

# APPLIED ELASTICITY

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Week 8

## Lecture 37: Torsion Problems II



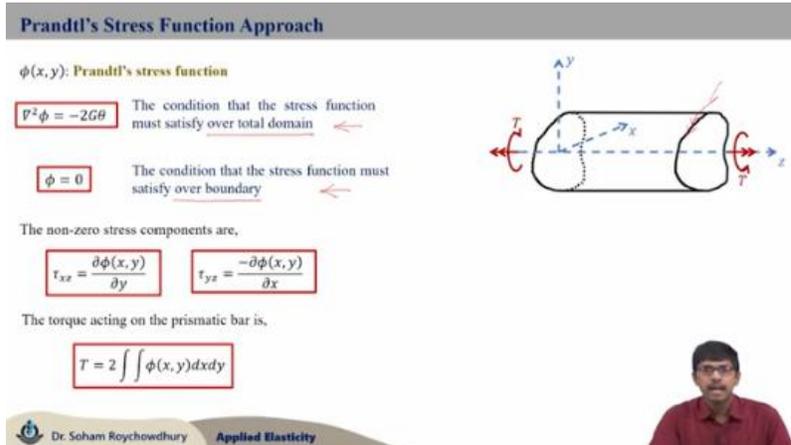
COURSE ON:  
APPLIED ELASTICITY

Lecture 37  
TORSION PROBLEMS II

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The slide features a central portrait of Dr. Soham Roychowdhury. To his left, there are diagrams of a beam under torsion and a 3D grid with axes labeled  $i, j, k$  and  $m$ . To his right, there are logos of IIT Bhubaneswar and a circular diagram showing stress distribution. The text 'COURSE ON: APPLIED ELASTICITY' and 'Lecture 37 TORSION PROBLEMS II' is on the left. The text 'Dr. Soham Roychowdhury School of Mechanical Sciences Indian Institute of Technology Bhubaneswar' is at the bottom right.

Welcome back to the course on Applied Elasticity. In today's lecture, we will continue our discussion on the torsion problems.



**Prandtl's Stress Function Approach**

$\phi(x, y)$ : Prandtl's stress function

$\nabla^2 \phi = -2G\theta$  The condition that the stress function must satisfy over total domain

$\phi = 0$  The condition that the stress function must satisfy over boundary

The non-zero stress components are,

$\tau_{xx} = \frac{\partial \phi(x, y)}{\partial y}$        $\tau_{yy} = -\frac{\partial \phi(x, y)}{\partial x}$

The torque acting on the prismatic bar is,

$T = 2 \iint \phi(x, y) dx dy$

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The slide contains a diagram of a prismatic bar with a cross-section in the  $x-y$  plane and a longitudinal  $z$ -axis. Torque  $T$  is applied at both ends. The stress function  $\phi(x, y)$  is shown as a dashed line within the cross-section. The slide includes the governing equation  $\nabla^2 \phi = -2G\theta$ , boundary condition  $\phi = 0$ , stress component formulas  $\tau_{xx} = \frac{\partial \phi}{\partial y}$  and  $\tau_{yy} = -\frac{\partial \phi}{\partial x}$ , and the torque formula  $T = 2 \iint \phi dx dy$ . A small portrait of Dr. Soham Roychowdhury is in the bottom right corner.

In the previous lecture, we started our discussion on the torsion problem. We had a quick recap on the simple theory of torsion and then started discussing the Saint-Venant theory

of torsion, which is valid for non-circular cross-sections. This can be solved using the Prandtl stress function approach, which we discussed in detail.

So,  $\phi(x, y)$  being a Prandtl stress function, this must satisfy two conditions. One is  $\nabla^2\phi = -2G\theta$ . This condition should be satisfied by the Prandtl stress function over the entire domain, over the entire cross-section, where  $\theta$  is the angle of twist per unit length of the bar and  $G$  is the modulus of rigidity or shear modulus. This is the first condition to be satisfied by the chosen Prandtl stress function, and the second condition which the Prandtl stress function should satisfy is  $\phi = 0$ . Actually, it should be  $\phi$  equals a constant, and for the singly connected boundary, meaning for solid shafts, which only have an outer boundary, we choose that constant to be 0. So,  $\phi = 0$  is the second constraint which should be satisfied by the Prandtl stress function  $\phi$  over the outer boundary. The first condition should be satisfied over the total domain for all values of  $x$  and  $y$  over the entire cross-section, whereas the second one must be satisfied only along the outer boundary, only along this boundary curve.

Now, the non-zero stress components  $\tau_{xz}$  and  $\tau_{yz}$  were defined with respect to the stress function  $\phi$  as:  $\tau_{xz}$  is  $\frac{\partial\phi}{\partial y}$ ,  $\tau_{yz}$  is  $-\frac{\partial\phi}{\partial x}$ , and the relation between torque and  $\theta$  was obtained like this. The torque acting is equal to the area integral of twice the Prandtl stress function.

For solving any torsion problem of a prismatic bar with a non-circular cross section, our first objective or goal is to choose a proper Prandtl stress function  $\phi(x, y)$  which satisfies both conditions: that is,  $\nabla^2\phi = -2G\theta$  over the entire domain, and  $\phi = 0$  over the outer boundary. Then, with respect to  $\phi$ , we can obtain the non-zero shear stress components. We can find out the point at which the maximum shear stress is generated.

Finally, we can find out the amount of torque in terms of that stress function, which would basically give us a  $T$ - $\theta$  relation - the relation between the applied torque and the resulting angle of twist. Now we will take different non-circular cross sections.

**Torsion of an Elliptical Bar**

Equation of elliptical boundary:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Total cross-sectional area is  $A = \pi ab$

Choice of stress function:

$$\phi = m \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

where  $m$  is a constant

The assumed form of  $\phi(x, y)$  satisfies the condition  $\phi = 0$  along the boundary.

$\nabla^2 \phi = -2G\theta$  over total domain  
 $\phi = 0$  along boundary

$a$ : Semi-major axis length  
 $b$ : Semi-minor axis length

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In today's lecture, we are going to start with an elliptic cross-section bar. We are going to consider a prismatic bar with a cross section being an ellipse, and we will apply this Prandtl stress function approach for solving the torsion problem of that elliptical bar. Torsion of an elliptical bar - which we are trying to solve - our aim is to get the stress distribution and also find out the torque  $T$  and angle of twist  $\theta$ , relation between them.

Let us consider this cross section. An elliptical cross section is considered for the prismatic bar, which is subjected to end torque  $T$ . The cross section is constant or uniform along the  $z$ -axis. Along the length, the cross section is not varying.  $T$  is independent of  $z$ .  $2a$  is the total semi-major axis. Semi-major axis length is taken as  $a$ , semi-minor axis length is taken as  $b$ . So, total minor axis is  $2b$  for this chosen ellipse.

The equation of this elliptic boundary, equation of this curve we can write as  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . This is the standard equation describing the elliptical boundaries or elliptic curve:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $a$  is the length of the semi-major axis,  $b$  is the length of the semi-minor axis. Based on this, we are going to choose our Prandtl stress function. And for this ellipse, the total cross-sectional area, the formula for cross-sectional area for an ellipse is equal to  $\pi ab$ . Total area is  $\pi$  times semi-major axis length times semi-minor axis length;  $\pi ab$  is equal to total cross-sectional area  $A$ .

Now, we will go for our choice of stress function. What are the conditions the stress function  $\phi$  should satisfy?  $\nabla^2 \phi$  or Laplacian of  $\phi$  is  $-2G\theta$  over total domain. This is one condition for all values of  $x$  and  $y$ , this should be satisfied, and  $\phi = 0$  along boundary.

The initial choice of  $\phi$  or Prandtl stress function is always with respect to or starts with the help of the second condition. We must have  $\phi$  to be 0 along the boundary and how is boundary defined? Boundary is defined by this ellipse equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Hence, we can choose our stress function of this particular form: some unknown constant  $m$  times  $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$  and you can see as this  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  along the boundary, if you substitute that here, this term goes to 0.

So,  $\phi = 0$  along the boundary, this second condition is automatically satisfied if you are going with a choice of  $\phi$  like this. Always for the Prandtl stress function approach the stress function is chosen like some unknown constant  $m$  times the equation of the outer boundary of the given cross section that will satisfy the second condition  $\phi = 0$  along the boundary automatically.

With this, we are starting our solution. Our Prandtl stress function  $\phi$  is  $m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ , where  $m$  is some unknown constant which we need to determine and that will be obtained with the help of this first boundary condition. If you replace  $\phi$  here, in the first condition, over entire domain,  $\nabla^2\phi = -2G\theta$ , replacing  $\phi$  there, we will get some idea of what the value of  $m$  should be.

**Torsion of an Elliptical Bar**

$\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

As the stress function must satisfy  $\nabla^2\phi = -2G\theta$  condition over total domain,

$$\nabla^2\phi = -2G\theta \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

$$\Rightarrow 2m\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta$$

$$\Rightarrow m = -\frac{G\theta a^2 b^2}{(a^2 + b^2)}$$

$$\therefore \phi = -\frac{G\theta a^2 b^2}{(a^2 + b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$


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Moving forward, substituting this  $\phi$  in the Laplacian condition:  $\nabla^2\phi = -2G\theta$ , we are substituting this  $\phi$  here and finding its Laplacian. Laplacian in the Cartesian coordinate is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}, \text{ and here that should be equal to } -2G\theta.$$

If you start from the given  $\phi$ ,  $\frac{\partial \phi}{\partial x}$  - what will that be? That will be  $\frac{2mx}{a^2}$  and then,  $\frac{\partial^2 \phi}{\partial x^2}$  will be I think  $\frac{2m}{a^2}$ . We can substitute this  $\frac{\partial^2 \phi}{\partial x^2}$  as  $\frac{2m}{a^2}$  and similarly,  $\frac{\partial^2 \phi}{\partial y^2}$ , that can be obtained as  $\frac{2m}{b^2}$ . Substituting those here, we will be getting left hand side as  $2m \left( \frac{1}{a^2} + \frac{1}{b^2} \right)$  and that should be equal to  $-2G\theta$ .

Now,  $a, b$  are known quantities. We can write  $m$  in terms of  $a, b$  as  $m = -\frac{G\theta a^2 b^2}{(a^2 + b^2)}$ . We got the expression for this unknown constant  $m$  and this equation of  $m$ , we can replace here in the expression of the Prandtl stress function  $\phi$  and can write the complete Prandtl stress function  $\phi$  as, with  $-m$  replaced by  $\frac{G\theta a^2 b^2}{(a^2 + b^2)}$ , times this equation of ellipse. That is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ . This is the total Prandtl stress function which is required to be used for solving the torsion problem of the prismatic bars with elliptical cross-section, with  $a$  being the semi-major axis length,  $b$  being the semi-minor axis length,  $G$  being the modulus of rigidity, and  $\theta$  being the angle of twist per unit length.

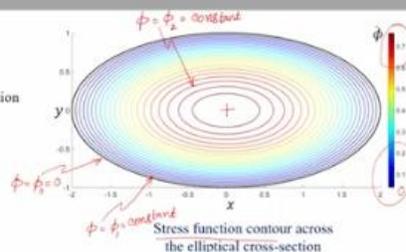
We have obtained an expression for  $\phi$ , or the Prandtl stress function, which satisfies both conditions. One is  $\nabla^2 \phi = -2G\theta$ , which is satisfied, and also  $\phi = 0$  over the boundary, which is satisfied due to the presence of this ellipse equation over the boundary. This part would go to 0. So,  $\phi$  will be 0 over the boundary. Hence, this Prandtl stress function satisfies both conditions.

**Torsion of an Elliptical Bar**

$$\phi = m \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

As the stress function must satisfy  $\nabla^2 \phi = -2G\theta$  condition over total domain,

$$\begin{aligned} \nabla^2 \phi = -2G\theta &\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \\ &\Rightarrow 2m \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = -2G\theta \\ &\Rightarrow m = -\frac{G\theta a^2 b^2}{(a^2 + b^2)} \end{aligned}$$

$$\therefore \phi = -\frac{G\theta a^2 b^2}{(a^2 + b^2)} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$


Stress function contour across the elliptical cross-section



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If I plot this Prandtl stress function over  $x$  and  $y$ , here you can see over the elliptic boundary I have plotted  $\phi$  for different sets of values of  $x$  and  $y$ . All these are equations of an ellipse. You can see as you keep on choosing different values of  $x$  and  $y$ , you will be getting these stress function contours. All these ellipses have stress function contours. For each of these ellipses,  $\phi$  is constant.

Let us say here,  $\phi = \phi_1$ , which is a constant. If you take another one, let us say slightly on the inner side, there  $\phi$  equals another constant  $\phi_2$ . So, all these elliptic lines are basically the stress contours.  $\phi$  is set to different constant values, and all those stress function contours are basically ellipses. You can see the set of concentric ellipses starting from the outer boundary and shrinking towards the center. And note that at the outer boundary,  $\phi$  should be 0.

So, on the outer boundary, let us say that is  $\phi = \phi_0$  and should be 0. And as you are going towards the inner side, the value of  $\phi$  should increase; the stress function contour value will increase, which you can see by looking at the color codes here. Blue side colors are closer to 0 on the outer boundary; values are on the lesser side, which are referred to with the blue contours.

As you are going towards the center, the values are increasing, and the color is shifting towards the red zone. The stress function value is increasing on the stress function contour near the center. This is the plot of the stress function, a pictorial representation of the stress function contour across the elliptical cross-section.

**Torsion of an Elliptical Bar**

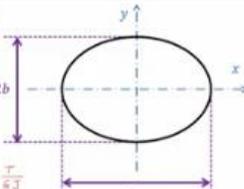
Torque acting on elliptical bar:  $\phi = -\frac{G\theta a^2 b^2}{(a^2 + b^2)} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$

$$T = 2 \iint \phi(x, y) dx dy = -\frac{2G\theta a^2 b^2}{(a^2 + b^2)} \iint \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) dx dy$$

$$= -\frac{2G\theta a^2 b^2}{(a^2 + b^2)} \left( \frac{I_{yy}}{a^2} + \frac{I_{xx}}{b^2} - A \right)$$

$$= -\frac{2G\theta a^2 b^2}{(a^2 + b^2)} \left( \frac{\pi a^4 b}{4a^2} + \frac{\pi a b^4}{4b^2} - \pi ab \right) = \frac{\pi G\theta a^3 b^3}{(a^2 + b^2)}$$

$\Rightarrow T = G\theta J'$  where  $J' = \frac{\pi a^3 b^3}{a^2 + b^2}$  is the effective polar second moment of area  
 The true polar second moment of area of ellipse is  $J = \frac{\pi ab}{4} (a^2 + b^2) = I_{xx} + I_{yy}$   
 For circular bars ( $a = b$ ),  $T = G\theta J$  as  $J = J' = \frac{\pi a^4}{2}$



$\iint x^2 dx dy = I_{yy} = \frac{\pi a^3 b}{4}$   
 $\iint y^2 dx dy = I_{xx} = \frac{\pi a b^3}{4}$   
 $\iint dx dy = A = \pi ab$   
 circular:  $\theta = \frac{T}{GJ}$   
 $J \neq J'$  for ellipse.



Moving forward to the torque acting on the elliptical bar, we had derived the equation of torque in terms of Prandtl stress function as  $T = 2 \iint \phi dx dy$ .  $\phi(x, y)$  is the Prandtl stress function, which for the present problem of elliptic cross-section is obtained like this. We will replace this expression of  $\phi$  here and we will try to obtain the relation between applied torque  $T$  and  $\theta$ .

You can see  $\phi$  is a function of  $\theta$ , the angle of twist. If I replace  $\phi$  within the integral, then the right-hand side will have  $\theta$  and some integral of  $x$  and  $y$  over the entire domain, and the left-hand side is torque. This equation will give us a relation between the applied torque and the resulting angle of twist.

Substituting this  $\phi$  in the integration  $T$ . And keeping that  $-\frac{G\theta a^2 b^2}{(a^2+b^2)}$ , this constant term out of the integral, we will have  $-\frac{2G\theta a^2 b^2}{(a^2+b^2)}$ , then the area integral of only this part, which is a function of  $x$  and  $y$ . So,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$  area integral - this is nothing but the equation of an ellipse.

If you try to recall the properties of any area, what is the double integral or area integral of  $x^2 dx dy$ , and what is the area integral of  $y^2 dx dy$ ? These are nothing but the two second moments of area,  $I_{yy}$  and  $I_{xx}$ . We can write this  $\iint x^2 dx dy$  as  $I_{yy}$ , and the  $\iint y^2 dx dy$  as  $I_{xx}$ , and the last term, the integral of  $dx dy$ , is the total area. So, this term is  $\iint \frac{x^2}{a^2} dx dy$ . The double integral of  $x^2 dx dy$  over the total area is  $I_{yy}$ , by definition of the second moment of area about the  $y$ -axis; this numerator is  $I_{yy}$ . This is  $\frac{I_{yy}}{a^2}$ , as written here. Similarly, the second term,  $\iint \frac{y^2}{b^2} dx dy$ , will be  $I_{xx}$ , divided by  $b^2$ , and the last term is  $\iint dx dy$ , which is the total area.

We can remove the integral and write this  $T$  as the constant times  $\frac{I_{yy}}{a^2} + \frac{I_{xx}}{b^2} - A$ . For the elliptical cross-section,  $I_{yy} = \frac{\pi a^3 b}{4}$ ,  $I_{xx} = \frac{\pi a b^3}{4}$ , and the total area  $A$  is  $\pi a b$ . So, for any ellipse with semi-major axis length  $a$  and semi-minor axis length  $b$ , we can write  $I_{xx}$ ,  $I_{yy}$ , and total area  $A$  using these standard formulae.

Now, I am going to replace all those forms in this equation of  $T$ . If I do so,  $T$  will be like this, where  $I_{yy}$  is written as  $\frac{\pi a^3 b}{4}$ ,  $I_{xx}$  is written as  $\frac{\pi a b^3}{4}$ , and area  $A$  is written as  $\pi a b$ . Now, we can simplify the term within the bracket. These  $a^2$  and  $b^2$  will get cancelled. We will be having  $\frac{\pi a b}{4} + \frac{\pi a b}{4} - \pi a b$ ; this would be  $-\frac{\pi a b}{2}$ . Multiplying this with the external constant, in total  $T$  would be  $\frac{\pi G \theta a^3 b^3}{(a^2 + b^2)}$ . This is the torque and the angle of twist  $\theta$  relation:  $T = \frac{\pi G \theta a^3 b^3}{(a^2 + b^2)}$ , for the elliptical cross-section.

Now, if you recall about the simple theory of torsion,  $\theta$  was defined as  $\frac{T}{GJ}$ . So, the angle of twist is normally written as torque divided by the modulus of rigidity times  $J$ , where  $J$  was known as the polar moment of area. From here, you can think  $T = G\theta J$ . Now, here, in this particular elliptic cross-section, this was valid for the circular cross - this is for the circular bar from the simple theory of torsion. If I compare that here, let us say  $T = G\theta J'$ , where  $J' = \frac{\pi a^3 b^3}{(a^2 + b^2)}$ .

If I compare this St. Venant's theory of torsion torque- $\theta$  relation with the simple theory of torsion torque- $\theta$  relation, then we can define a quantity  $J'$ , which is known as the effective polar second moment of area, and that is equal to  $\frac{\pi a^3 b^3}{(a^2 + b^2)}$ . We are simply deriving this by defining the expression of  $T = G\theta J'$ , where  $J'$  should be this.

This is not the actual polar second moment of area of the elliptic cross-section, because the effective polar second moment of area is obtained from this  $T$ - $\theta$  relation based on St. Venant's theory. But, from the properties of area, the actual polar second moment of area for the ellipse is equal to  $J = \frac{\pi a b}{4} (a^2 + b^2)$ . This is simply the summation of  $I_{xx}$  and  $I_{yy}$  by definition. Whatever  $I_{xx}$  and  $I_{yy}$  we had, if you sum them up, you will get the actual polar second moment of area of this elliptic cross-section.

You can see this  $J$  is not equal to  $J'$  for an ellipse. Thus, the simple theory of torsion is not valid if you try to use this formula; we will end up with a wrong result, but for a circle, that should be the same. Let us verify that; how can we verify for the circle? We

must have  $a = b$ . If you impose  $a = b$ , both  $J$  and  $J'$  would come out to be the same as  $\frac{\pi a^4}{2}$ .

Imposing  $a = b$  in this equation of  $J$  and also in this equation of  $J'$ , you will get both  $J$  and  $J'$  as  $\frac{\pi a^4}{2}$ . Thus, for the circular bar,  $T = G\theta J$ , which equals  $G\theta J'$ . This is valid for both the St. Venant's theory as well as the simple theory of torsion. So, this is a validation, by using this or by reducing the elliptic cross-section to a circular cross-section, we can verify that St. Venant's theory matches with the result given by the simple theory of torsion.

**Torsion of an Elliptical Bar**

Nonzero stress components:

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -\frac{G\theta a^2 b^2}{(a^2 + b^2)} \left(\frac{2y}{b^2}\right) = -\frac{2G\theta a^2 y}{(a^2 + b^2)} = -\frac{2Ty}{\pi a b^3}$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = \frac{G\theta a^2 b^2}{(a^2 + b^2)} \left(\frac{2x}{a^2}\right) = \frac{2G\theta b^2 x}{(a^2 + b^2)} = \frac{2Tx}{\pi a^3 b}$$

The resultant shear stress in  $x$ - $y$  plane can be obtained as,

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{2T}{\pi a b} \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}} \quad \left[ \begin{array}{l} -a \leq x \leq a \\ -b \leq y \leq b \end{array} \right]$$

With  $a > b$ , the maximum shear stress occurs at  $x = 0$ , and  $y = \pm b$ , i.e., at the end points of minor axes.

$$\tau_{max} = \tau \Big|_{(0, \pm b)} = \frac{2T}{\pi a b^2}$$

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Moving forward to the shear stress components. The non-zero stress components  $\tau_{xz}$  and  $\tau_{yz}$  generated due to the application of this torque  $T$  were defined in terms of the Prandtl stress function  $\phi$  as:  $\tau_{xz} = \frac{\partial \phi}{\partial y}$ ,  $\tau_{yz} = -\frac{\partial \phi}{\partial x}$ , where  $\phi$  we had obtained. The  $T$  and  $\theta$  relation was also obtained. Replacing this stress function in these expressions  $\frac{\partial \phi}{\partial y}$  and  $-\frac{\partial \phi}{\partial x}$ , and then simplifying it further, we will get  $\tau_{xz}$  as  $-\frac{2G\theta a^2 y}{(a^2 + b^2)}$ , and  $\tau_{yz}$  as  $\frac{2G\theta b^2 x}{(a^2 + b^2)}$ .

At any  $x$  and  $y$ , if you replace the values of  $x$  and  $y$  in these two equations at any particular point, we can get both the stress components  $\tau_{yz}$  and  $\tau_{xz}$ . These are the only two non-zero stress components; all the rest are zero. By putting these values of  $x$  and  $y$  at any point within the domain within the ellipse, we can find out the stress distribution using these two equations. And if you write this equation in terms of  $T$ , these would be

like  $\tau_{xz} = -\frac{2Ty}{\pi ab^3}$  and  $\tau_{yz} = \frac{2Tx}{\pi a^3 b}$ . This is obtained by using this  $T$ - $\theta$  relation. Here, writing  $\theta$  in terms of  $T$ , these equations  $\tau_{xz}$  and  $\tau_{yz}$  in terms of the applied torque  $T$  can be obtained.

Now, finding the resultant shear stress, the total shear or resultant shear acting at any point  $P$  will be the square root of the summation of the squares of  $\tau_{xz}$  and  $\tau_{yz}$ . Thus, this resultant shear  $\tau$  at any point within the elliptic cross section is  $\sqrt{\tau_{xz}^2 + \tau_{yz}^2}$ .

Substituting  $\tau_{xz}$  and  $\tau_{yz}$  in terms of these equations, we can write the resultant shear stress as:  $\frac{2T}{\pi ab} \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}$ , where  $x$  is varying between  $-a$  to  $+a$ ,  $y$  is varying between  $-b$  to  $+b$ . We are interested to find out the maximum value of the shear stress, and the point at which this shear stress would be maximum.

Now,  $x$  can vary between  $-a$  to  $+a$ ,  $y$  can vary between  $-b$  to  $+b$ . We also know for any torsion problem, the maximum shear stress occurs at the outer boundary. Outer boundary has 4 critical points, 2 endpoints of the same major axis, 2 endpoints of the minor axis. These are the 4 points where there is a possibility of having the maximum shear. The endpoints of the major axis are defined like  $(a, 0)$  and  $(-a, 0)$ . Endpoints of the minor axis are defined like  $(0, b)$  and  $(0, -b)$ .

If you check the value of  $\tau$  at all these 4 points and compare them as  $a > b$ , we can clearly see that maximum shear stress is going to occur at  $x = 0, y = \pm b$ . These are the points corresponding to the maximum shear stress. At the end points of the minor axis, we are going to get the maximum shear stress.  $\tau_{max}$  which is  $\tau$  at  $(0, \pm b)$  can be obtained as  $\frac{2T}{\pi ab^2}$ .

The value of the  $\tau$  at two end points of the major axis will be lesser than this. Hence, all the endpoints of the major axis are not critical points as long as the maximum shear stress is concerned. Only the endpoints of the minor axes will be the critical point corresponding to maximum shear stress and that shear stress value is  $\frac{2T}{\pi ab^2}$  for this elliptic cross-section.

**Displacement Formulation for Torsion**

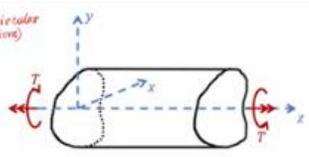
**Displacement fields:**  
 $u = -\theta yz$      $v = \theta xz$      $w = w(x, y)$

*( $w$  known  $\neq 0$  for non-circular sections)*

**Stress fields:**  
 $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$   
 $\tau_{xz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = G \left( \frac{\partial w}{\partial x} - \theta y \right)$   
 $\tau_{yz} = G \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = G \left( \frac{\partial w}{\partial y} + \theta x \right)$

**Equilibrium equation:**  
 $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$

$\Rightarrow \nabla^2 w = 0$     Governing equation in terms of out of plane displacement component




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Moving to the displacement formulation of torsion, we discussed the torque- $\theta$  relationship and stress. Now we are interested in finding out the displacement - how the body is deforming with the St. Venant's theory of torsion for non-circular cross-sections.

Displacement fields were assumed like this:  $u = -\theta yz$ ,  $v = \theta xz$ , and  $w$  is some function of  $x$  and  $y$ . This is an unknown function and is not equal to 0 for non-circular cross-sections, as warping is allowed for non-circular sections.

Using this, if you obtain the strain fields and stress fields, we will find that four stress components are 0. We have already discussed this:  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ , and  $\tau_{xy}$ , these four stresses are 0. Only two non-zero stresses are  $\tau_{xz}$  and  $\tau_{yz}$ . These can be written in terms of corresponding strain or displacement components as:  $\tau_{xz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$ , and  $\tau_{yz} = G \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$ . These are obtained using the simple strain-displacement relations.

Substituting  $u$ ,  $v$ , and  $w$  into these two equations, you will get  $\tau_{xz}$  as  $G \left( \frac{\partial w}{\partial x} - \theta y \right)$  and  $\tau_{yz}$  as  $G \left( \frac{\partial w}{\partial y} + \theta x \right)$ . Note that  $w$  is a function of  $x$  and  $y$  only, independent of  $z$ .

Coming to the equilibrium equation for the torsion problem. We were left with only one equilibrium equation to be satisfied, as we discussed in the previous lecture. That equilibrium equation, in the absence of any body force, is  $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$ . If I replace  $\tau_{xz}$  and  $\tau_{yz}$  with these expressions in the equilibrium equation and simplify it, It would have

only two terms:  $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ , meaning  $\nabla^2 w = 0$ . This is the governing equation for the out-of-plane displacement component  $w$ , which must be non-zero for non-circular cross-sections:  $\nabla^2 w = 0$ .

So, instead of using  $\phi$ , the stress function, we can also solve the problem by using this displacement formulation. This is also called the warping function approach.  $w$  is directly proportional to the warping function;  $\theta$  times the warping function  $\psi$  equals  $w$ . So, if  $\nabla^2 w = 0$ , the Laplacian of the warping function is also 0. This is the alternate approach of St. Venant's theory of torsion for non-circular cross-sections.

**Displacement Formulation for Torsion of an Elliptical Bar**

$T = \frac{\pi G \theta a^3 b^3}{(a^2 + b^2)} \Rightarrow \theta = \frac{T(a^2 + b^2)}{\pi G a^3 b^3}$

Nonzero stress components:

$$\tau_{xz} = G \left( \frac{\partial w}{\partial x} - \theta y \right) = G \left( \frac{\partial w}{\partial x} - \frac{T(a^2 + b^2)y}{\pi G a^3 b^3} \right)$$

$$\tau_{yz} = G \left( \frac{\partial w}{\partial y} + \theta x \right) = G \left( \frac{\partial w}{\partial y} + \frac{T(a^2 + b^2)x}{\pi G a^3 b^3} \right)$$

$\Rightarrow \frac{\partial w}{\partial x} = \frac{T(b^2 - a^2)y}{\pi G a^3 b^3}$   
 $\Rightarrow \frac{\partial w}{\partial y} = \frac{T(b^2 - a^2)x}{\pi G a^3 b^3}$

Integrating,  $w = \frac{T(b^2 - a^2)xy}{\pi G a^3 b^3}$

This form of  $w(x, y)$  satisfies  $\nabla^2 w = 0$ , i.e., governing equation of displacement formulation of torsion.

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Let us apply this to the elliptical cross-section, the problem we are discussing in this lecture. We know the expressions of  $\tau_{xz}$  and  $\tau_{yz}$  in terms of  $T$ . Here, for the displacement formulation, we will avoid  $\phi$ . The stress function is not involved. We will try to solve for  $w$  in the displacement formulation.

So,  $\tau_{xz}$  and  $\tau_{yz}$  resultant shear stresses were already obtained in terms of applied torque, and the torque- $\theta$  relation is also known, which are written here. Using the obtained torque- $\theta$  relation, we can express  $\theta$ , the angle of twist per unit length, in terms of  $T$  as  $\theta = \frac{T(a^2 + b^2)}{\pi G a^3 b^3}$ . Writing the non-zero stress components in terms of  $w$ , as discussed in the previous line,  $\tau_{xz} = G \left( \frac{\partial w}{\partial x} - \theta y \right)$ , and  $\tau_{yz} = G \left( \frac{\partial w}{\partial y} + \theta x \right)$ .

In terms of  $T$ , we know these two expressions of  $\tau_{xz}$  and  $\tau_{yz}$ . We are going to compare these two, and also, this particular  $\theta$  is replaced by using the  $\theta$ - $T$  relationship. If you replace this  $\theta$  here, obtain  $\tau_{xz}$  and  $\tau_{yz}$  from these two big expressions, and then compare these two equations with the previous ones. Previously obtained expressions of  $\tau_{xz}$  and  $\tau_{yz}$  from the first equation: from  $\tau_{xz}$ , you can write  $\frac{\partial w}{\partial x}$  as a function of the applied torque. From the second equation of  $\tau_{yz}$ , you can write  $\frac{\partial w}{\partial y}$  as a function of the applied torque as this:  $\frac{\partial w}{\partial x}$  will be  $\frac{T(b^2-a^2)y}{\pi G a^3 b^3}$ , and  $\frac{\partial w}{\partial y}$  will be  $\frac{T(b^2-a^2)x}{\pi G a^3 b^3}$ .

Combining these two, we get the partial derivative of the out-of-plane displacement component  $w$  with respect to  $x$  and  $y$ . If I combine these two and then integrate, that would result in the solution of  $w$ , which will come out to be  $\frac{T(b^2-a^2)xy}{\pi G a^3 b^3}$ . The integration constant is set to 0 because at the center point  $(0, 0)$ , the displacement should be 0. At one of the ends, we will have 0 displacement at the boundary. Keeping the center point non-moving at  $x = 0, y = 0, w = 0$ , we can eliminate the integration constant, and this will be the function of  $w$ . You can see  $w$  is proportional to  $x, y$  and also directly proportional to  $T$ . This is the solution for the displacement formulation for torsion of elliptic bars.

If you substitute this  $w$  into the Laplacian equation, you can verify that the obtained  $w$  satisfies the Laplacian equation, which is the governing differential equation for the displacement formulation of torsion.

Plotting this  $w$  (displacement contours) across the elliptic cross-section, you will get a plot like this. Similar to the stress function contour, here I am plotting the displacement contours.

This total elliptic cross-section is divided into four quadrants like this. This is along the positive  $x$ -axis, and this is along the positive  $y$ -axis. Note that for different quadrants, you will get different signs of  $w$ . If  $x$  and  $y$  are both positive (first quadrant), then the quantity  $x, y$  is positive; both  $x$  and  $y$  are positive. Since  $b < a$ , the term  $(b^2 - a^2)$  is negative, which results in  $w$  being negative in the first quadrant. Similarly, if you check

further,  $w$  is positive in the second quadrant, negative in the third quadrant, and once again positive in the fourth quadrant.

So, one-fourth of the cross-section is going inside (first quadrant), another one-fourth is coming out (second quadrant), the third quadrant is going inside (similar to the first quadrant), and the fourth quadrant is once again coming out (similar to the second quadrant). You can see the color code for first and third, you are having a blue region which is negative warping. Those cross-sectional points are going inside the plane. For second and fourth, that is coming out, and this clearly shows the warping of the cross-section. So, some part of the elliptic cross-section is coming out, some part is going in, and this shows the distortion of the cross-section for this non-circular elliptic bar.

#### Summary

- Torsion of an Elliptical Bar:

- Stress Function
- Relation between Torque and Angle of Twist
- Maximum Stress
- Displacement Field



In this lecture, we discussed the torsion of an elliptic bar using both the stress function approach and the displacement formulation approach. We obtained the expression of maximum stress, which occurs at the two endpoints of the minor axis, and we also obtained the torque and angle of twist relation.

Thank you.