APPLIED ELASTICITY

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WEEK: 01
Lecture- 04

Welcome back to the course of applied elasticity. In today's lecture we are going to discuss about the tensor calculus. In three previous lectures we were mostly talking about the tensor algebra where the indicial notations were first introduced and then followed by definition of second order tensor and different tensor algebraic manipulations. Now, we are going to discuss about the calculus related to the tensor quantities.

So, first we are going to start with the tensor valued function of a scalar and its derivative. So, let us say $\tilde{\tilde{T}}$ is a second order tensor valued function for a scalar small t. Then derivative of capital $\tilde{\tilde{T}}$ with respect to small t is defined to be another second order tensor which is given by this particular expression. $\frac{d}{dt}$ of this second order tensor valued function capital $\tilde{\tilde{T}}$ is equals to limit Δt tending to 0. $\tilde{\tilde{T}}$ evaluated at time t plus Δt minus capital $\tilde{\tilde{T}}$ evaluated at initial time t divided by Δt. So, this is similar to the formal definition of derivative of any function.

Similar to that or following that the definition of the derivative of a tensor valued function can be given. through this particular expression. $\frac{d}{dt}$ of second order tensor $\tilde{\tilde{T}}$ is equals to $\lim\limits_{\Delta t \to 0}$ difference between the value of $\tilde{\tilde{T}}\,(t + \Delta t)$ and at $\tilde{\tilde{T}}(t)$ divided by change in t that is Δt. Now, what are the properties related to this derivative operator for tensor? So, if you are taking derivative $\frac{d}{dt}\left(\tilde{\tilde{T}}\tilde{a}\right)$ where t is a second order tensor $\tilde{a}$ is a vector. So, for that if you are taking the derivative we will apply the chain rule of differentiation.

So, first derivative will be applied on t and keeping $\tilde{a}$ as it is. So, $\frac{d\tilde{\tilde{T}}}{dt}\tilde{a}$ and plus the second term in the second term we will apply the derivative on vector $\tilde{a}$. So, tensor capital $\tilde{\tilde{T}}\frac{d\tilde{a}}{dt}$ this is the first property. same property is valid if we are having product of 2 tensors $\tilde{\tilde{T}}$ $\tilde{\tilde{T}}$ and capital S. So, $\frac{d}{dt}\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)$ is equals to $\frac{d\tilde{\tilde{T}}}{dt}$ times $\tilde{\tilde{S}}$ plus capital $\tilde{\tilde{T}}$ times $\frac{d\tilde{\tilde{S}}}{dt}$.

So, we are taking chain wise differentiation of both the tensors capital $\tilde{\tilde{T}}$ and capital $\tilde{\tilde{S}}$ one after another. And coming to the third property, if the tensor $\tilde{\tilde{T}}$ is multiplied with a scalar valued function of time t, $\alpha(t)\tilde{\tilde{T}}$ and then we are taking its time derivative, then there will be two terms. First, the derivative of the scalar valued function $\frac{d\alpha(t)}{dt}$followed by tensor capital $\tilde{\tilde{T}}$ plus $\alpha(t)$ scalar function remaining as it is into $\frac{d\tilde{\tilde{T}}}{dt}$.

So, all these chain rule of differentiations are valid for the derivative of tensor valued function as well. Now, we are going to define the gradient of a scalar function which will be coming across this kind of quantities like this kind of calculus operations such as gradient, curl, divergence, stress, Laplacian in the elasticity problems very often. So, that is why we need to first introduce all these quantities and their formal definition. So, we are starting with phi

which is a scalar field or scalar valued function of $\tilde{x}$. $\tilde{x}$ vector is the position vector and this scalar valued function can be any scalar function may be density of the material or electric potential or any other field which is defined within the body. and that is function of location of the point or position vector of any point of the body. So, ϕ is a function of $\tilde{x}$ vector this is called a scalar valued function for the position vector $\tilde{x}$. Now, gradient of the scalar ϕ is defined to be a vector and that is denoted by this. So, this particular operator is called the gradient operator.

So, $\tilde{\nabla}\phi$ is a vector and that is defined through this particular equation. $\tilde{\nabla}\phi$ vector dotted with $d\tilde{x}$ vector, small change in the position vector is equals to $\phi(\tilde{x} + \mathrm{d}\tilde{x})$ minus ϕ evaluated at initial $\tilde{x}$. So, through this expression, we can define the gradient of any scalar function ϕ. So, for the rectangular Cartesian coordinate system with base vector $\tilde{e}_i$ we can expand $\phi(\tilde{x} + \mathrm{d}\tilde{x})$ this term in this particular form. So, $\phi(\tilde{x} + \mathrm{d}\tilde{x})$ is having 3 components $\phi(x_1 + dx_1,\ x_2 + dx_2, x_3 + dx_3)$ this is on the left hand side and this can be expanded in the series solution

$\phi(\tilde{x})$ that is $\phi(x_1, x_2, x_3)$ at initial $\tilde{x}$, the value of scalar function at initial location plus $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_1}dx_1$ plus $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_2}dx_2$plus $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_3}dx_3$, three first order terms which are contribution due to change in x_1 coordinate, change in x_2 coordinate, change in x_3 coordinate respectively. plus there will be second order term, third order term and so on involving dx_1 square, dx_2 square, dx_3 square and all. Now, considering dx_i to be small, the higher order terms are neglected and with that if we are taking this first term on the right hand side towards left. So, it will be

$\phi(\tilde{x} + d\tilde{x})$ minus $\phi(\tilde{x})$ equals to $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_1} dx_1$ plus $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_2} dx_2$ plus $\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_3} dx_3$ which using summation convention can be written as $\frac{\partial\phi(\tilde{x})}{\partial x_i} dx_i$ and by definition of the gradient, this right hand side is equals to $\tilde{\nabla}\phi . d\tilde{x}$. So, $(\tilde{\nabla}\phi)_i$ component into dx_i or $\tilde{\nabla}\phi$ vector dot $d\tilde{x}$ vector is equals to $\phi(\tilde{x} + \mathrm{d}\tilde{x})$ minus $\phi(\tilde{x})$. So, comparing this $\tilde{\nabla}\phi$ and $\frac{\partial\phi}{\partial x_i}$, we can write that $(\tilde{\nabla}\phi)_i$ is $\frac{\partial\phi}{\partial x_i}$ which is derivative of phi with respect to xi. So, here this comma i phi comma i refers to the first derivative of ϕ scalar valued function ϕ with respect to x_i and in total this gradient vector of ϕ can be written as $\phi_{,i}\tilde{e}_i$ where

i refers to derivative with respect to x_i and gradient operator is $\frac{\partial}{\partial x_i}\tilde{e}_i$. This is one operator which can be operated, which can act on any scalar valued function i. Now, similar to the gradient of a scalar, we can also find the gradient of a vector function. Now, how is that defined? Gradient of a scalar was defined to be a vector.

Gradient of a vector is defined to be a second order tensor. So, as we are applying gradient operator on any quantity, its order is getting increased by 1. Gradient of scalar was a vector, gradient of a vector is a second order tensor. Now, if we are considering any vector field $\tilde{V}$, which is function of position vector $\tilde{x}$, gradient grad $\tilde{V}$ is defined through this particular transformation grad $\tilde{V}$ acting over $d\tilde{x}$ grad

$\tilde{V}$ is a second order tensor that is acting over $d\tilde{x}$ is equals to that vector field $\tilde{V}$ evaluated at $\tilde{x}$ plus $d\tilde{x}$ minus vector field $\tilde{V}(\tilde{x})$. Now in the in the component form if you are writing the $\tilde{\nabla}\tilde{V}$. So, gradient of V i j is equals to by definition of tensor component this is equals to $\tilde{e}_i$ dot $\tilde{\nabla}\tilde{V}$ acting over $\tilde{e}_j$. Now, $\tilde{V}$ vector is nothing but $V_1\tilde{e}_1$ plus $V_2\tilde{e}_2$ plus $V_3\tilde{e}_3$. V_1, V_2, V_3 are the components of $\tilde{V}$ vector along $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ base vectors or unit vectors and grad operator is defined as $\frac{\partial}{\partial x_j}\tilde{e}_j$.

So, using this definition, this particular expression within the bracket this can be written as $\frac{\partial\tilde{V}}{\partial x_j}$ and in total this is $\tilde{e}_i$ dot $\frac{\partial\tilde{V}}{\partial x_j}$. if you are taking the dot product of $\tilde{e}_i$ unit vector with another vector $\frac{\partial\tilde{V}}{\partial x_j}$ only the ith component will be remaining. So, thus this would become $\frac{\partial\tilde{V}}{\partial x_j}$ that is basically the derivative of the ith component of $\tilde{V}$ with respect to jth component of the position vector this is written as $V_{i,j}$. j means the derivative is with respect to jth component of the position vector and the subscript before comma that is i refers to ith component of the vector field $\tilde{V}$.

So, thus the ij component of second order tensor gradient of $\tilde{V}$ is equals to $V_{i,j}$ this is valid for gradient of any vector field. Now, after gradient we are going to the trace operator. Trace of any second order tensor. that we are going to define now second order tensors can be written in terms of its component with the help of dyadic product thus we will first define the trace of dyadic product of two vectors a and b

so trace of dyadic product $\tilde{a}\otimes\tilde{b}$ is defined to be a scalar and that is nothing but dot product of $\tilde{a}$ and $\tilde{b}$ so $\tilde{a}.\tilde{b}$ $\mathrm{tr}(\tilde{a}\otimes\tilde{b})$ is $\tilde{a}.\tilde{b}$. Now, coming to any second order tensor $\tilde{\tilde{T}}$, trace of that tensor $\tilde{\tilde{T}}$ can be written in terms of its components as trace of T_{ij} $\tilde{e}_i\otimes\tilde{e}_j$. So, we already know that $\tilde{\tilde{T}}$ tensor in terms of its component can be written as $T_{ij}\tilde{e}_i\otimes\tilde{e}_j$. So, that is replaced here. Now, by definition of trace of two dyadic product, this trace of dyadic of $\tilde{e}_i$ $\tilde{e}_j$ is nothing but $\tilde{e}_i.\tilde{e}_j$ and T_{ij} being a scalar that would come out of the trace.

So, $\tilde{e}_i.\tilde{e}_j$ dot product of 2 unit vectors is basically δ δ_{ij}. So, T_{ij} into δ_{ij} and by using the properties of δ this is nothing but T_{ii}. So, trace of any second order tensor $\tilde{\tilde{T}}$ is equals to T_{ii} that is basically summation of all the diagonal terms of the second order tensor $\tilde{\tilde{T}}$. now coming to divergence of a vector field after gradient and trace now we are going to divergence of a vector field now divergence of any vector field $\tilde{V}$ is defined to be a scalar field so you note that when we are taking the gradient of a vector field

that is resulting a tensor order is increasing by 1 whereas when we are taking the divergence of a vector field that is resulting a scalar so order is reducing by 1 while taking the divergence operator so divergence operator I am writing it here with div, this is the divergence operator. Divergence of any vector field $\tilde{V}$ is defined as trace of the gradient of $\tilde{V}$. We had already defined trace and gradient. So, divergence of $\tilde{V}$ is equals to trace of gradient of $\tilde{V}$. Now, gradient of $\tilde{V}$, this quantity is a second order tensor. If we take

rectangular cartesian coordinate system and try to write the components of this particular ah right hand side trace of gradient of $\tilde{V}$ so trace of any tensor can be written as T_{ii} trace of any tensor $\mathrm{tr}\left(\tilde{\tilde{T}}\right)$ is equals to T_{ii} here the tensor is grad $\tilde{V}$ so trace of grad $\tilde{V}$ is ii component of grad $\tilde{V}$ and we also know that gradient of $\tilde{V}$ this was defined as $V_{i,j}$. So, basically this would result $V_{i,i}$ we are taking i and j to be same as both the indices are same here gradient of $\tilde{V}$ is $V_{i,j}$ trace of that would result $V_{i,i}$. So, divergence of $\tilde{V}$ divergence of any vector field $\tilde{V}$ is equals to $V_{i,i}$.

Now, coming to the divergence of a tensor field, this would be resulting a vector, divergence of a vector is a scalar, divergence of a tensor field $\tilde{\tilde{T}}$ is defined to be a vector and for any such vector $\tilde{a}$, this is defined through this transformation law. $\mathrm{div}\, \tilde{\tilde{T}}$ is a vector dotted with $\tilde{a}$, another vector, any arbitrary vector $\tilde{a}$ equals to $\mathrm{div}\left(\tilde{\tilde{T}}^{T}\right)\tilde{a}$ minus $\mathrm{tr}\left(\tilde{\tilde{T}}^{T}\tilde{\nabla}\tilde{a}\right)$. Through this particular transformation, the divergence of our transfer field is defined for any arbitrary vector $\tilde{a}$. Now, for simplifying this, we will be taking all these terms one by one and then simplify them using the indicial notations.

So let us consider left hand side $\left(\mathrm{div}\, \tilde{\tilde{T}}\right).\tilde{a}$. So this is dot product of two vectors $\left(\mathrm{div}\, \tilde{\tilde{T}}\right)$ is one vector $\tilde{a}$ is another vector. So that will be resulting a scalar. So as left hand side is a scalar both the terms on the right hand side must result to different scalars as well. So, let us first take the first term $\mathrm{div}\left(\tilde{\tilde{T}}^{T}\tilde{a}\right)$. Here, we are assuming a new quantity. So, this $\tilde{\tilde{T}}^{T}\tilde{a}$ is assumed to be a new vector $\tilde{b}$.

So, first term is basically becoming divergence of this vector $\tilde{b}$. Divergence of any vector $\tilde{b}$ is defined to be $b_{i,i}$ as discussed in just previous slide. So, $\mathrm{div}(\tilde{b})$ is $b_{i,i}$ where $\tilde{b}$ is $\tilde{\tilde{T}}^{T}\tilde{a}$. So, replacing that $\left(\tilde{\tilde{T}}^{T}\tilde{a}\right)_{i,i}$. Now, $\left(\tilde{\tilde{T}}^{T}\tilde{a}\right)_{i,i}$ is further simplified as $T^{T}_{ij}a_j$ and then T^{T}_{ij} is equals to T_{ji} transpose is removed and both the indices i and j are flipped.

So, thus this is equals to $T_{ji}a_j$ entire thing, i, i means the differentiation with respect to x_i. So, now if you are applying the chain rule of differentiation, this would result two terms. Once we are differentiating T_{ji}, keeping a_j constant which results the first term $T_{ji,i}a_j$ then we are differentiating second term a_j keeping T_{ji} constant which results the second term $T_{ji}a_{j,i}$ so this is the these two terms are coming from the first term of right hand side expression given now coming to the second term which is $\mathrm{tr}\left(\tilde{\tilde{T}}^{T}\tilde{\nabla}\tilde{a}\right)$

This one is defined to be a second order tensor. $\tilde{\tilde{T}}$ transpose is a second order tensor. $\tilde{\nabla}\tilde{a}$ vector is another second order tensor. Product of these two second order tensors is defined to be a new tensor $\tilde{\tilde{A}}$, capital $\tilde{\tilde{A}}$. Now, trace of any tensor capital $\tilde{\tilde{A}}$ is basically A_{ii}. So, this is equals to $\tilde{\tilde{T}}^{T}\tilde{\nabla}\tilde{a}$.

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Now, it is product of two tensor this is first tensor this is second tensor with the definition of product of tensors we can write $\left(\tilde{\tilde{T}}^T\right)_{ik}\left(\tilde{\nabla}\tilde{a}\right)$ by using the product of definition of product of two tensors. $\left(\tilde{\tilde{T}}^T\right)_{ik}$ can be written as T_{ki} removing transpose sign and flipping the indices whereas, gradient of $\left(\tilde{\nabla}\tilde{a}\right)$ by definition of gradient of a vector this is equals to $a_{k,i}$. Thus this particular term by replacing k with another dummy index j this is $T_{ji}a_{j,i}$. Now, we had simplified both the terms on the right hand side.

So, if you combine them in the given fashion. So, this term is the first term on the right hand side $\mathrm{div}\left(\tilde{\tilde{T}}^T\tilde{a}\right)$ minus $\mathrm{tr}(\tilde{\tilde{T}}^T\tilde{\nabla}\tilde{a})$ is this term. Now, if you put this, you can see $T_{ji}a_{j,i}$, this term will get cancelled and this will be having a single non-zero term $T_{ji,i}a_j$. Now, if you go for the left hand side, $\left(\mathrm{div}\,\tilde{\tilde{T}}\right).\tilde{a}$, we can write that as dot product of two vectors $\left(\mathrm{div}\,\tilde{\tilde{T}}\right)_j a_j$ and comparing it with the right hand side.

So, we are comparing these two. This is right hand side and this is left hand side. a_j is common. So, $\mathrm{div}\,\tilde{\tilde{T}}$ jth component should be same as $T_{ji,i}$. So, divergence of any tensor $\tilde{\tilde{T}}$ $i\ th$ component of that is defined to be $T_{im,m}$ where m is a dummy index and i is the free index.

Here, in the previous equation, we were having j as the free index, i as the dummy index. Here, I have changed the names to be i as the free index and m as the dummy index. So, ith component of divergence of $\tilde{\tilde{T}}$ is $T_{im,m}$. This is divergence of a tensor field which is basically resulting a vector. Now, coming to curl of a vector field. im,m where m is a dummy index and i is the free index.

Here in the previous equation we were having j as the free index i as the dummy index here I have changed the names to be i as the free index and m as the dummy index. So, ith component of $\mathrm{div}\,\tilde{\tilde{T}}$ is $T_{im,m}$ this is divergence of a tensor field which is basically resulting a vector. Now, coming to curl of a vector field, curl of a vector field is another vector field and curl of vector $\tilde{V}$ is defined to be twice of the dual vector of the anti-symmetric part of gradient of $\tilde{V}$ and denoted as this. So, gradient operator cross product $\tilde{V}$, this refers to the curl of vector $\tilde{V}$. It is denoted through this particular form and it is defined as twice of the dual vector $\tilde{t}^A$.

So, in one of the previous lecture, we had defined the dual vector which is existing for the case of anti-symmetric tensors. So, here $\tilde{t}^A$ is the dual vector for the anti-symmetric part of gradient of $\tilde{V}$. So, any vector can be written as summation of a symmetric tensor and any tensor can be written as summation of a symmetric tensor and an anti symmetric tensor. So, gradient of $\tilde{V}$ tensor can have can be written as summation of a symmetric tensor and an anti-symmetric tensor. Here that anti-symmetric part is chosen which is which can be written as half of $\tilde{\nabla}\tilde{V}$ minus $\left(\tilde{\nabla}\tilde{V}\right)^T$ transpose and dual vector of this anti-symmetric tensor is this $\tilde{t}^A$.

Twice of that dual vector defines the curl. So, in the rectangular Cartesian system $\text{Curl}\,\tilde{V}$ which is gradient operator cross product $\tilde{V}$ this is the method by which the way through which the curl is denoted. This is $2t_i^A$ component of the anti-symmetric dual vector times $\tilde{e}_i$, $\tilde{e}_i$ is the unit vector. Now, if you recall the definition of the dual vector for any anti-symmetric tensor capital $\tilde{\tilde{T}}$ we had defined t_1^A as minus T_{32} or T_{23} where for this particular case $\tilde{\tilde{T}}$ is equals to half of grad $\tilde{V}$ minus transpose of $\tilde{\nabla}\tilde{V}$.

So, half of grad $\tilde{V}$ minus transpose of grad $\tilde{V}$ is $\tilde{\tilde{T}}$ and minus T_{32} was defined as the first dual vector component t_1^A by definition of dual vector. Similarly, second dual vector component was defined as T_{13}, third one was defined as T_{21}. So, here we are putting those definition of dual vector which we had derived earlier and the first term t_1^A becomes half of $V_{3,2}$ minus $V_{2,3}$ which with the help of permutation symbol can be written as minus half of $e_{1jk}V_{j,k}$. Similarly, second and third component can also be written in terms of permutation symbol and in total if you combine all three t_i^A, the ith component of the dual vector of the anti-symmetric tensor is equals to $-\frac{1}{2}e_{ijk}V_{j,k}$.

Now, definition of curl is equals to double of the dual vector and with that this half will get cancelled with the 2 existing here and curl of $\left(\tilde{\nabla}\times\tilde{V}\right)_i$ component of curl of $\tilde{V}$ vector is $-e_{ijk}V_{j,k}$. Now, we know that e_{ijk} can be written as $-e_{ikj}$ by definition of the permutation symbol. So, minus sign can be dropped and we can write this as $e_{ijk}V_{k,j}$. Now, flipping the indices j and k, interchanging the names of j and k, this particular thing can be replaced by interchanging j with k and k with j. Thus, curl of $\tilde{V}$, ith component of that becomes $e_{ijk}V_{k,j}$.

And if we want to use the minus sign, then the first definition can be used, which is minus $e_{ijk}V_{j,k}$. So, with this, we can define the curl of a vector field and curl of vector field is another vector field. While using curl operator, there is no change in the order of the system. Now, coming to curl of a tensor field, this is defined to be another second order tensor so that for any constant vector $\tilde{a}$ note that constant vector $\tilde{a}$ not any variable vector for any constant vector $\tilde{a}$ the $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right) . \tilde{a} = \tilde{\nabla} \times \left(\tilde{a} . \tilde{\tilde{T}}\right)$.

So, through this particular transformation the curl of a tensor field is defined. Now, if you consider $\tilde{a} . \tilde{\tilde{T}}$, $\left(\tilde{a} . \tilde{\tilde{T}}\right)_j$ component is written as $a_k T_{kj}$ which is equals to $T_{lj} a_l$. Here, I am defining or changing the name of the dummy index k with l. Thus, $\left(\tilde{a} . \tilde{\tilde{T}}\right)_{j}$ will be $T_{lj} a_l$. Now, we are going to take the curl of this particular vector. So, this is a vector quantity.

If you are considering right hand side of the given identity, curl of $\tilde{a} . \tilde{\tilde{T}}$, this ith component with the help of the curl of a vector definition as discussed in the previous slide, curl of $\tilde{V}$ is minus of $e_{ijk} V_j$ comma k, where $\tilde{V}$ is is basically this quantity here $\tilde{a} . \tilde{\tilde{T}}$. So, $\left(\tilde{\nabla} \times \left(\tilde{a} . \tilde{\tilde{T}}\right)\right)_i$ component is minus $-e_{ijk} \left(T_{lj} a_l\right)_{,k}$. Now, this comma k means derivative with respect to k component of the position vector x_k. Now, we are going to expand this

and if you are taking derivative it would be minus $e_{ijk} a_l T_{lj,k}$. Note that here as $\tilde{a}$ is a constant vector $a_{l,k}$ while taking the time derivative derivative with respect to x_k this term $a_{l,k}$ should also be coming T_{lj} times this using the chain rule of differentiation, but that is taken to be 0 as a is a constant vector. Now, coming to the left hand side of the given equation $\tilde{\nabla} \times \tilde{\tilde{T}}$ dot $\tilde{a}$ this is equals to $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right)_{il} a_l$ $\tilde{\nabla} \times \tilde{\tilde{T}}$ is a tensor that is acting over $\tilde{a}$. Now, comparing both right hand side and left hand side expressions.

so this is the right hand side expression this is the left hand side expression if we compare both of them a_l is the common term which will get cancelled and then $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right)_{il}$ is equals to minus $-e_{ijk} T_{lj,k}$. Thus, the curl of a tensor field can be simplified and written in terms of positive permutation symbol. This minus sign is dropped e_{ijk} is changed to e_{ikj} with the help of property of the permutation symbol. So, $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right)_{il}$ is equals to $e_{ikj} T_{lj,k}$ Changing the names of the dummy indices k to m and j to n,

we can write this as $e_{imn} T_{ln,m}$. So, $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right)_{ij}$ component is $e_{imn} T_{jn,m}$. Through this, the curl of a tensor field is defined which is another second order tensor. So, in total in this particular lecture we had discussed about gradient of a scalar and vector function, trace of second order tensor and this divergence and curl operators which are acting over the vector and tensor field and their corresponding expressions. Thank you.