APPLIED ELASTICITY

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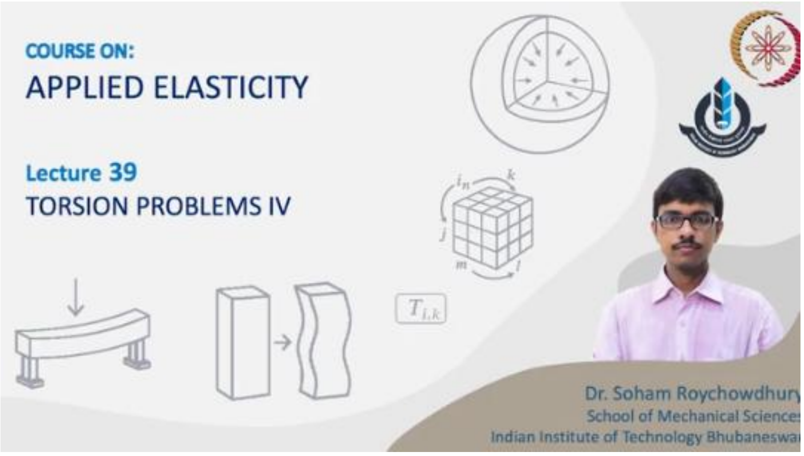
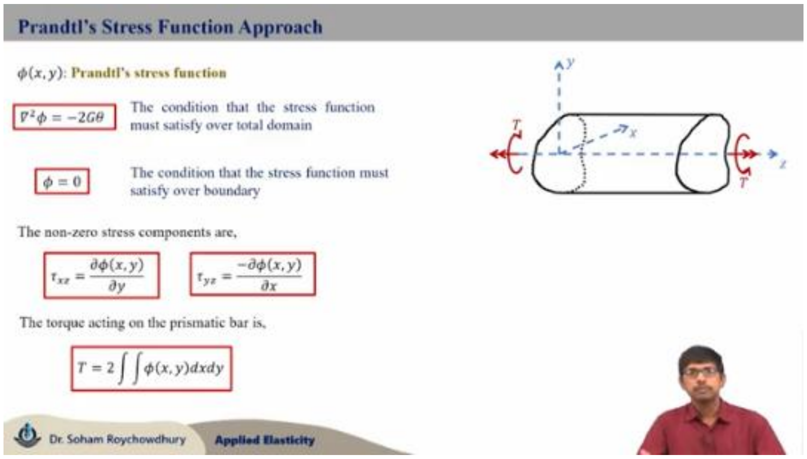
Week 8

Lecture 39: Torsion Problems IV



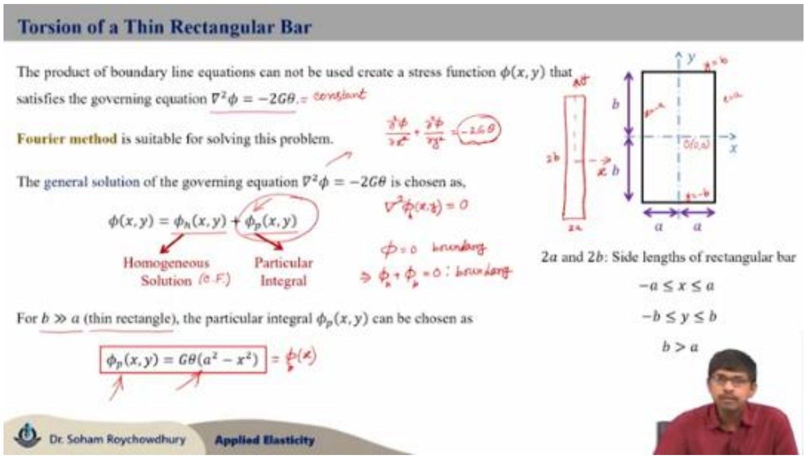
Welcome back to the course on Applied Elasticity. We are going to continue our discussion on the torsion problems of elasticity in this particular lecture. We started our discussion on the torsion problem in the previous lectures, where we considered two different cross-sectional bars. One is an equilateral triangular cross-section, and another is an elliptical cross-section, both solved with the help of the Prandtl stress function approach. In today's lecture, we are going to talk about the torsion of rectangular bars.



To have a quick recap on the Prandtl stress function approach, $\phi(x, y)$ is the Prandtl stress function, which is used to solve the torsion problems of prismatic bars with non-circular cross-sections. This ϕ must satisfy two conditions. The first condition is $\nabla^2\phi = -2G\theta$. This must be satisfied over the entire domain of the problem for all values of x and y, where θ is the angle of twist per unit length and G is the shear modulus of the material of the bar. The second condition that the Prandtl stress function must satisfy is $\phi = 0$ over the entire external boundary. We should have $\phi = 0$ over the boundary. The first condition is over the total domain; the second condition is only over the boundary.

By choosing a stress function that satisfies both conditions, with the help of that stress function, we can define the non-zero stress components: τ_{xz} as $\frac{\partial\phi}{\partial y}$ and τ_{yz} as $-\frac{\partial\phi}{\partial x}$. The total torque acting on the prismatic bar of non-circular cross-section can be related to the stress function as $T = 2\iint \phi dxdy$. The area integral of 2 times the stress function gives the total torque.

Using this particular approach, we are solving the torsion problems of different prismatic bars with different non-circular cross-sections. We had considered the elliptical cross-section bar and equilateral triangular cross-section bar in the last two lectures. In this lecture, the cross-section which we are going to consider for the prismatic bar undergoing torsion is a rectangle.



Let us consider a thin rectangular bar which is shown here. The center of the bar is chosen as the origin, which is $(0, 0)$. Along the x-axis, the side length of the rectangle is $2a$, and along the y-axis, the side length of the rectangle is $2b$. $2a$ and $2b$ are the side lengths of the rectangle where b is taken to be greater than a; x varies between $-a$ to $+a$, and y varies between $-b$ to $+b$.

If we are trying to find out the stress function for this particular problem, the approach used in the last two lectures for the elliptic cross-section and for the triangular cross-section is not going to work for the rectangular cross-section. What was the approach used for the elliptic cross-section? We chose the stress function as the equation of the ellipse multiplied by some unknown factor m, which we later obtained by using $\nabla^2\phi = -2G\theta$.

For the triangular cross-section, there are three different equations for the three sides of the triangle. The product of those three side equations of the triangle was taken as the choice of stress function with an unknown factor m, and like that, we proceeded and obtained the value of m. However, for this particular problem, that approach is not going to work.

Here, what are the equations of the different sides? For these two sides which are parallel to the x-axis, the equations are $x = a$ and $x = -a$. For the top and bottom, the equations are $y = b$ and $y = -b$. Using the previous approach, if we try to write that $(x + a)(x - a)(y + b)(y - b)$, that is the equation which is the product of the equations of all four different edges of this rectangle. With that, if we proceed as our choice of ϕ, this will never satisfy $\nabla^2\phi = -2G\theta$ equation.

Here, the product of boundary lines of four different sides cannot be used to define a stress function because that will be having fourth degree polynomial of x and y and $\nabla^2\phi$ should be $-2G\theta$ which is basically a constant. We want $\nabla^2\phi$ to be a constant. That is possible only if the highest order of the polynomial of x, y in ϕ is restricted to 3. Here, it will be 4, if you are taking ϕ as product of four different sides of this rectangle. With that, it is impossible to satisfy this governing equation of ϕ over the entire domain. We cannot ensure that $\nabla^2\phi$ is equal to a constant which is equal to $-2G\theta$, if ϕ is chosen as product of four sides. Thus, different solution approach is required to be adopted.

The solution method which we will be choosing here is the Fourier form of solution. Instead of ϕ being chosen as a polynomial form, which was motivated from the equations of the boundary lines for the previous problem of triangle, here, we will be choosing a Fourier form of solution defined by sine and cosine terms. Fourier method of solution is suitable for solving the torsion problem of the prismatic bars with rectangular cross section; polynomial form cannot be used. The general solution for this governing equation: $\nabla^2\phi = -2G\theta$, this is chosen in this particular form.

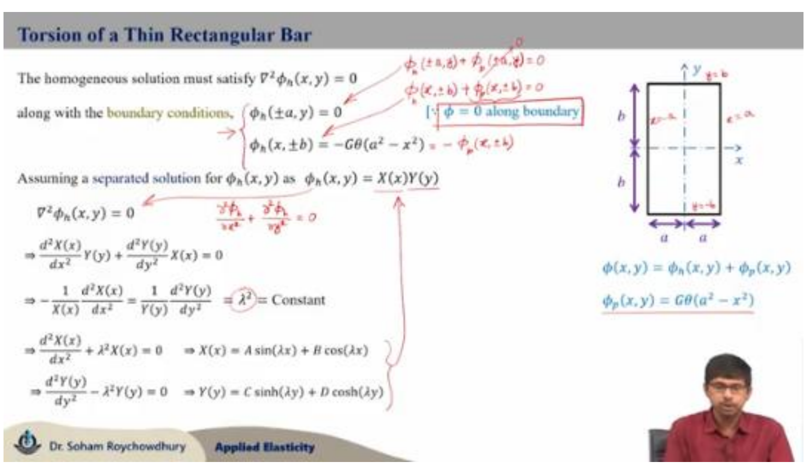
If you look at this, what is this equation? If you expand, you get $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$. This is a PDE (partial differential equation) of ϕ involving a non-zero constant on the right-hand side. Hence, it is a non-homogeneous differential equation, and the solution should have two components: one is the homogeneous solution, and the other is the particular integral.

The homogeneous solution is also called the complementary function. It is named as ϕ_h and the particular integral is named as ϕ_p. The effect of the right-hand side non-zero term, $-2G\theta$, would be present in the particular integral. The homogeneous solution is basically the solution of this equation. The Laplacian of the homogeneous solution, or $\nabla^2\phi_h(x, y) = 0$, which comes from the homogeneous solution part.

The particular solution must be chosen to satisfy the non-homogeneous equation $\nabla^2\phi = -2G\theta$, and it is chosen as follows. The particular integral of ϕ is chosen as $\phi_p = G\theta(a^2 - x^2)$, which satisfies the $\phi = 0$ constraint over the boundary. This choice is applicable only if b is much larger than a, that is, for a thin rectangular cross-section. For thin rectangular bars (where b is much larger than a), with dimensions $2a$ and $2b$, we can choose the particular integral as the product of the two sides $(x + a)(x - a)$. Here, ϕ is taken to be a function of x only, independent of y, because the two side lengths are very small compared to the other two side lengths. This approximation is valid only for a thin rectangular bar.

We are trying to choose the particular integral, motivated by the solution for a thin rectangular bar, using a polynomial form. Thus, one part of the solution, ϕ_p, is chosen as this. Our objective is to find out the homogeneous solution using the Fourier form. The overall solution is valid for the total rectangular body, not only for the thin rectangular part.

The particular integral we have chosen is valid only for the thin rectangular part, and now this ϕ must be 0 over the boundary. As ϕ must be 0 at the boundary, both ϕ_h and ϕ_p, they should be individually 0 along the boundary. These should be satisfied by both terms. This ϕ_p, particular term, along $x = \pm a$, is already 0. However, at $y = \pm b$, this is non-zero because it does not contain any function of y.



Moving forward, the homogeneous solution must satisfy that the Laplacian of homogeneous ϕ is equal to 0, as discussed in the previous slide. Along with the boundary condition that $\phi = 0$ along the boundary. $\phi = 0$ along the boundary can be written in this fashion.

Boundaries are defined by two curves. One is $x = \pm a$, that is $x = a$ and $x = -a$. The top face is $y = +b$, and the bottom is $y = -b$. The overall boundary would be 0: $\phi_h(\pm a, y) + \phi_p(\pm a, y)$ is 0. That is the boundary condition for the $x = a$ edge and $x = -a$ edge. The choice of ϕ_p is this, which is $G\theta(a^2 - x^2)$. So, for $x = \pm a$, $\phi_p(\pm a, y)$ is automatically 0. This is directly satisfied. Hence, the equation which we get for this $x = \pm a$ is that the boundary equation, $\phi_h(\pm a, y)$ should be 0.

Coming to the top and bottom face, defined by $y = \pm b$, we should have a similar boundary condition: $\phi_h(x, \pm b) + \phi_p(x, \pm b)$ should be equal to 0. Here, as ϕ_p is independent of y, we cannot have any value of this for $y = \pm b$. This is just a function of x. Thus, this would result in this particular boundary condition: $\phi_h(x, \pm b) = -\phi_p(x, \pm b)$, and ϕ_p for all values of y, whether it is $\pm b$ or any y, that is equal to $G\theta(a^2 - x^2)$. $-\phi_p$ means $-G\theta(a^2 - x^2)$.

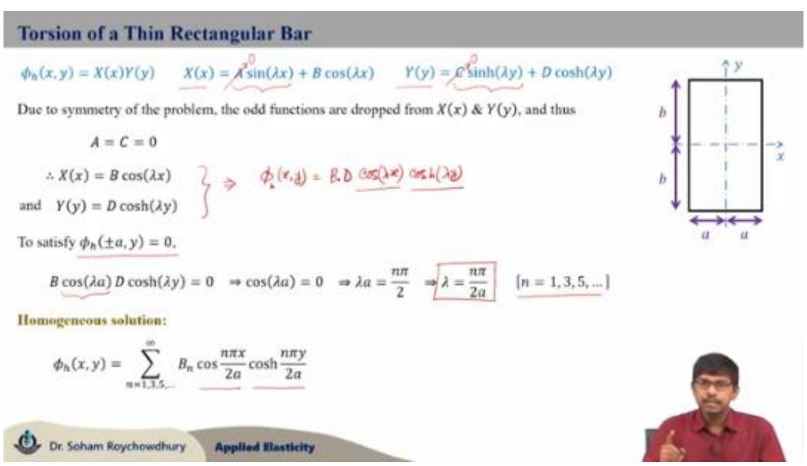
These are the two boundary conditions which are required to be satisfied to ensure $\phi = 0$ along all four edges of the rectangular bar. If you are able to satisfy these two boundary conditions on ϕ_h, that is, $\phi_h(\pm a, y)$ is 0, and $\phi_h(x, \pm b)$ is $-G\theta(a^2 - x^2)$. Then the total ϕ, that is the summation of ϕ_h and ϕ_p, would go to 0 along all four boundaries.

We have three particular equations to be satisfied by ϕ_h. One is $\nabla^2 \phi_h = 0$, and then these two boundary conditions over ϕ_h. Using this, we need to get the solution of ϕ_h, and for that, we are going to use the Fourier form of solution. Assuming the separated solution of $\phi_h(x, y)$ as XY following the Fourier form solution approach. We are assuming the stress function to be the product of two separated solutions of two functions: one is X, a function of x only, and another is Y, which is a function of y only. So, ϕ_h is written as the product of one function, X, and another function, Y.

And now, we will replace this ϕ_h back in the harmonic equation: $\nabla^2\phi_h = 0$. $\frac{\partial^2\phi_h}{\partial x^2} + \frac{\partial^2\phi_h}{\partial y^2} = 0$. Substituting this ϕ_h in this equation, for the first term, Y will be treated like a constant that will come out. So, $Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2}$ will be 0. And this can be rearranged in this form, and we can write that $-\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$.

Now, the left-hand side is a function of x, and the right-hand side is a function of y. For all values of x and y, these two can be equal only if both sides are equal to a constant, and let us choose that constant to be λ^2. One function of x can be equal to another function of y for all values of x and y only if both functions are equal to a constant, and we are choosing that constant as λ^2. Hence, the two equations in terms of X and Y can be obtained as $\frac{d^2X}{dx^2} + \lambda^2 X = 0$, and in Y, it is $\frac{d^2Y}{dy^2} - \lambda^2 Y = 0$.

From these two equations, we will be getting the general solution of X and Y as: $X = A\sin(\lambda x) + B\cos(\lambda x)$, and $Y = C\sinh(\lambda y) + D\cosh(\lambda y)$. So, we got these two separable solutions: X and Y. We can replace it back, and we will be getting the ϕ_h, the homogeneous solution of the stress function ϕ, which would be the product of these two: $[A\sin(\lambda x) + B\cos(\lambda x)][C\sinh(\lambda y) + D\cosh(\lambda y)]$.



Moving forward, looking at the symmetry of the problem. As the problem is symmetric about the x-axis, the problem is also symmetric about the y-axis. Both X and Y can only have even terms of x and y. We have one odd function, and one even function of x and y

in both X function and Y function. So, we are dropping the odd functions of x and y to ensure the symmetry. $\sin(\lambda x)$ is an odd function, and $\sinh(\lambda y)$ is another odd function. These two functions are required to be set to 0, which can be done only if we enforce the corresponding coefficients A and C to 0.

X becomes $B\cos(\lambda x)$, and Y becomes $D\cosh(\lambda y)$. With this only the symmetry of the obtained stress can be achieved for a rectangular cross-section, which is symmetric about both the x-axis and the y-axis. So, X and Y can only be even functions of x and y, or the functions are neglected by setting $A = 0$ and $C = 0$.

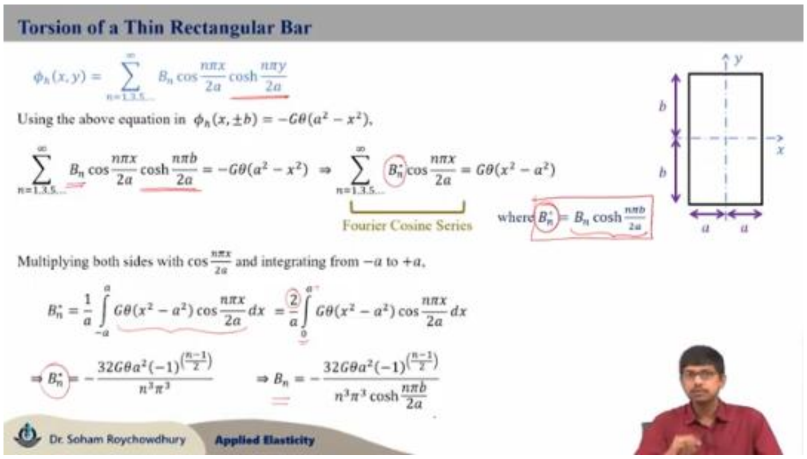
Now, we will be going for the boundary condition. We had two boundary conditions on ϕ_h. The first one was $\phi_h(\pm a, y)$ is equal to 0, and let us substitute that here. With this, our $\phi_h(x, y) = BD\cos(\lambda x)\cosh(\lambda y)$.

BD can be taken as a new constant; some other name may be given. So, at $x = \pm a$ for all values of y, this equation would be like this: $BD\cos(\lambda a)\cosh(\lambda y)$. For all values of y, this must be satisfied. We must have $\cos(\lambda a)$ to be 0, and thus, λa should be $\frac{n\pi}{2}$, where n are the odd integers: 1, 3, 5, 7, and so on. From this, we can get the value of λ as $\frac{n\pi}{2a}$. So, we got the λ as $\frac{n\pi}{2a}$, where a is half of the length of the edge parallel to the x-axis, and n can only be odd integers here.

Coming to the homogeneous solution, we can explicitly write the homogeneous solution as a series solution. Why series? Because this n is coming, *i.e.*, λ is dependent on n. We have to write the homogeneous solution ϕ_h as a series solution over n, n are only odd integer 1, 3, 5, 7, 9 and varying till ∞.

So, we are having two terms. One was $\cos(\lambda x)$, and λ is $\frac{n\pi}{2a}$. So, this term is $\cos\left(\frac{n\pi x}{2a}\right)$, and second term was $\cosh(\lambda y)$, and λ is $\frac{n\pi}{2a}$. So, second term is $\cosh\left(\frac{n\pi y}{2a}\right)$ and this constant is just taken to be a single constant B_n, which is product of previous B and previous D. And n, I am attaching because for different values of n, this constant would be different. This is a series solution.

This is our homogeneous solution. This satisfies the Laplacian. This also satisfies one of the boundary condition. But we are still left with another boundary condition. And using that boundary condition, we should be able to solve the value of B_n. Let us proceed for that.



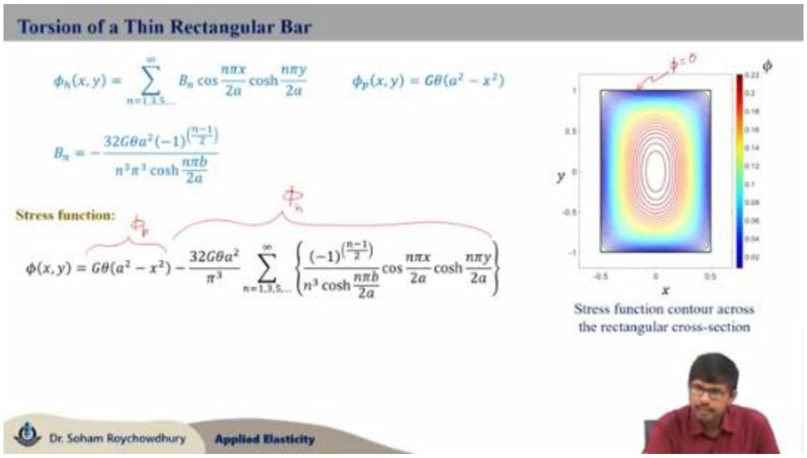
ϕ_h is obtained like this. Substituting this ϕ_h in the remaining boundary condition, which is $\phi_h(x, \pm b) = -G\theta(a^2 - x^2)$. If we substitute it here, it would be something like this. y is substituted with $\pm b$ on the left hand side. In this term, it is like cosh $\left(\frac{n\pi b}{2a}\right)$, and the rest of the terms are the same. The right-hand side is $-G\theta(a^2 - x^2)$. We can further simplify it and rewrite this B_n in terms of a new constant B_n^*. Note that B_n is a constant, and this cosh $\left(\frac{n\pi b}{2a}\right)$ is also constant, independent of x and y. The product of these two, I am writing as a new constant B_n^*.

The left-hand side becomes summation over odd numbers of n, B_n^* cos $\left(\frac{n\pi x}{2a}\right)$. The right-hand side is $G\theta(x^2 - a^2)$. The minus sign I have taken within the bracket, rewriting $(a^2 - x^2)$ as $(x^2 - a^2)$. This particular left-hand side term is basically a Fourier cosine series term, where B_n^* can be easily obtained by multiplying both sides with cos $\left(\frac{n\pi x}{2a}\right)$ and then integrating over $-a$ to $+a$.

Just by following the method of finding the constants or coefficients of Fourier cosine series, we are multiplying both sides with cos $\left(\frac{n\pi x}{2a}\right)$ and then integrating over $-a$ to $+a$. With that, the left-hand side will only be the B_n^* constant, the coefficient of the cosine term for the Fourier cosine series. The right-hand side would be like $\frac{1}{a}\int_{-a}^{+a} G\theta(x^2 - a^2)\cos\left(\frac{n\pi x}{2a}\right)dx$.

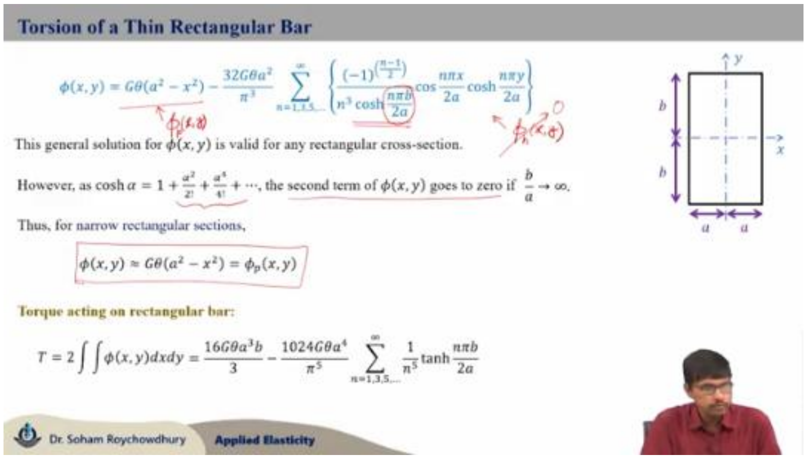
Now, this particular function, whatever is there within the integral, is an even function. This integral of the even function from $-a$ to $+a$ can be rewritten as the integral of the same function between 0 to a multiplied with 2. This 2 is coming, and the integral limit is changed to 0 to a instead of $-a$ to $+a$.

If you evaluate this integral, B_n^* can be obtained as $-\frac{32G\theta a^2(-1)^{\left(\frac{n-1}{2}\right)}}{n^3\pi^3}$, where n can only be odd numbers: 1, 3, 5, and so on. Using the relation between B_n^* and B_n, with the help of this B_n^*, we can also express B_n as this. This is the expression of B_n, which we need to substitute in the expression of ϕ_h to get the total homogeneous solution.



Now, we have the overall homogeneous solution. We had already chosen one particular integral. Substituting all of them, the overall stress function can be obtained like this. This is the summation of the particular integral. This is ϕ_p, and this is ϕ_h, where n varies between 1 to ∞ and can only take odd values.

If you plot this stress function contour for different values of x and y, it will look like this. On the outer periphery at the external edge, $\phi = 0$. As you move towards the inner side, the value of ϕ continuously increases, similar to the previous stress function contours of elliptical or triangular cross-sectional bars.



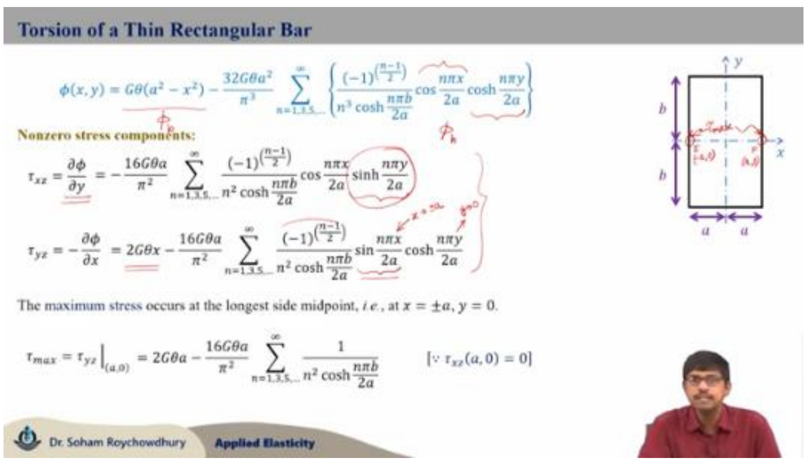
Moving forward, this general solution of ϕ is valid for any rectangular cross-section, not only for thin rectangular cross-sections but for any rectangular cross-section. Only the particular integral part was chosen for a thin rectangular cross-section, but for finding the homogeneous solution, we had not imposed any such condition. Hence, this general solution is valid for any rectangular cross-section for all values of b and a, but b must be greater than a.

Then, let us try to see how it can be reduced to a thin rectangular bar with b much greater than a. The cosine hyperbolic term, which is in the denominator of ϕ_h, is the second term in ϕ_h; the first term is ϕ_p. This cosh term - let us say $\cosh\alpha$ - can be written as $1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots$, and so on, for the higher-order terms.

Here in this term, α equals $\frac{n\pi b}{2a}$. As b becomes very large compared to a, for a very thin rectangular bar, $\frac{b}{a}$ theoretically tends to ∞ or practically goes to a very large value. α being very large, all these terms will shoot up, and hence, the denominator term in ϕ_h will be very large, resulting in the overall ϕ_h value going to 0. The second term of ϕ, that is ϕ_h, will go to 0 if you are going for a thin rectangular cross section, a very narrow rectangular cross section with b much greater than a. The second term of ϕ, that is ϕ_h, will go to 0, and approximately ϕ will be equal to only the first term, ϕ_p, which is $G\theta(a^2 - x^2)$.

Note that we started with this. The solution of the particular integral was chosen as $G\theta(a^2 - x^2)$, which is valid for the rectangular cross section. Then we obtained a general solution, which is valid for all rectangles. Then, once again, if you impose b much greater than a, we are going back to our initial assumption of ϕ_p, and thus, the solution method is validated.

Now, finding the torque acting on the rectangular bar, which is $2\iint \phi dxdy$. If you substitute this expression of ϕ here and integrate it, this would be the expression for T, the torque, which relates T and θ.



Moving to the non-zero stress components. τ_{xz} and τ_{yz} can be obtained as $\frac{\partial \phi}{\partial y}$ and $-\frac{\partial \phi}{\partial x}$, respectively. By substituting the expression of ϕ, τ_{xz} will be obtained as this, and τ_{yz} will be obtained as this.

So, τ_{xz} involves $\frac{\partial \phi}{\partial y}$. If you look at the first term of ϕ, it is independent of y, so there is no contribution from the ϕ_p term. This is the ϕ_p term in τ_{xz}. In ϕ_h, we have only one term as a function of y, which is the cosine hyperbolic term. If you take the partial derivative $\frac{\partial \phi}{\partial y}$, that cosine hyperbolic term changes to a sine hyperbolic term. Similarly, for τ_{yz}, $-\frac{\partial \phi}{\partial x}$ is taken, where the ϕ_p term gives some contribution because ϕ_p was a function of x.

In ϕ_h, the function of x is $\cos\left(\frac{n\pi x}{2a}\right)$, which results in $-\frac{\partial}{\partial x}$ as $\sin\left(\frac{n\pi x}{2a}\right)$. So, this cosine term changes to a sine term in the τ_{yz} expression. The resultant τ can be obtained by taking the resultant of both these non-zero shear stresses, τ_{xz} and τ_{yz}. If you look at the maximum shear stress points, the maximum shear stress occurs at the midpoint of the longer sides. At these two points, let us say points E and F, we will get the maximum shear stress.

We can verify that and obtain it as follows: at $x = \pm a$, $y = 0$, these points are given as $(a, 0)$ for F and $(-a, 0)$ for E. If you substitute $y = 0$, this particular term will vanish. So, τ_{xz} for this $x = \pm a$, $y = 0$ is 0. Overall the resultant τ will only be τ_{yz} at $(\pm a, 0)$. If you substitute that here, this x is replaced with $\pm a$ and this y is replaced with 0. This will be $\cosh 0$ and this will be $\sin\left(\pm\frac{n\pi}{2}\right)$, which is once again $(-1)^{\left(\frac{n-1}{2}\right)}$. That -1 and this -1 term will be coupled, resulting in 1.

Hence, the τ_{max} will be given by this equation: $2G\theta a - \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^2\cosh\left(\frac{n\pi b}{2a}\right)}$. This is the maximum stress generated in the rectangular bar, which occurs at the midpoints of the two longer sides, that is at points E and F.

In this lecture, we discussed the torsion of the rectangular bar. We obtained the expression of the stress function, then the maximum shear stress, and the relation between the applied torque and the resulting angle of twist per unit length.

Thank you.