APPLIED ELASTICITY

Dr. Soham Roychowdhury

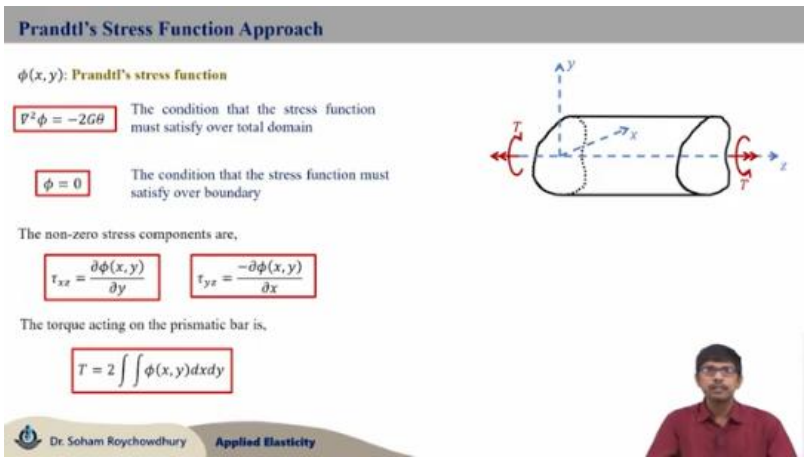
School of Mechanical Sciences

IIT Bhubaneswar

Week 8

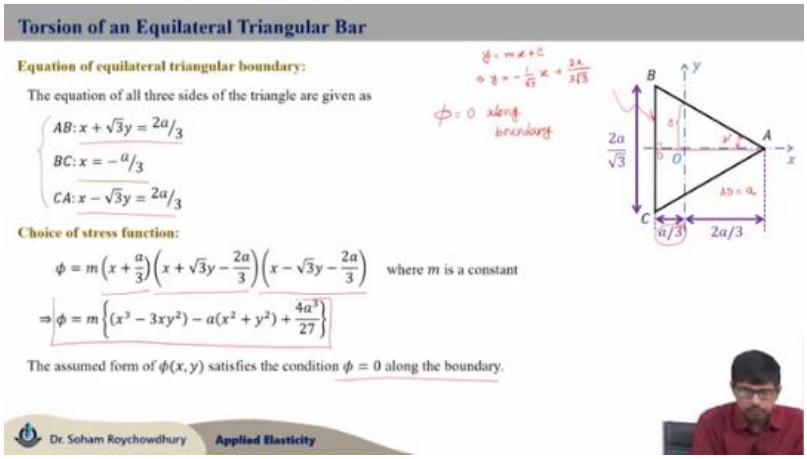
Lecture 38: Torsion Problems III

Welcome back to the course on Applied Elasticity. We are going to continue our discussion on torsion problems, which we started in the previous lectures. In today's lecture, we are going to talk about the torsion of a bar with an equilateral triangular cross-section.



Before that, let's have a quick recap on the theory of the Prandtl stress function approach, which is used for solving torsion problems of non-circular cross-sections. First, we need to define a function called the Prandtl stress function. This function must satisfy two conditions. The first condition is that the Laplacian of this Prandtl stress function, ϕ, should be equal to $-2G\theta$, and this condition should be satisfied over the entire domain. For all values of x and y, the condition $\nabla^2\phi = -2G\theta$ should be satisfied. The next condition is that $\phi = 0$ over the outer boundary. This condition should be satisfied over the outer boundary.

Based on these two conditions, we need to choose a proper ϕ depending on the different types of cross-sections. Then, the non-zero stress components, τ_{xz} and τ_{yz}, can be obtained in terms of that stress function as $\tau_{xz} = \frac{\partial\phi}{\partial y}$ and $\tau_{yz} = -\frac{\partial\phi}{\partial x}$. The applied torque, T, can be expressed in terms of the Prandtl stress function ϕ as $2\iint \phi dxdy = T$. Using this stress function approach - the Prandtl stress function approach - we are going to solve the torsion problem of a prismatic bar with an equilateral triangular cross-section.



Let us consider this particular cross-section of the bar. We are considering a prismatic bar subjected to a torque T. The cross-section of the bar is uniform along its length and is given by this particular equilateral triangular cross-section, as shown in this figure. Points A, B, and C are the three corners of the triangle, and the side length is equal to $\frac{2a}{\sqrt{3}}$.

For this particular triangle, as you can see from point B to C, the side length is equal to $\frac{2a}{\sqrt{3}}$ and length of the median means from one of the tip if you drop a perpendicular to the opposite tip, let us say length of this A to D, is equal to a. So, we are choosing the length of the median to be a, and then, the length of the side of this equilateral triangle will be $\frac{2a}{\sqrt{3}}$. All these internal angles between these three A, B, C points would be $60°$ for the equilateral triangle.

Now, we will try to choose a proper stress function for predicting the torsion of this triangular bar. The equation of equilateral triangular boundary that we are going to consider first and with respect to this equation of the triangular boundary, we will be choosing our stress function in such a way that stress function vanishes that goes to 0, $\phi = 0$ along the boundary.

Here, boundary is defined by three straight lines, one is A to B, then B to C and then A to C. Out of these three straight lines, AB and AC are two lines which are inclined to x and y axes. However, B to C is parallel to y-axis. This BC, this side boundary is having a constant value of x coordinate as $-\frac{a}{3}$ and y value is varying continuously from $-\infty$ to $+\infty$.

We are going to write the equation of these three sides of the triangle as: AB expression is $x + \sqrt{3}y = \frac{2a}{3}$, BC line equation is $x = -\frac{a}{3}$, and AC line expression is $x - \sqrt{3}y = \frac{2a}{3}$. So, you can take all of these three lines separately using the simple geometry you can obtain the equation for all three of them.

It is like straight line is y equals to some $mx + c$. For A to B, that particular one, this angle is equal to half of $60°$, or $30°$. Slope m is equal to $\tan 30°$ with a negative sign because this AB line is having negative slope, which will be $-\frac{1}{\sqrt{3}}$ and you can also obtain this intersection from geometry which will give you the value of c for equation AB. Putting those values back the expression of AB you can obtain as $y = -\frac{1}{\sqrt{3}}x$ and then it

will be I think $+\frac{2a}{\sqrt{3}}$. I have taken $\sqrt{3}$ on the left hand side and then written this equation as $x + \sqrt{3}y = \frac{2a}{3}$.

Similarly, equation of CA can be obtained as $x - \sqrt{3}y = \frac{2a}{3}$, and equation of this BC which is parallel to y-axis at a constant x magnitude of $-\frac{a}{3}$, the equation is given as $x = -\frac{a}{3}$. These are the three equations of three sides of the triangle and based on these, we are going to choose our stress function.

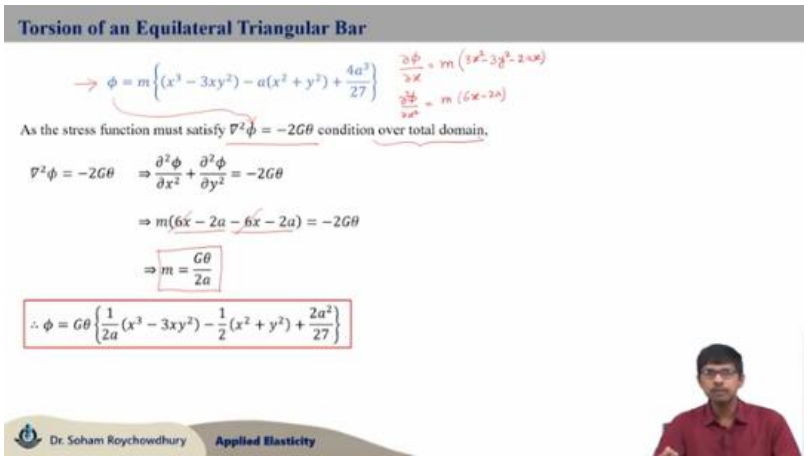
The condition that the stress function should satisfy is $\phi = 0$ along the boundary. Here, boundary is a discrete boundary, means boundary is mathematically described by three discrete equations of AB side then BC side and then CA side. ϕ must be 0 along all three sides.

So, the most simple or the suitable choice of ϕ is product of these three equations. If we take ϕ to be product of these three equations, then along all the sides AB, BC, and CA, ϕ will go to 0. We are choosing our stress function as ϕ equals to some unknown constant m times product of equation of three sides of the triangle: equation of AB multiplied with equation of BC multiplied with equation of CA. So, $\phi = m\left(x + \frac{a}{3}\right)\left(x + \sqrt{3}y - \frac{2a}{3}\right)\left(x - \sqrt{3}y - \frac{2a}{3}\right)$.

This is a cubic polynomial of x, You can note that three terms are there containing linear function of x; their product would be resulting a cubic polynomial of x. m is the unknown constant. You can explicitly multiply all these three terms within bracket and expand this ϕ in a polynomial form of x, y as: $\phi = m\left\{x^3 - 3xy^2 - a(x^2 + y^2) + \frac{2a^3}{27}\right\}$. This is the choice of stress function, which satisfies this $\phi = 0$ condition along the boundary because for AB part, this term will go to 0, for BC part, this term will go to 0, for CA part, this term will go to 0.

For all three sides of the boundary, whatever we choose, the total ϕ will go to 0 because either one of the term is going to 0 along the boundary. So, $\phi = 0$ along the boundary satisfied with these particular choice of stress function. For this particular equilateral

triangular bar torsion problem, we are going to choose this ϕ, that is the Prandtl stress function for this triangular bar.

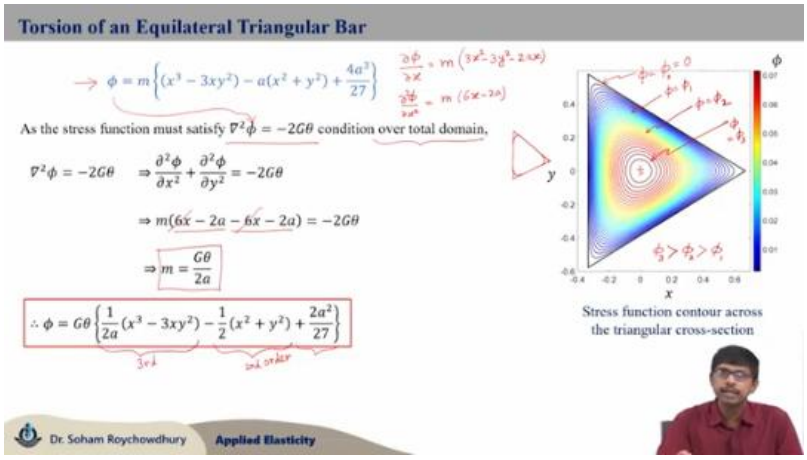


Moving forward with this chosen ϕ, we need to evaluate the unknown constant m, and for that we are having another condition that is $\nabla^2\phi = -2G\theta$ over the entire domain. Replacing this expression of ϕ on the left hand side, that is in the Laplacian operator and expanding it, $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ is the Laplacian of ϕ. If we replace the given ϕ there, and then try to simplify these, we would be getting something like this.

Here, you can see from the given ϕ, $\frac{\partial\phi}{\partial x}$ would be $m(3x^2 - 3y^2 - 2ax)$ and then, $\frac{\partial^2\phi}{\partial x^2}$ would be $m(6x - 2a)$. So, these first two terms are coming from $\frac{\partial^2\phi}{\partial x^2}$. Similarly, $\frac{\partial^2\phi}{\partial y^2}$ will give these two terms: $-6x - 2a$. You can see this $6x$ and $-6x$ terms will get cancelled, and left hand side will be $-4ma$, right hand side is $-2G\theta$. Minus sign and one 2 we can cancel and thus, m would be $\frac{G\theta}{2a}$. So, the value of the unknown constant is obtained in terms of θ as $\frac{G\theta}{2a}$, where a is the length of the median of this equilateral triangle, G is shear modulus, θ is angle of twist per unit length.

You can replace this obtained m back in this expression of ϕ. Thus, the total stress function ϕ, Prandtl stress function for the torsion problem of equilateral triangular bar becomes: $\phi = G\theta\left\{\frac{1}{2a}(x^3 - 3xy^2) - \frac{1}{2}(x^2 + y^2) + \frac{2a^3}{27}\right\}$.

This is a third order polynomial of x, y, and this particular Prandtl stress function satisfies both the condition. One is $\nabla^2\phi = -2G\theta$ over total domain for all values of x and y. Second one is $\phi = 0$ across all three boundary sides of the equilateral triangle. With this ϕ, we will proceed for finding the torque and the shear stresses generated within the equilateral triangular bar.



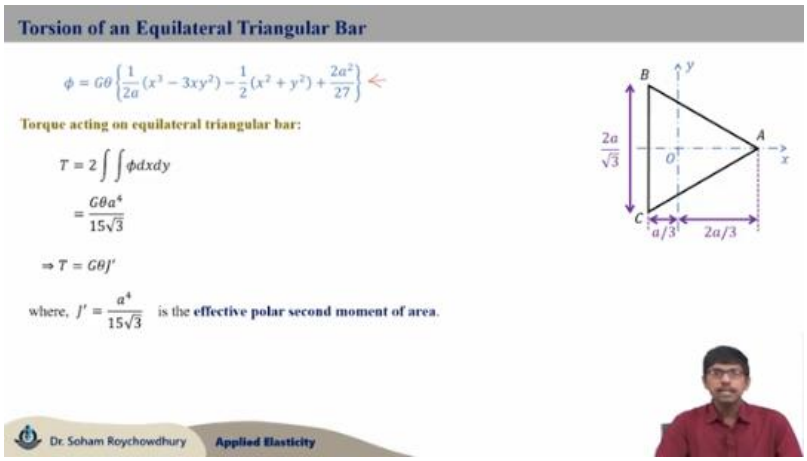
Moving forward, we will plot the stress function contour for this triangular cross section. This phi is plotted for different values of x and y, and all these contours are basically giving us the lines along which the stress function value is constant. We know that at the outer boundary, on all three outer surfaces, $\phi = \phi_0$ which is basically 0. We must have the stress function to be vanishing at the external boundary.

If you are taking some internal one, let us say here, $\phi = \phi_1$, then if you further go inside, $\phi = \phi_2$, almost at the centre, ϕ equals let us say ϕ_3. And if you look at the colour codes towards the centre, the value of the stress function is clearly increasing as compared to the value of the stress function near the periphery. The blue colour refers to the smaller values of stress function which occurs around the boundary, while the red colour refers to the higher value of stress function which occurs towards the centre.

Note one more thing that the outer boundary was triangular. So, near the outer boundary, the stress function contours are also similar to the triangle with the rounded edges. Instead of a pure triangle, we are having this rounded edges triangle as the stress function contour near the boundaries. However, as you are going towards the centre, this is reducing close to a circle.

Why so? Look at the ϕ expression: the first term $x^3 - 3xy^2$, this is a cubic order term. Second term $x^2 + y^2$, this, and last term is a constant. So, second and third $\frac{(x^2+y^2)}{2} + \frac{2a^3}{27}$, this is basically giving the equation of the circle. As you are going closer to the centre, both x and y values drop, and thus, these third order polynomial terms becomes negligible as compared to second order terms.

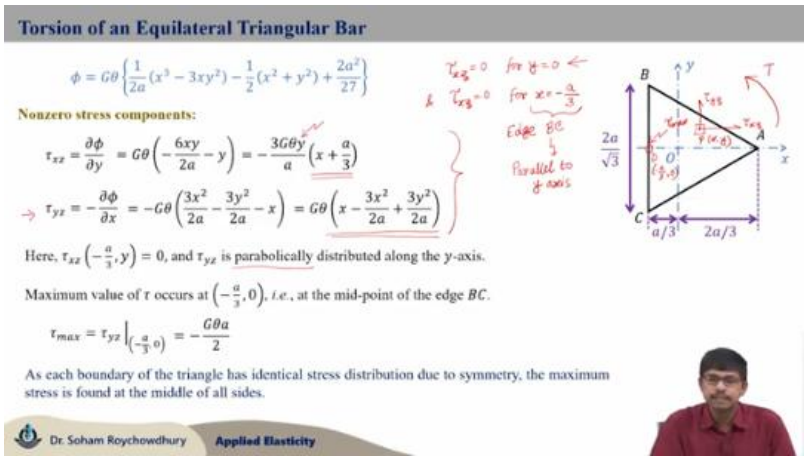
Hence, the third order term effect is going away and second order, that is circular term, dominates and thus, as you are going closer to the centre of the triangle, that is the cross section point between three medians of this equilateral triangle, towards the centroid of the triangle, we get the stress function contours to be almost circular, and the value of ϕ for those stress function contours are also increasing.



Going to the torque acting on the triangular bar, we can obtain the torque as the area integral of 2ϕ. We had already obtained ϕ as $G\theta\left\{\frac{1}{2a}(x^3 - 3xy^2) - \frac{1}{2}(x^2 + y^2) + \frac{2a^3}{27}\right\}$, which is given by this equation. The torque is obtained by integrating this ϕ multiplied by 2 over the entire area, over the entire domain.

If you substitute this expression of ϕ here, and then integrate it over this triangular region, that is a slightly lengthy calculation. You can do it, or you can also take help from some software like MATLAB. With that, if you integrate it, the result would be $T = \frac{G\theta a^4}{15\sqrt{3}}$. This is the relation between torque and the angle of twist θ for the equilateral triangular bar. So, $T = \frac{G\theta a^4}{15\sqrt{3}}$.

We can once again define some J', the effective polar moment of area for the equilateral triangular cross-section, and that J' is $\frac{a^4}{15\sqrt{3}}$. So, if we define J' as the effective second polar moment of area as $\frac{a^4}{15\sqrt{3}}$, then, this $T = G\theta J'$ equation can be used for the St. Venant's theory of torsion of equilateral triangular bars.



Moving forward, going to the non-zero stress components. As we know, there will be two non-zero stress components resulting from this kind of torsion problem: one is τ_{xz}, and another is τ_{yz}. The rest of the stress components are all zero. τ_{xz} and τ_{yz} can be written in terms of the partial derivative of the stress function ϕ as $\tau_{xz} = \frac{\partial \phi}{\partial y}$, $\tau_{yz} = -\frac{\partial \phi}{\partial x}$.

We had already obtained the Prandtl stress function, ϕ. Replacing this ϕ here, the expression of τ_{xz} can be obtained as $-\frac{3G\theta y}{a}\left(x + \frac{a}{3}\right)$. Similarly, substituting ϕ in the τ_{yz} expression, that can be simplified and obtained as $G\theta\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right)$. So, if you take any point P somewhere within the domain, that will be subjected to two non-zero stress components. One is τ_{yz} along the y-axis, and another is τ_{xz} along the x-axis. And those two expressions, the stress expressions τ_{xz} and τ_{yz}, can be obtained by putting the the x and y coordinates of that point P, here in these equations.

P being any point within the equilateral triangular cross-section within the domain, you can substitute the x and y values here, and get the values of the the non-zero shear stress components τ_{xz} and τ_{yz}. The resultant of these two shear stresses will give us the overall

shear stress, which is $\sqrt{\tau_{xz}^2 + \tau_{yz}^2}$. If you carefully look at the expression of τ_{xz}, this particular term will go to 0 if $x = -\frac{a}{3}$, and this will also go to 0 if $y = 0$. τ_{xz} is 0 for $y = 0$, and $y = 0$ means the x-axis.

So, along the complete x-axis, the τ_{xz} term is 0. Also, the τ_{xz} term is 0 for $x = -\frac{a}{3}$. If x is constant and set as $-\frac{a}{3}$, then τ_{xz} will also be 0. This $x = -\frac{a}{3}$ refers to one of the edges of the triangle, which is this vertical edge B to C. Along that edge, this stress component τ_{xz} would be 0. So, τ_{xz}, for $x = -\frac{a}{3}$ and whatever value of y, will go to 0, which is basically the equation of one particular side of this triangle. And also, if you look at the τ_{yz} equation, here, this τ_{yz} is proportional to y^2. So, if you have any axis which is the y-axis or parallel to the y-axis, meaning the x value is constant, for such cases, τ_{yz} is going to have a parabolic variation along y. τ_{yz} has a parabolic variation over y for the y-axis or any other axis parallel to the y-axis.

If you consider B to C, this edge is also parallel to the y-axis. Hence, if you put $x = -\frac{a}{3}$ here in the expression of τ_{yz} along this BC line, which is parallel to the y-axis, we will get τ_{yz}, and that will also be proportional to y^2. We will have a parabolic distribution of τ_{yz} along that B to C edge.

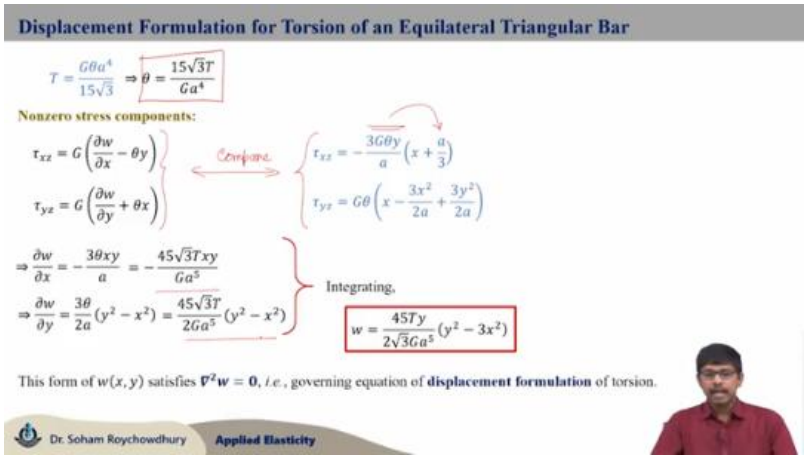
Combining all these discussions, we can conclude that the maximum value of τ within that BC edge is going to occur at the midpoint. Let us say the midpoint of the BC edge is given by D, which is $\left(-\frac{a}{3}, 0\right)$. At this particular point, we are going to have τ_{max}. As we know, for torsion, when this bar is subjected to torque T, the maximum shear stress can occur along the external boundaries.

One of the boundary edges we are considering is B to C. Why this particular one? Because it is parallel to the y-axis, it is easy to conclude if you choose this BC as compared to AB or AC. The same conclusion can be obtained for other edges as well. Choosing this BC as one of the edges of our interest, we can show that the midpoint of BC, point D, is the point of the maximum value of τ. At point D, as it is on the x-axis, $y = 0$. So, τ_{xz} is 0.

Total τ at point D is only coming due to the contribution of τ_{yz}. Total τ at point D which is τ_{max} is equal to τ_{yz} at that point D, which is denoted by $\left(-\frac{a}{3}, 0\right)$, and substituting that in the expression of τ_{yz}, we can get the maximum shear stress as $-\frac{G\theta a}{2}$. So, maximum shear stress within the equilateral triangular boundary is occurring at the midpoint of this vertical edge BC and the maximum value of the stress is $-\frac{G\theta a}{2}$.

As this thing is symmetric, instead of choosing this BC edge to be vertical, we can simply rotate the cross section and choose AB edge to be parallel to y-axis, or we can choose AC edge to be parallel to y-axis; that is our choice - x and y axes choice, we can keep changing.

With that, it can be proved that the midpoint of all three sides of the equilateral triangle are the critical points which is subjected to maximum shear stress. So, each side of the triangle is going to have identical stress distribution due to symmetry of this equilateral triangle Thus, the maximum shear stress is obtained at middle point of all three sides of the equilateral triangular bar.



Moving to the displacement formulation. We are now trying to find out the out-of-plane displacement or distortion of this equilateral triangular cross section bar whenever it is subjected to torque T. T-θ relation is written here: $T = \frac{G\theta a^4}{15\sqrt{3}}$, from which, we can obtain theta in terms of T as $\theta = \frac{15\sqrt{3}T}{Ga^4}$.

If you recall the displacement formulation equations discussed in the previous lecture, we know the non-zero stress components τ_{xz} and τ_{yz} can be written in terms of the out-of-plane displacement component w as: τ_{xz} is $G\left(\frac{\partial w}{\partial x} - \theta y\right)$, τ_{yz} is $G\left(\frac{\partial w}{\partial y} + \theta x\right)$. These are the two equations of non-zero stress components in terms of the out-of-plane displacement component w. w is displacement along the z direction, i.e., along the longitudinal axis.

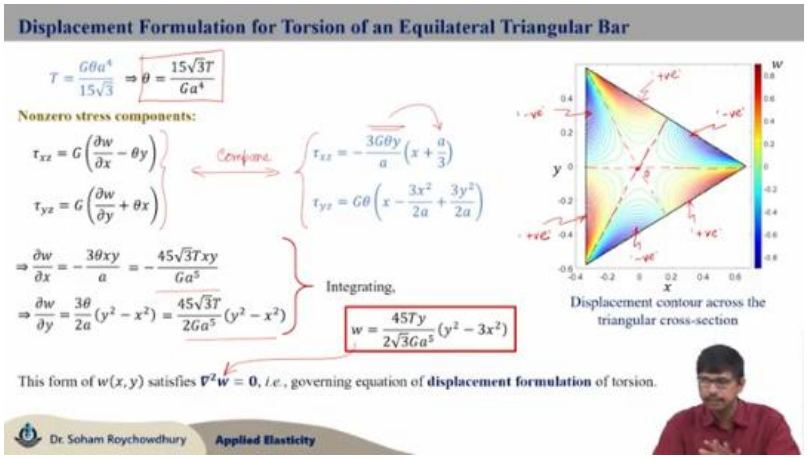
Now, τ_{xz} and τ_{yz} - these two shear stresses we had already obtained in terms of the stress function, and the angle of twist, θ. Those equations obtained in the previous slide were like this. These two are the expression of the non-zero shear stresses in terms of displacement component w. These are the expressions of non-zero shear stresses in terms of θ.

Comparing these two, equating τ_{xz} and τ_{yz} in both the forms from the first equation, you can see there is a term like $-G\theta y$. Here also, this first term multiplied with $\frac{a}{3}$, that will be give us the same thing: $-G\theta y$, that will get cancelled and from that, we can get $\frac{\partial w}{\partial x}$ to be $-\frac{3\theta xy}{a}$. Replacing theta in terms of T using this equation, $\frac{\partial w}{\partial x}$ would be $-\frac{45\sqrt{3}Txy}{Ga^5}$.

Similarly, comparing the expression of τ_{yz} in both the forms, cancelling the same term, $\frac{\partial w}{\partial y}$ would come out to be $\frac{3\theta}{2a}(y^2 - x^2)$. Replacing θ in terms of T, $\frac{\partial w}{\partial y}$ would be $\frac{45\sqrt{3}T}{2Ga^5}(y^2 - x^2)$. We got the two expressions which are partial derivatives of out-of-plane displacement component w in terms of x and y. Combining them and integrating, we can get the expression of the out-of-plane w as $\frac{45Ty}{2\sqrt{3}Ga^5}(y^2 - 3x^2)$.

Integration constant is taken to be 0 by imposing that at $x = 0$, $y = 0$, w is 0. At the centroid of the triangle, there is no distortion: w is 0 at $x = 0$, $y = 0$ and with that, the integration constant is said to be 0. Using this w, you can verify $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ - these two expressions can be satisfied. This gives us the out-of-plane displacement component obtained due to torsion or twist of equilateral triangular bar. You can verify that this w

satisfies the governing equation of displacement formulation which is Laplacian of this obtained w is equal to 0.



Plotting this displacement contour for the equilateral triangular bar, the figure will be looking something like this. Here you can see, there are six regions. If you try to recall the elliptic bar, there were four regions. Here, let us draw the medians that is the perpendicular from one of the tip toward the opposite base of the triangle. By these three medians, this is some centroid O, let us not just fix $(0, 0)$ because I think we fixed origin somewhere else. About that centroid point O, if we draw these medians, that divides the total triangle into six components, and the alternate components are different signs of w.

In this one, w is positive, in this one w is positive, and in this one w is positive. Those three portions are trying to come out of the plane, whereas, this one is having negative w, this one is also having negative w, and this one is having negative value of w because those are marked by blue colours, while red colours are positive region. Total section is divided into six parts, three parts are trying to come out and alternate three parts are trying to go in. This clearly shows the distortion of the cross-section of the equilateral triangular bar when it is subjected to constant uniform twisting moment T over the entire length.

In this lecture, we discussed the torsion of an equilateral triangular bar. We discussed the stress formulation, stress function approach, and the displacement formulation. We obtained the expression for maximum non-zero shear stresses generated, and the relation between the applied torque and the resulting angle of twist.

Thank you.