APPLIED ELASTICITY

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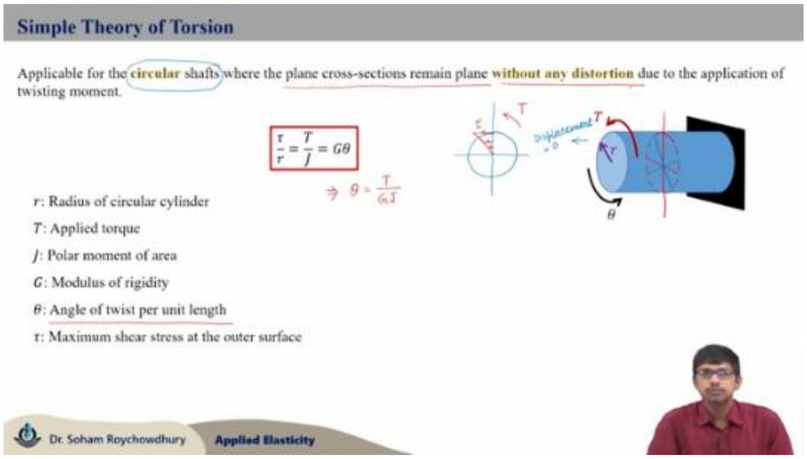
Week 8

Lecture 36: Torsion Problems I

Welcome back to the course on Applied Elasticity. In today's lecture, we are going to start a new topic: torsion problems in elasticity.



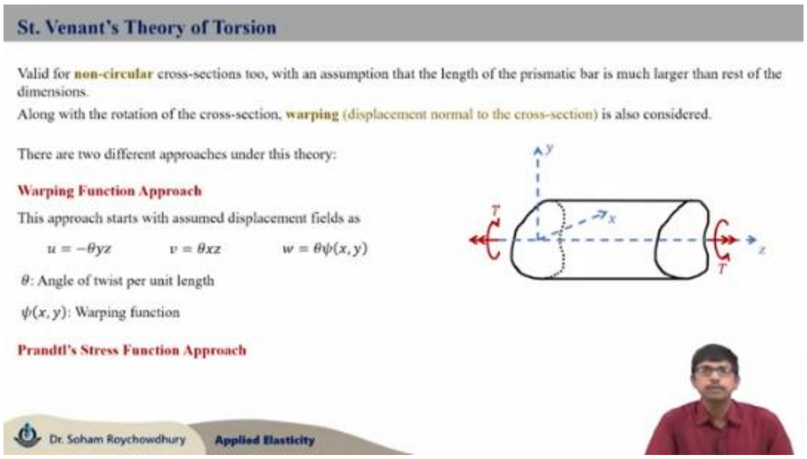
First, we will briefly discuss the simple theory of torsion, which is valid for shafts with circular cross-sections. The simple theory of torsion, normally covered in undergraduate solid mechanics courses, can only deal with prismatic bars or shafts with circular cross-sections. This theory assumes that plane cross-sections remain plane without any distortion during torsion.

If you consider a circular bar or a prismatic bar with a circular cross-section of radius r, which is free at one end and fixed at the other end, as shown in this figure. Let us apply a twisting moment or torque at the free end of this prismatic bar, where the magnitude of the torque is T. Due to the application of this torque, the bar will twist. Let us say θ is the measure of the twist. So, θ is the angle of twist per unit length of this circular bar.

This assumption states that plane cross-sections remain plane without any distortion. If you take any section here, you will get one circular cross-section before twisting. After twisting, the circular cross-section will remain a circle. Let us say this was before twisting; after twisting, these lines will simply shift, with no other difference observed. It will still be a circle, just rotated with respect to the centroidal axis. This is the meaning of that plane section remaining plane without any distortion, and as there is no distortion of the surface, the displacement component along this direction is equal to 0. So, displacement is equal to 0 along the axis of twist along the longitudinal axis of the bar. If these are true, then we can say the simple theory of torsion is applicable.

The simple theory of torsion can be used for solving torsion problems of the bar only with circular cross-sections. For any other cross-section, the distortion of the cross-section is unavoidable, and thus the simple theory of torsion cannot be used. For this particular theory, the relation between the applied torque, the resulting angle of twist per unit length θ, and the generated shear stress τ is like this: $\frac{\tau}{r} = \frac{T}{J} = G\theta$, where r is the radius of the circular cylinder, T is the applied torque, θ is the angle of twist generated per unit length due to the torsion, G is the modulus of rigidity or shear modulus of the material, J is called the polar moment of area, and τ is the maximum shear stress generated at the outer surface.

If this twist is applied on the circular cross-section, the maximum shear stress is expected at the outer surface. Let us say we are plotting the maximum shear stress. This will be the variation of τ. So, at radius equal to r at the outer radius, we are going to have maximum shear stress generated, and for the circular cross-section, which can be obtained by the simple theory of torsion, that $\frac{\tau}{r} = \frac{T}{J} = G\theta$, and from this, we can get a relation between θ and T as $\theta = \frac{T}{GJ}$, where note that θ is not the total angle of twist. This is the angle of twist per unit length. This particular theory is known to you. The limitation of this theory is that it is valid only for circular cross sections where distortion can be avoided.



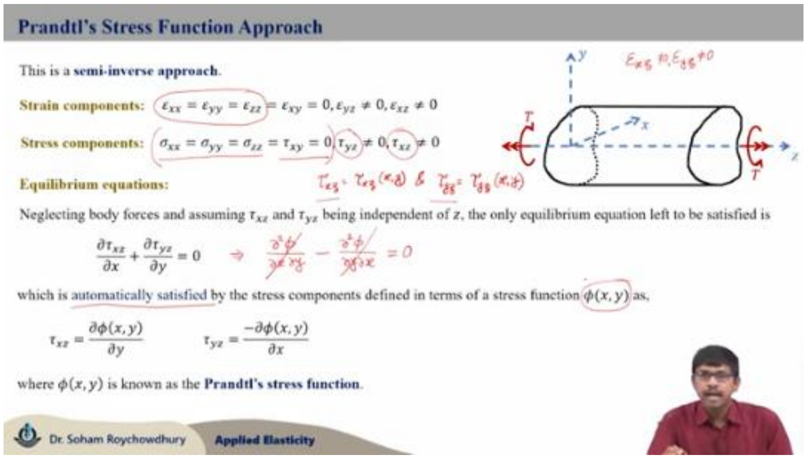
For any other cross section, there would be distortion, and thus this theory cannot be used. For that, we will be using Saint-Venant's theory of torsion. It is valid for non-circular cross sections, and allow out-of-plane displacement. This theory has one assumption: that the length of the prismatic bar is much larger compared to the other two dimensions. So, the cross-sectional dimensions are much smaller. They are much smaller compared to the length of the bar. Under that assumption, this particular Saint-Venant's theory of torsion is applicable, and it is valid for circular as well as non-circular cross sections, such as elliptic, triangular, rectangular cross-section bars, or any other arbitrary cross-sectional bars.

Let us consider a prismatic bar with z being the longitudinal axis, and it has some arbitrary cross section. You can clearly see this cross section is arbitrary, not varying along the length; the cross section is uniform, but the shape is not a simple circle - it has some arbitrary curve. This is subjected to a twisting moment T at both ends about the longitudinal axis z. This is the twisting of a prismatic bar with a general non-circular cross section, which we will try to solve with the help of Saint-Venant's theory of torsion. Here, this theory allows out-of-plane displacement normal to the cross section.

For the simple theory of torsion, as I told, displacement along the z-axis is not allowed. Here, displacement along the z-axis, that is normal to the cross-section, is allowed. It is allowed to have a non-zero value, and this particular phenomenon is called the warping of the cross-section. So, for a circular cross-section, there would be no warping, resulting in zero out-of-plane displacement w along the z-direction. Whereas, for a non-circular shaft, w along the z-direction (out-of-plane displacement) will be non-zero, and hence we can say warping is occurring along with the twisting of the bar.

There are two different approaches for solving the problem of twist or torsion of non-circular bars. The first approach is called the warping function approach, whereas the second one is called the Prandtl stress function approach. For the warping function approach, it starts with the assumption of the displacement field, where the out-of-plane displacement w is chosen to be a function known as the warping function. The in-plane displacements u and v are chosen like this: $u = -\theta yz$, $v = \theta xz$, and $w = \theta\psi(x, y)$. $\psi(x, y)$ is called the warping function, which depends only on the cross-sectional variables x and y, and θ is the same angle of twist per unit length.

Using this theory, starting with the displacement field, which includes this unknown warping function ψ, the torsion problem will be reduced to a problem of finding this warping function for the first approach, the warping function approach. Whereas, the second approach is based on the elasticity solution theory, where we will involve a stress function.



In this lecture, we will talk about the Prandtl stress function approach for solving the Saint-Venant's theory of torsion, which is valid for non-circular cross-sections. In the Prandtl stress function approach, this uses a semi-inverse approach where the equilibrium equation, we will try to satisfy automatically by defining the stress components as partial derivatives of a stress function, which is known as the Prandtl stress function.

First, let us start with the strain components. For this particular type of torsion problem, when the twist is given about the z-axis and no axial forces are there along x, y, and z, all the axial normal strains - ε_{xx}, ε_{yy}, ε_{zz} - should be 0. Applied torque about the z-axis can cause only two non-zero strains - non-zero shear strains: ε_{yz} and ε_{xz}. These two are only not equal to 0 as T is applied about z-axis and another in-plane shear strain ε_{xy} should also be 0.

With this, we have 4 stress components to be 0, and only 2 out-of-plane shear strains ε_{xz} and ε_{yz} are non-zero. As no axial forces (no normal forces) are there, all the normal strains are 0 and in-plane normal shear strain ε_{xy} is also 0, and only 2 out-of-plane shear strains will be resulted due to application of the torque about z-axis. With the help of these strain components, we can obtain the stress components.

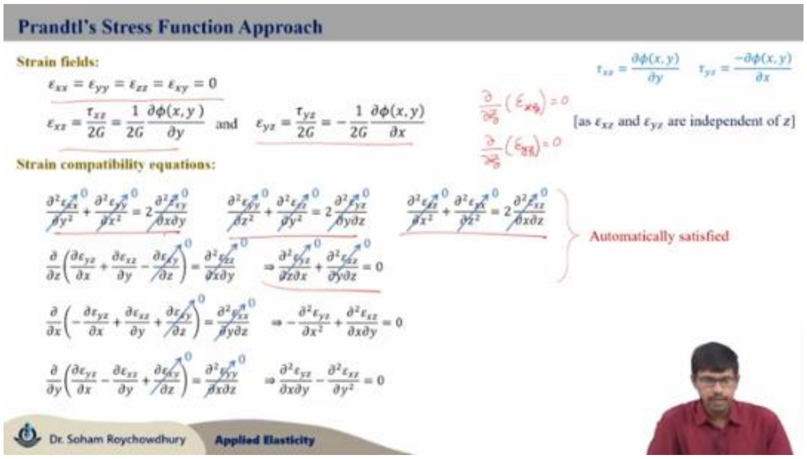
Now, all three normal strains being 0, we must have all three normal stresses to be 0. As ε_{xy} is 0, the corresponding shear stress τ_{xy} would also be 0. However, these two - τ_{yz} and τ_{xz}, the out of plane shear stresses - would be non-zero as the corresponding strain components are non-zero.

Coming to the equilibrium equation, as four of the stress components are 0 and we are going to neglect the body forces, with that, out of three equilibrium equations, we will be left with only one equilibrium equation. σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy} are set to 0, body forces are neglected, and we are also assuming that τ_{xz} is function of x and y only, and similarly, τ_{yz} is also function of x and y only.

These are independent of z because the cross section is uniform; along z, the cross section is not changing. The applied torque is not changing, that is also constant along z. So, this resultant shear stresses τ_{xz} and τ_{yz} must be independent of z; they are non-zero but not dependent on z. So, the $\frac{\partial \tau_{xz}}{\partial z}$ term is 0, and the $\frac{\partial \tau_{yz}}{\partial z}$ term is also 0. Putting all these assumptions into the equilibrium equation, the two equilibrium equations would be automatically satisfied along the x and y axes.

The z-axis equilibrium equation would be the only remaining one, as $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$. This equation can be automatically satisfied if we define these two non-zero stress components, τ_{xz} and τ_{yz}, in terms of a stress function of this particular form: $\tau_{xz} = \frac{\partial \phi}{\partial y}$, and $\tau_{yz} = -\frac{\partial \phi}{\partial x}$. So, if you substitute τ_{xz} as $\frac{\partial \phi}{\partial y}$, the first term will be $\frac{\partial^2 \phi}{\partial x \partial y}$, and the second term, substituting τ_{yz} as $-\frac{\partial \phi}{\partial x}$, would be $-\frac{\partial^2 \phi}{\partial y \partial x}$. These two would cancel each other, and thus, this equation is automatically satisfied.

So, this particular choice of stress function, ϕ, is called the Prandtl stress function, which is used for solving torsion problems of non-circular bars. We have reduced our problem to a single unknown problem where $\phi(x, y)$, the Prandtl stress function, is the only unknown we need to solve. And with respect to that, using these two equations, the shear stress components are obtained, which all automatically satisfy the equation of equilibrium.



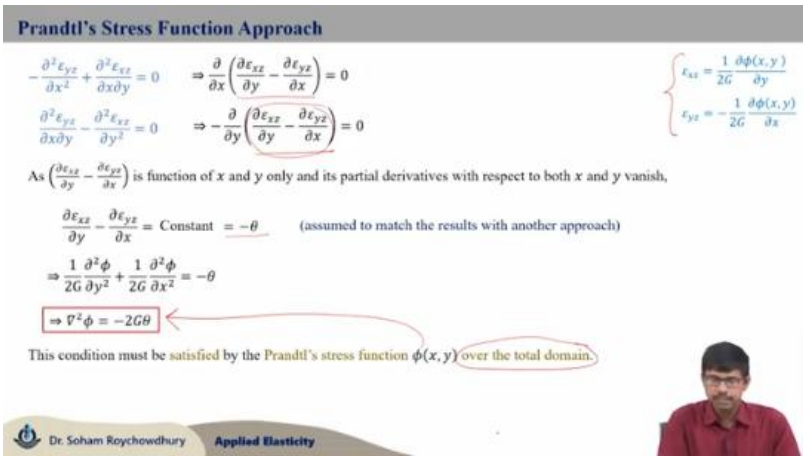
Moving forward, we will try to derive the strain fields. So, these two τ_{xz} and τ_{yz} are defined in terms of ϕ as $\frac{\partial\phi}{\partial y}$ and $-\frac{\partial\phi}{\partial x}$, respectively. The two corresponding non-zero strain components, ε_{xz} and ε_{yz}, can be written as follows: $\varepsilon_{xz} = \frac{1}{2G}\frac{\partial\phi}{\partial y}$, $\varepsilon_{yz} = -\frac{1}{2G}\frac{\partial\phi}{\partial x}$, and the rest of the strain components (ε_{xx}, ε_{yy}, ε_{zz}, and ε_{xy}) are all 0, as we had already discussed. These are the two non-zero strain components we are left with.

Now, these two non-zero strains must satisfy the strain compatibility equation to ensure a unique displacement field. Since four of the strain components are already zero, many of the strain-displacement equations would be automatically satisfied. First, let us write down all six strain-displacement compatibility equations. Here, you can see the six strain-displacement compatibility equations are written. The first 3 are of similar nature, whereas these 3 are of similar nature.

Now, on these 6 equations, we will impose these 4 strain constant components to be 0: ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy} to be 0. If we enforce that, these many terms will go to 0. After that, the last 3 equations will only be left with 3 terms: 2 terms on the left-hand side, and the right-hand side we are having a 0 term. Here, once again, we will check this expression of the non-zero strain components: ε_{xz}, ε_{yz}. These 2 are dependent on ϕ only, where ϕ is a function of x and y.

So, ϕ is independent of z, meaning ε_{xz} and ε_{yz} are also independent of z. Thus, this term: $\frac{\partial^2\varepsilon_{xz}}{\partial x\partial z}$ will also go to 0. Whenever you are having $\frac{\partial}{\partial z}$ term of this ε_{xz}, that would be 0.

Similarly, $\frac{\partial \varepsilon_{yz}}{\partial z}$ would also go to 0. Setting all those to 0 because ϕ is independent of z, we will see the first 4 equations. These four strain compatibility equations are automatically satisfied. We are only left with these two strain compatibility equations, which our displacement and strain fields should satisfy to ensure unique displacement components.



Moving forward, starting with these two, these two can be rewritten in this form. The first equation would be $\frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}\right) = 0$, and the second term is $-\frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}\right) = 0$.

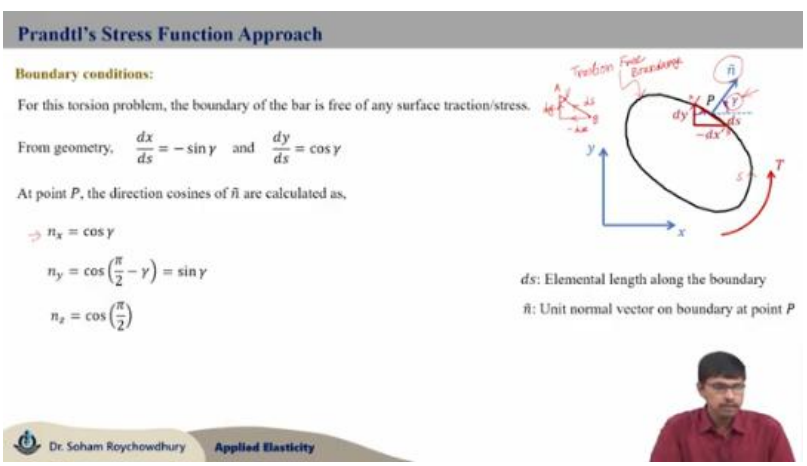
You can see the term remaining within the bracket is the same for both equations. So, $\frac{\partial}{\partial x}$, the partial derivative with respect to x of that term, is 0, and the partial derivative with respect to y of that term is 0, and this term is only a function of x and y. This can be true only if this entire term within the bracket is a constant. So, if the partial derivative of this function, $\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}$, is vanishing for both partial derivatives with respect to x and with respect to y, that means this term must be a constant.

Let us choose that constant to be $-\theta$. Why minus θ? This is our choice; we can assign any name to this constant. If I choose this as $-\theta$, we will be able to match the obtained equation with the equations derived from another approach, the warping function approach. You can take any other name, like c or something, but if you do, you will eventually have to relate this solution to the warping function solution, and then we will derive a relation with θ, the angle of twist per unit length. So, to match the results of the two different approaches, we are choosing this constant to be $-\theta$, and the angle of twist is obviously a constant since the applied torque is constant.

Substituting ε_{xz} and ε_{yz} in terms of ϕ. We know ε_{xz} and ε_{yz} in terms of ϕ. If you substitute it back here, this equation would be $\frac{1}{2G}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-\theta$. So, $\frac{1}{2G}\nabla^2\phi=-\theta$. And $\nabla^2\phi$ can be written as $-2G\theta$. This is the governing equation we need to solve or use to choose ϕ.

ϕ must satisfy this equation: $\nabla^2\phi=-2G\theta$, and this should be satisfied over the entire domain. Over the total problem domain, over the entire cross-section, the Prandtl stress function ϕ must satisfy this particular equation. If you recall for the bending problem, the governing equation was the bi-harmonic equation. The choice of ϕ must satisfy the bi-harmonic condition. Here, that is not so.

The chosen stress function, the Prandtl stress function, must satisfy this equation: $\nabla^2\phi=-2G\theta$. Only then can this be used as a possible stress function for the torsion problem because only then the strain compatibility equation would be satisfied, and this should be satisfied over the entire domain for all values of x and y. After this, let us see if we are supposed to impose any more conditions on the choice of ϕ.



That should be imposed on the basis of the boundary conditions of the problem. For the torsion problem, as the the lateral surface of the prismatic bar is free of any kind of surface fractions or stresses, that boundary should be free of stress. This condition is required to be imposed.

Let us consider this particular boundary. We are looking from the z-axis, so, the boundary curve is given. This is a traction free boundary. We are not having any surface traction present along this boundary. As this is a traction free boundary, we should have all the surface traction components t_x, t_y, t_z should be 0. Now, we are choosing a point P on the boundary, and along that, ds is a small elemental length; s being a path coordinate here. Along the boundary, the coordinate is taken to be s.

So, we are taking a small length of ds from here to here around point P and $\hat{n}$ is the unit normal at point P on the boundary curve Γ. This ds, as you are going along the boundary, along s direction, this instead of going along the boundary we can say we are going like this. First we are going along the negative x-axis by amount $-dx$, then, we are going on the along the positive y-axis by amount dy.

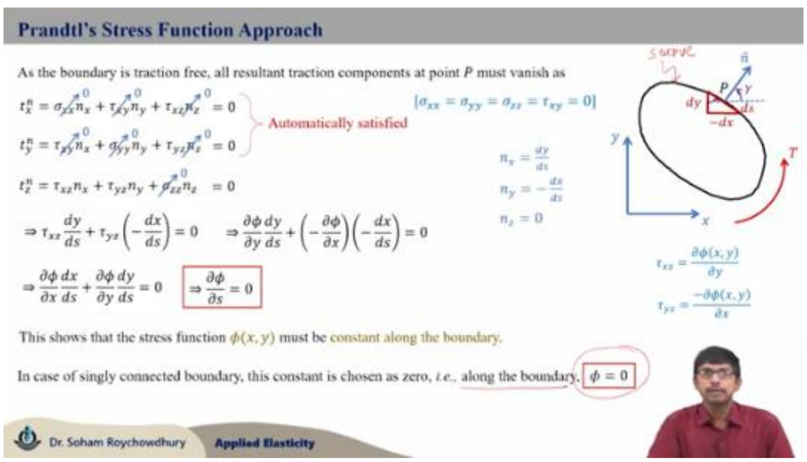
Let us say this is a point A, this is a point B, and we are starting from point B going to point A. One possible option is going along the boundary ds which is basically arc, not a straight line, but if ds is very small, we can assume this arc to be almost a straight line. Another alternate path instead of going from B to A along ds, we can go along $-dx$, along the negative x-axis and then along the y-axis, we can go by an amount dy. With that also, we can reach point A. So, the $d\tilde{s}$ vector is nothing but $-dx\hat{\imath} + dy\hat{\jmath}$.

Like that, we can also write the $d\tilde{x}$ vector. Now, from geometry, considering this angle being γ, γ is the angle that the unit normal $\hat{n}$ is making with the x-axis. From this triangle, this angle is γ. From there, we can write $\frac{dx}{ds}$ as $-\sin\gamma$ and $\frac{dy}{ds}$ as $\cos\gamma$. The base of the triangle is $-dx$, the height is dy, and with that, $\sin\gamma$ is $-\frac{dx}{ds}$, and $\cos\gamma$ is $\frac{dy}{ds}$.

Now, if you are writing the direction cosines for this unit normal $\hat{n}$, the cosine of the angle this unit normal is making along the x, y, and z axes at point P, If we write those, we are naming them as n_x, n_y, and n_z. These are the direction cosines of the unit normal

$\hat{n}$ at boundary point P. n_x is the cosine of the angle that the $\hat{n}$ vector is making with the x-axis, which is nothing but γ.

So, n_x is $\cos\gamma$. n_y is the cosine of the angle that the $\hat{n}$ vector is making with the y-axis, which is $90° - \gamma$. So, $\cos(90° - \gamma)$, which is nothing but $\sin\gamma$, and n_z is the angle the $\hat{n}$ vector is making with the z-axis. The z-axis is perpendicular to the xy-plane on which the $\hat{n}$ vector is lying. So, this n_z should be $\cos(90°)$, the z-axis being orthogonal to the plane containing $\hat{n}$, and thus, this would be 0. Substituting $\sin\gamma$ as $-\frac{dx}{ds}$ and $\cos\gamma$ as $\frac{dy}{ds}$, we can write n_x as $\frac{dy}{ds}$, n_y as $-\frac{dx}{ds}$, and n_z as 0.



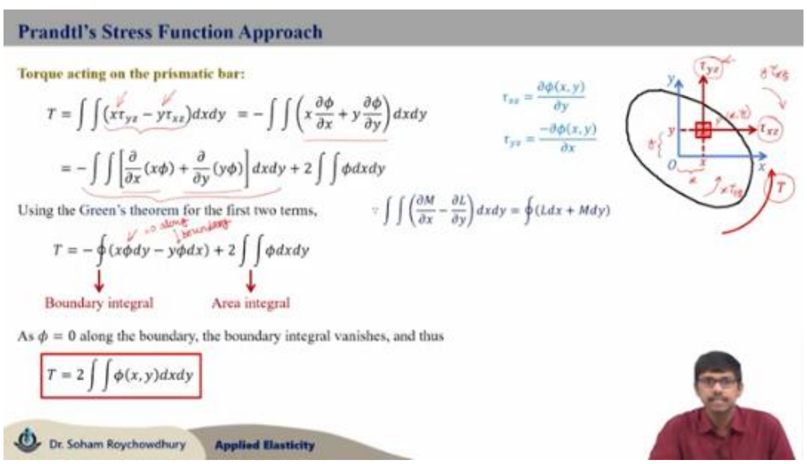
Moving forward, as this entire boundary is free of any kind of traction, all the surface traction components t_x^n, t_y^n, and t_z^n must be 0 over the boundary. For the entire boundary, we must have all three equations to be satisfied: $t_x^n = 0$, $t_y^n = 0$, $t_z^n = 0$. Using the relation between the Cauchy stress components and surface traction components, like $\tilde{t} = \tilde{\tilde{\sigma}}\tilde{n}$ or $\tilde{\tilde{\sigma}}^T\tilde{n}$. And $\tilde{\tilde{\sigma}}$ and $\tilde{\tilde{\sigma}}^T$ are the same due to the symmetry of $\tilde{\tilde{\sigma}}$.

Expanding this into the x, y, and z directions, we will be getting these three equations. We already know that σ_{xx}, σ_{yy}, σ_{zz} and τ_{xy} are 0 for the torsion problems. Those terms will go to 0 and also n_x, n_y, n_z we had evaluated out of which n_z is 0, n_x is $\frac{dy}{ds}$, n_y is $-\frac{dx}{ds}$. Setting n_z to be 0, the first two equations would be automatically satisfied, and we will be left with only one equation which is the last equation $\tau_{xz}\frac{dy}{ds} + \tau_{yz}\left(-\frac{dx}{ds}\right) = 0$.

Here, we are substituting τ_{xz} and τ_{yz} in terms of stress function ϕ, and if you simplify this, cancel the minus sign, this equation would be coming out as $\frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} = 0$.

Now, ϕ is function of x and y, and ds can be written in terms of dx and dy. Thus, this equation is nothing but $\frac{\partial\phi}{\partial s} = 0$, or we can even write $\frac{d\phi}{ds} = 0$. So, along the path coordinate s, ϕ cannot change if $\frac{\partial\phi}{\partial s} = 0$ means ϕ should be constant along s. This boundary curve is defined by this s curve. Along the s curve, ϕ must be constant. This means the Prandtl stress function, ϕ, must be constant along the boundary. This is another constraint we are having. $\nabla^2\phi = -2G\theta$ over entire domain for all values of x and y, and ϕ must be constant along the boundary.

For the case of singly connected boundary, like this, means we are not having any hollow pipe, only outer boundary is existing, for such cases we choose that constant to be 0. The condition $\phi = 0$ should be satisfied along the boundary. The Prandtl stress function must satisfy these two conditions: $\phi = 0$ along the boundary, and $\nabla^2\phi = -2G\theta$.



Coming to the relation between the torque and the angle of twist. Let us try to derive the expression for torque. The torque acting on the system with a non-circular cross-section can be written in terms of the generated stress components as follows: $T = \iint(x\tau_{yz} - y\tau_{xz})dxdy$. Let us consider this particular cross-section figure, where I am taking an element at point P, which is subjected to τ_{xz} acting on the z-plane, along the x-axis, and

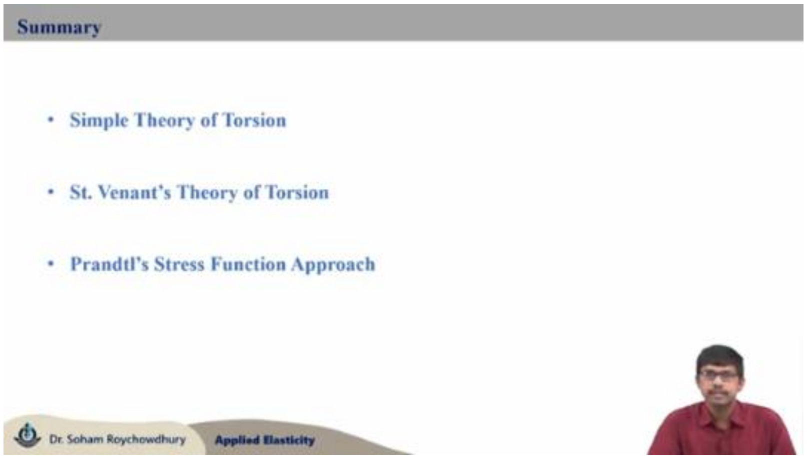
τ_{yz} acting upward at point P on the z-plane, along the y-axis. Now, both of them will create some torque about point O.

If you first consider τ_{yz}, This point P has coordinates (x, y). This distance is x, and this distance is y, thus, the torque created by τ_{yz} is $x\tau_{yz}$ created in the counterclockwise direction due to this component. Similarly, τ_{xz} creates a clockwise torque of $y\tau_{xz}$ about point O. Since it is clockwise - opposite to the previous torque due to τ_{yz} - we add a minus sign. So, this first term is the torque due to τ_{yz}. The second term is the torque due to τ_{xz}. They are opposite in nature. We are taking counterclockwise torque to be positive.

This is net torque due to stress distribution for the small element around point P of dx, dy side lengths. So, total torque over the entire domain entire cross section will be $\iint (x\tau_{yz} - y\tau_{xz}) dxdy$ and that should be or must be equal to this applied torque T from the balance of the torque because this torque results this particular stresses the non-zero shear stresses - τ_{xz} and τ_{yz}. Substituting τ_{xz}, τ_{yz} in terms of ϕ, this equation will be like this.

We can further simplify this in this particular fashion where $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}$ is written as two sets of integral: 2ϕ we are adding, and 2ϕ we are subtracting. So that the first term becomes $\iint \left(-\frac{\partial}{\partial x}(x\phi) + \frac{\partial}{\partial y}(y\phi)\right) dxdy + \iint 2\phi dxdy$. If you expand these two partial derivatives using the chain rule, then one term will get cancelled and we will be getting this initial one back. Using the Green's theorem, the first area integral is converted into the boundary integral as: $\iint \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy = \oint (Ldx + Mdy)$. This first area integral term is converted into a boundary integral term of $-\oint (x\phi dy - y\phi dx) + 2\iint \phi dxdy$.

If you recall, over the entire boundary, this term ϕ must be 0. This is equal to 0 along the boundary and hence, this boundary integral term would vanish and total T will just be $2\iint \phi dxdy$. Applied torque is twice integral of the Prandtl stress function. As the boundary integral term vanish, we will only be left out with area integral term.



In this lecture, we discussed the simple theory of torsion and then, introduced the Saint-Venant's theory of torsion, which is valid for both circular and non-circular cross-section. And then, in detail, we discussed the solution technique for the Prandtl stress function approach to torsion problems.

Thank you.