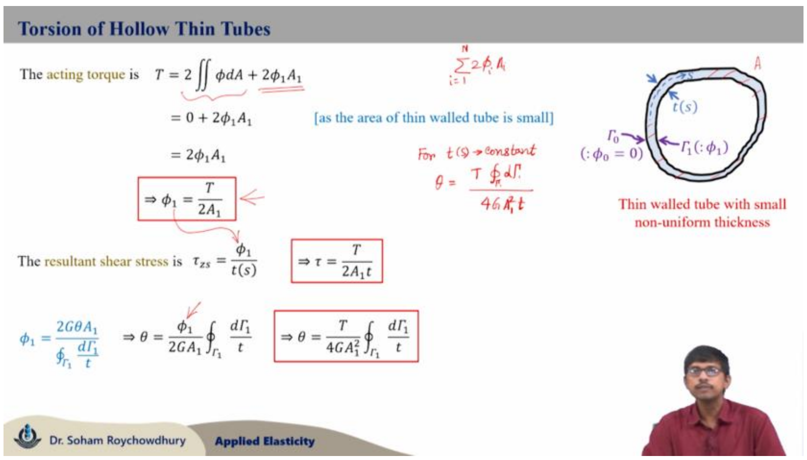
Now, ϕ being a linear function of thickness, this condition is valid only for the thin-walled tube when $t(s)$ is small. If $t(s)$ is large, then this particular assumption is not valid. So, this theory is applicable only for the torsion of thin-walled hollow tubes with a single cavity.

With this assumption of ϕ being a linear function of thickness, we can write τ_{zs} as $\frac{\phi_1}{t(s)}$, as ϕ is directly proportional to $t(s)$. ϕ_1 is a constant, that is the stress function for the inner boundary, and that is equal to $t(s)\tau_{zs}$, where τ_{zs} is the shear stress along the s direction, along the direction of the path coordinate.

Moving forward, for the inner boundary, using the single-value displacement function condition to ensure the single-value w on the inner boundary, we must have $\oint_{\Gamma_1} \tau_{zs} d\Gamma_1 = 2G\theta A_1$. Substituting τ_{zs} as $\frac{\phi_1}{t(s)}$; this form of τ_{zs} is substituted here, and ϕ_1 is a constant that is not changing over Γ_1. Thus, I am taking ϕ_1 out of the integral, but t is a function of s; thus, this is not a constant and cannot be taken out of the integral. So, $\phi_1 \oint_{\Gamma_1} \frac{d\Gamma_1}{t} = 2G\theta A_1$. And from this, we can write ϕ_1 as $2G\theta A_1$ divided by the boundary curve integral of $\frac{d\Gamma_1}{t}$.

Using this equation, we can choose the stress function ϕ_1, where t is within the integral because this is a variable thickness problem. If t is a constant, for such cases, we can take t out of the integral, and the bottom one will simply be the integral of $d\Gamma_1$. So, for constant thickness or t being constant cases, ϕ_1 will be $\frac{2G\theta t A_1}{\oint_{\Gamma_1} d\Gamma_1}$. This is simply the periphery, the perimeter of the inner boundary. If it is a circle, this will just be $2\pi r_i$, where r_i is the radius of the inner circle, something like that.

Here, for the thin-walled tube, Γ is taken as the centerline or mid-thickness curve of the hollow section. If you are considering this centerline curve along the path coordinate s, that can be considered as the equation for Γ, as this is a hollow section and t is small, this particular approximation can be done for finding ϕ, the stress function.



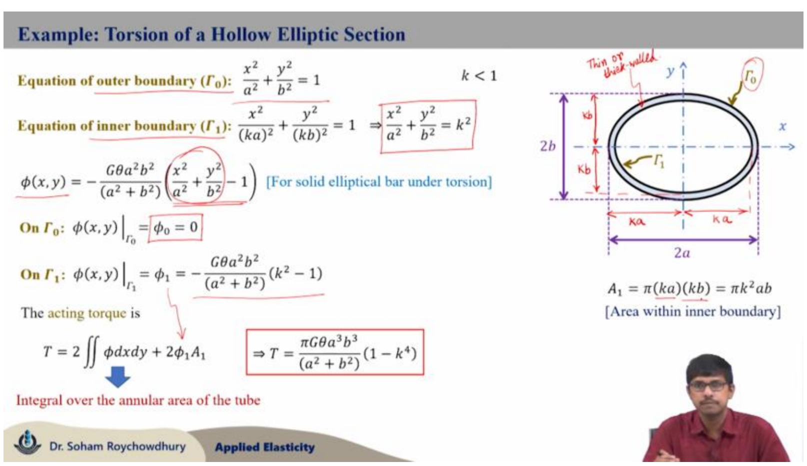
Moving forward, the torque can be obtained for this section by using this formula: $T = 2\iint \phi dxdy + 2\sum_{i=1}^{N} \phi_i A_i$, where i varies from 1 to N. Here, we have only one tubular cavity. So, $N = 1$, the single term $2\phi_1 A_1$ is present. ϕ_1 we had already obtained, and the first term is $2\iint \phi dxdy$. What is the area here? Area means the area of this thin hollow tube.

Now, this particular area is very small as the thickness is taken to be small or negligible. This first area integral will vanish for the hollow thin tubes. Thus, T will just be $2\phi_1 A_1$ for the thin wall tube. Area being small for this area integral, the first term of T is neglected, and T becomes $2\phi_1 A_1$. As ϕ_1 is known, from that, we can write down the expression of T or, alternately, we can write ϕ_1 as $\frac{T}{2A_1}$, which is the alternate form of ϕ_1. Using this also, we can obtain the stress function for the inner boundary.

Coming to the resultant shear stress, τ_{zs}. As we had used the linear distribution of the stress function, within the thin-walled hollow tube, between the outer boundary ϕ_0 and the inner boundary ϕ_1. So, $\tau_{zs} = \frac{\phi_1}{t(s)}$, and ϕ_1, I have written in terms of T as $\frac{T}{2A_1}$. Substituting this ϕ_1 here, the resultant shear stress τ_{zs} can be written as $\frac{T}{2A_1 t}$. This will be the resultant shear stress generated for the thin-walled hollow tube, which is torque divided by 2 times the inner area times the thickness t.

Using $\phi_1 = \frac{2G\theta A_1}{\oint_{\Gamma_1} \frac{d\Gamma_1}{t}}$, we can get the expression of θ, the angle of twist, as $\frac{\phi_1}{2GA_1} \oint_{\Gamma_1} \frac{d\Gamma_1}{t}$. This is the relation between θ and ϕ_1. Here, we can replace the obtained form of ϕ_1 writing this ϕ_1 in terms of T as $\frac{T}{2A_1}$. In this, the angle of twist for the torsion of hollow thin tubes can be obtained as $\frac{T}{4GA_1^2} \oint_{\Gamma_1} \frac{d\Gamma_1}{t}$, and this is valid for the variable thickness.

For thin-walled tubes with constant thickness, we will have $\theta = \frac{T}{4GA_1^2 t} \oint_{\Gamma_1} d\Gamma_1$. This will be the angle of twist.



Moving forward, we will solve one example problem, which is the torsion of a hollow elliptic section. We are considering a bar with an elliptical cross-section and a single tubular cavity. That tubular cavity is also elliptical, and the centers of both ellipses coincide with each other. So, they are concentric ellipses. $2a$ and $2b$ are taken as the major and minor axis lengths for the outer ellipse, and the inner ellipse is defined with a factor k, where $k < 1$.

If we are writing the equation of the outer boundary, this outer boundary is given by Γ_0. The equation of the outer ellipse is written as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The inner boundary, defined by Γ_1, has the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k^2$, where $k<1$.

What we are doing is assuming this length to be ka. Similarly, the length of the semi-major axis of the inner ellipse is taken to be kb. So, the total major axis will be $2ka$ for the inner ellipse, and the total minor axis will be $2kb$ for the inner ellipse. k is a factor less than 1, and note that we are not imposing the constraint of a thin wall. It may be a thin wall or a thick wall, either one is valid for this particular solution.

We have the expression of Γ_0: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the outer boundary. We also have the equation of the inner boundary Γ_1, which is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k^2$.

The stress function, ϕ, for the solid elliptical bar undergoing torsion - this problem was solved using the Prandtl stress function approach, and the corresponding stress function ϕ was obtained as: $-\frac{G\theta a^2b^2}{(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$.

Now, for both the outer boundary Γ_0 and the inner boundary Γ_1, we need to choose the corresponding ϕ_0 and ϕ_1. That choice will be motivated by the form of ϕ derived for the torsion of the prismatic bar with a solid elliptical cross-section. For the outer boundary Γ_0, we choose ϕ to be ϕ_0 as 0. So, simply this part will go to 0, as the equation of the outer boundary is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. ϕ will go to 0 on the outer boundary; thus, for the outer boundary, $\phi_0=0$.

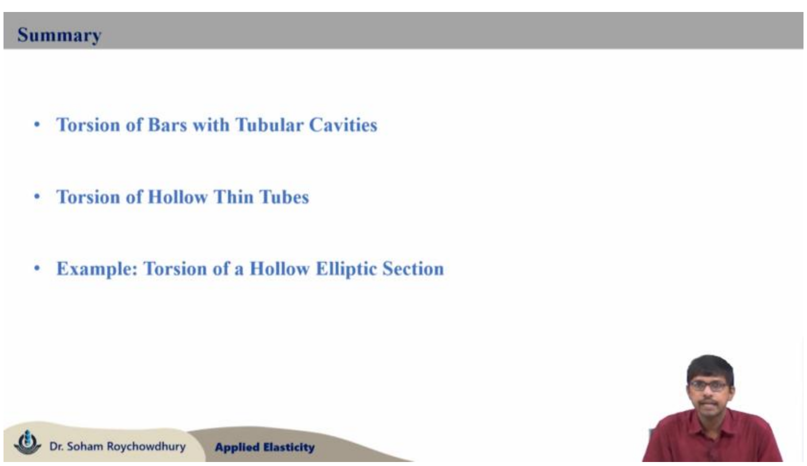
Coming to the inner boundary ϕ_1, we obtain it by substituting $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k^2$ as this is the condition for Γ_1. This term is replaced with k^2 when writing ϕ_1. Thus, ϕ_1, corresponding to Γ_1, would be - the outer constant remains - $-\frac{G\theta a^2b^2}{(a^2+b^2)}(k^2-1)$.

We got both, ϕ_0 as 0 and ϕ_1 as non-zero, which is explicitly obtained for this elliptic boundary, which is valid for both thin-walled hollow elliptic section as well as thick-walled hollow elliptic section. Now, we can obtain the torque acting on it by using this formula: $2\iint \phi dxdy + 2\phi_1 A_1$, where ϕ_1 is this. Note that this first area integral is integral over the actual area of the tube, and not over the total area of the solid tube; this is area of the annular elliptic tube, which is the hollow elliptic one, which may be having a thin-wall section or a thick-wall section.

In general, for thin-wall tubes, we neglect this particular first term stating that the thickness is small. Here, as we are solving a problem for both solid and hollow, the result should be valid for both, and thus, we cannot set the first term to 0. We need to put the expression of this ϕ, and then integrate it over this annular area which will give us the first term, and then for finding second term, ϕ_1 is already there, and A_1 is area within the inner boundary of the ellipse.

Area of any ellipse is π times semi-major axis length times semi-minor axis length. For the inner ellipse, semi-major axis is ka, and semi-minor axis is kb. So, area will be $\pi ka \times kb$, which is $\pi k^2 ab$. If you substitute that in the second term and integrate the first term for the annular area, the overall T would come out to be $\frac{\pi G\theta a^3 b^3}{(a^2+b^2)}(1-k^4)$, where k is a term less than 1 which defines the ratio of the semi-major axis of the inner boundary and semi-major axis of the outer boundary.

This solves the torsion problem of the hollow elliptic cross section which is not necessarily required to be a thin-walled elliptic section. It may be thin-walled or may be thick-walled tube as well.



In this lecture, we discussed about the solution strategy of torsion of bars with tubular cavity, which may have a single cavity or multiple cavities. Then, we extended that for the solution of the torsion of hollow thin tubes with variable thickness, and with the help of that theory, we solved the problem of torsion of a bar with hollow elliptic section.

Thank you.