APPLIED ELASTICITY

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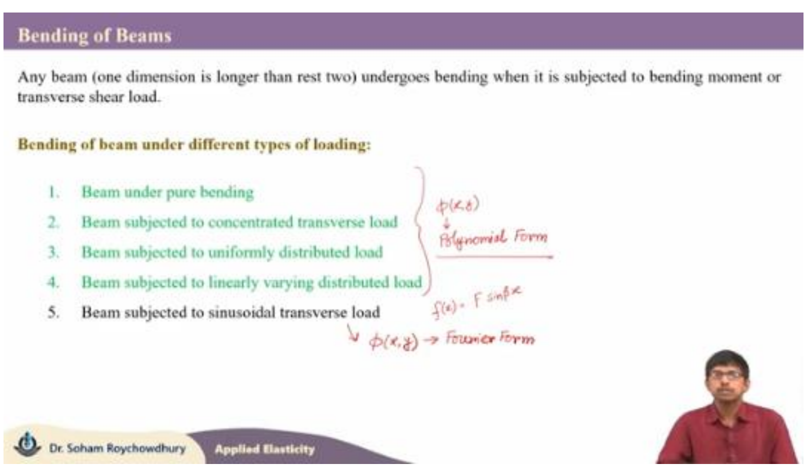
Week 7

Lecture 35: Bending of Beams V

Welcome back to the course on Applied Elasticity. In today's lecture, we are going to continue our discussion on the bending of beams. We are considering the bending of beams under different types of loading. Beams are one-dimensional continuum that undergo bending when subjected to either a bending moment or a transverse shear load.



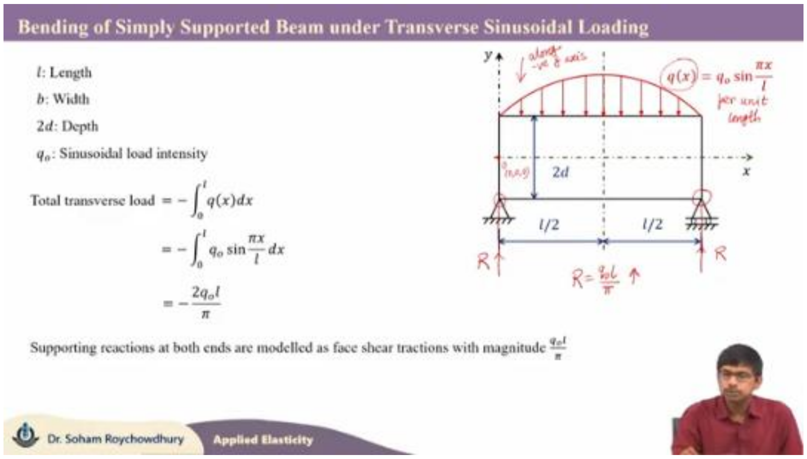
We have already discussed four different types of beam bending problems: when the beam is subjected to either a pure bending moment, a concentrated shear load at one particular section, a uniformly distributed load over a span, or a linearly varying distributed load over a span.

Coming to the last type of loading, we will consider, here the beam is subjected to a transverse load with sinusoidal intensity. The transverse load expression $f(x)$ is written as some constant F multiplied by $\sin(\beta x)$. If we are able to write the load, or if the load acting on the beam is of this particular form, $\sin(\beta x)$ times some intensity F, then we need to solve that problem using a different approach. That is what we are going to discuss now, along with all these previous five problems.

The stress function $\phi(x, y)$ was chosen in polynomial form. In one of the previous week's lectures, we discussed that there are two possible forms of stress function choice. One is called the polynomial form, and the other is called the Fourier form. So, in the x, y coordinate rectangular Cartesian coordinate system, the polynomial form can be chosen as a combination of different-order or degree polynomials - second, third, fourth, fifth - all of which we used for solving the first four types of beam bending problems.

Coming to the last one, if the loading is not just a point load, a uniformly distributed load, or a linear function, if this is a trigonometric function, here it is $\sin(\beta x)$, it may be $\cos(\beta x)$, $\sinh(\beta x)$, or $\cosh(\beta x)$. If the trigonometric or hyperbolic functions are required to be used for defining the transverse loading, for such cases this polynomial form of solution will fail, and we must use or choose $\phi(x, y)$ in the Fourier solution form.

And for a much more general type of loading, instead of a single sine wave, if it is any general periodic function of x, in that case, the Fourier series solution is required to be used. However, we are going to consider a simple sinusoidal transverse load for which the Fourier form of stress function $\phi(x, y)$ we will be using.

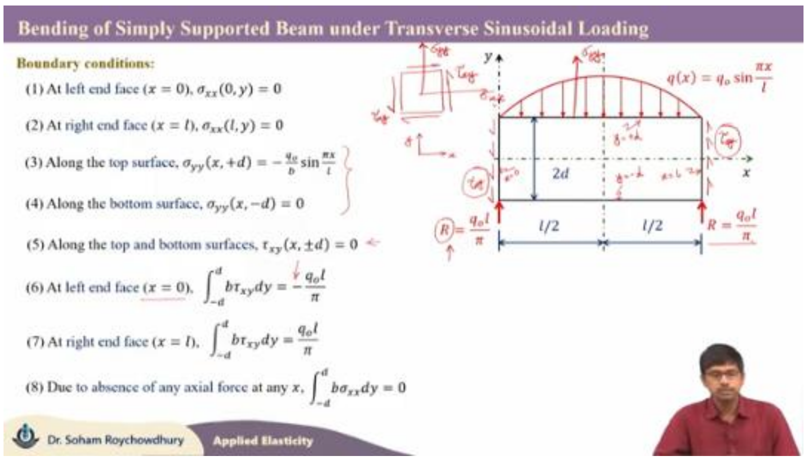


Let us consider a simply supported beam like this of length l and total width $2d$. The origin I had taken here is $(0, 0, 0)$; origin at the left edge face. y is varying between $-d$ to $+d$, x is varying between 0 to l. It is subjected to a sinusoidal transverse shear load on the top face in the downward direction with $q(x)$ being $q_0 \sin\left(\frac{\pi x}{l}\right)$, where q_0 is the sinusoidal load intensity. This is also defined per unit length; q_0 has the unit of force per unit length. l is length, b is width, and $2d$ is depth, total thickness of the beam.

Now, once the beam is subjected to this sinusoidal load intensity, for finding the reaction forces at both left hinge and right hinge, two simple support points, two hinge or pin joints, we will be having the vertical reaction forces, which will be obtained by using the symmetry of the problem and also by using the vertical force balance.

So, first, we need to find out the total downward vertical force acting on the beam due to the sinusoidal load intensity $q(x)$. That can be obtained by integrating this $q(x)$ over the entire length. $\int_0^l q(x)dx$, $q(x)$ is downward shear load per unit length. If you integrate that dx from 0 to l, total length of the beam, we get the total transverse load. A negative sign is added because the total transverse load is acting downward along the negative y-axis. We have the total downward transverse load as $\frac{2q_0 l}{\pi}$, which is the integral of $q(x)dx$ from 0 to l with a minus sign. By substituting $q(x)$ as $q_0 \sin\left(\frac{\pi x}{l}\right)$, we get the total transverse load acting on the beam as $-\frac{q_0 l}{\pi}$.

These would be supported by two reaction forces. Thus, the reaction at both ends - here at the left support and also at the right support - will be this reaction force R, and this R is equal to $\frac{q_0 l}{\pi}$, which is upward. So, the total R from both hinges will balance this transverse downward load acting on the top face. In the free body diagram of the beam, I have drawn R as $\frac{q_0 l}{\pi}$ upward at both hinge points. The hinge or pin supports (simple supports) are removed, and the top face is acting with $q(x)$, the sinusoidal load intensity of $q_0 \sin\left(\frac{\pi x}{l}\right)$ in the downward direction, defined per unit length of the beam.



Let us try to write down the boundary conditions. Starting with the left edge, the left edge is the $x = 0$ face, whereas the right edge is the $x = l$ face. So, at $x = 0$ (the left edge), there is no axial stress. Thus, σ_{xx} should be 0 for the left free edge; no axial stress would be present. However, shear stress would be present on the left edge because of this R; this R is causing a shear stress distribution on the left edge. The same applies to the right edge. Similarly the right edge is also free for any kind of normal traction. So, $\sigma_{xx}(l, y)$ is also 0 because right edge is defined as $x = l$ plane. 1 and 2 defines the left and right faces being free of any kind of normal surface fraction.

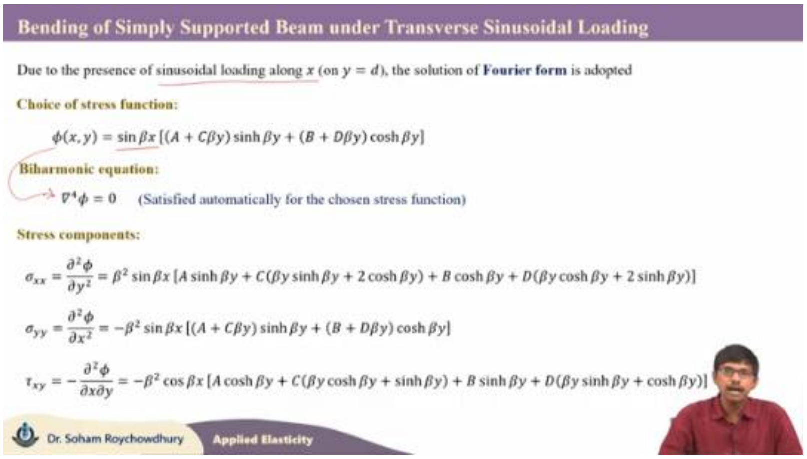
Coming to the top face, the top face is defined with $y = +d$, bottom face is defined with $y = -d$. The normal stress on the top face σ_{yy} is non-zero because of the sinusoidal load. σ_{yy} for the top face $(x, +d)$ is equal to $-q_0 \sin\left(\frac{\pi x}{l}\right)$. This $\frac{q_0}{b}$ term is coming to match the dimension. q_0 is defined as force per unit length, and σ_{yy} is stress. So, we need to divide the q_0 with the width of the beam at any particular section x, and thus, $\frac{q_0}{b}$ is the sinusoidal load intensity per unit area. σ_{yy} on the top face - $y = +d$ plane - should be equal to $-q_0 \sin\left(\frac{\pi x}{l}\right)$. Whereas, for the bottom face, that being free of any kind of normal or shear stress, σ_{yy} for the bottom face $(x, -d)$ would be 0. These two are the normal surface tractions on top and bottom face.

Similarly, we can write the shear surface traction on the top and bottom face to be 0 because no shear is acting on the top, and no shear force, external shear force is acting on the bottom. For both $y = \pm d$, the shear surface traction would be 0. So, with this 5 on all four edges, we have defined the external surface traction. Mostly, they are 0; only for the top face, we have σ_{yy} to be non-zero, which is $-q_0 \sin\left(\frac{\pi x}{l}\right)$.

Coming to the shear traction on the left edge. If you consider the left edge, there we have this R, equal to $\frac{q_0 l}{\pi}$, upward reaction force acting. Considering any 2D stress element with this being positive x and this being positive y, the τ_{xy} is positive on this positive x plane, and on the negative x plane downward τ_{xy} is positive. If you look at this left hand side plane, here, τ_{xy} should be downward by sign convention, whereas, on the right hand side plane, that being a positive x plane, τ_{xy} should be positive upward, that is along the positive y-axis. For the left hand side that is $x = 0$, left end face boundary condition 6 is $\int_{-d}^{+d} b\tau_{xy} dy$, that would give us the net downward reaction force because by convention τ_{xy} direction is downward here.

So, $\int_{-d}^{+d} b\tau_{xy} dy$ would be the net downward reaction, but external load is acting upward, that is $\frac{q_0 l}{\pi}$. As the directions are different, we have this minus sign. For left edge, $x = 0$, $\int_{-d}^{+d} b\tau_{xy} dy = -\frac{q_0 l}{\pi}$. Similarly, you can write for the right end face $x = l$. Here, τ_{xy} upward is positive, which is in the same direction as R. Here, there would be no negative sign. $\int_{-d}^{+d} b\tau_{xy} dy = \frac{q_0 l}{\pi}$ for the right end face; this is the shear traction distribution at $x = l$ face.

Finally, due to absence of any axial loading at any x, at any cross section, $\int_{-d}^{+d} b\tau_{xy} dy = 0$. Integral of the axial stress at any cross section over the total area is 0 as the problem is free of any kind of axial force along the x direction. These are the 8 boundary conditions available.

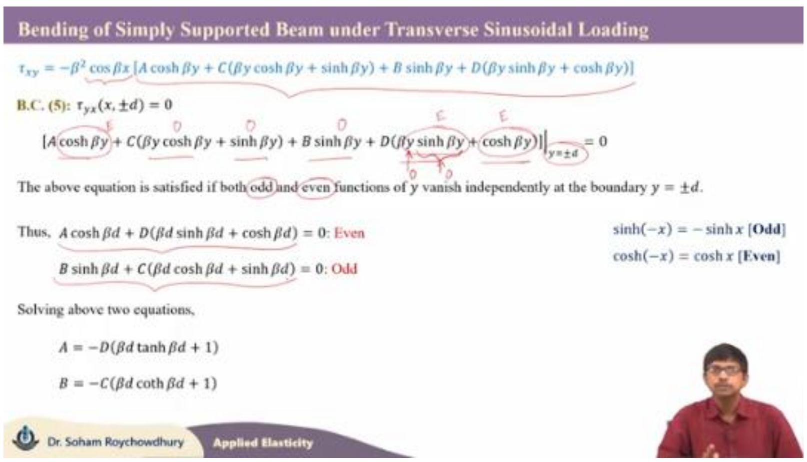


As I had already told, we are going to use the Fourier form of solution due to the presence of sinusoidal loading along x, for the top edge $y = d$. We are going to choose our stress function in this form. If you try to recall the last week's lecture where the general Fourier solution $\phi(x, y)$ was written which was having 5 set of terms. One is $\alpha = 0$, $\beta = 0$ term which is a polynomial. Then that was followed by one term with $\sin(\beta x)$, one with $\cos(\beta x)$, one with $\sinh(\beta x)$, another was $\cosh(\beta x)$.

Here, the transverse shear load is acting along the x-axis, on the top face, where it is function of some $\sin\left(\frac{px}{l}\right)$ type. As the sinusoidal loading is along x and that is represented by a sin trigonometric function, we are only considering the $\sin(\beta x)$ term in the Fourier form of solution. If a cosine loading was acting on the top face, then $\cos(\beta x)$ and the corresponding term should be used. Based on the type of external loading present, you have to choose the proper form of Fourier or proper term in the Fourier form of stress function solution with which the problem can be properly represented.

With this choice of stress function, if you substitute it in the biharmonic equation $\nabla^4\phi = 0$, you can see this is automatically satisfied. This is left as an exercise; you can substitute this form of ϕ here and verify that this particular Fourier form solution of stress function would automatically satisfy the biharmonic equation. That was already done while deriving the Fourier form solution. We had imposed this biharmonic condition for finding the Fourier form of stress function ϕ.

Coming to the stress component, with this form of ϕ, if we obtain σ_{xx}, σ_{yy} and τ_{xy}, we will get the three stresses like this. These three equations you will get for the three non-zero in-plane stress components: σ_{xx}, σ_{yy}, and τ_{xy}. These three stresses or stress functions involve four unknown constants: A, B, C, and D, which we need to obtain by using the boundary conditions.

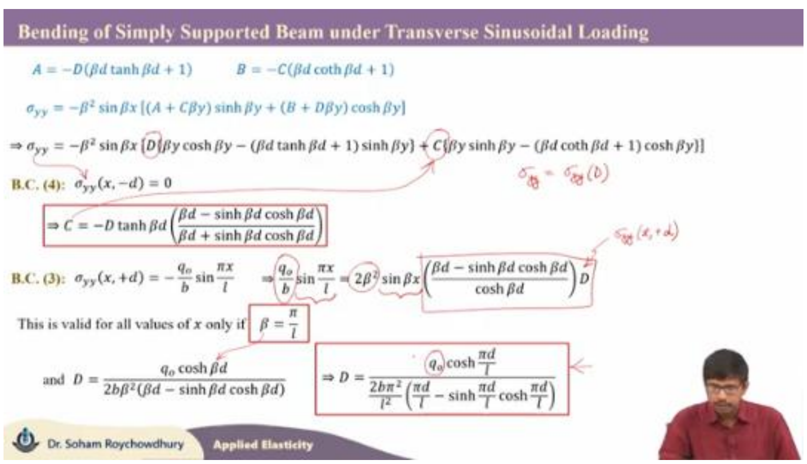


Starting with the fifth boundary condition: τ_{xy} or τ_{yx} at the top and bottom face is 0. $\tau_{xy}(x, \pm d) = 0$. I have written the general form of τ_{xy}, replacing y with $\pm d$ and x remaining as x, setting that to 0. As this $\cos(\beta x)$ term was there in τ_{xy} and this to be valid for all values of x with $y = \pm d$, $\cos(\beta x)$ cannot be 0. So, the coefficient of this $\cos(\beta x)$, this entire thing at $y = \pm d$, should be 0.

You can see here, we are having some terms which are even functions of y, and some terms which are odd functions of y. This equation can be satisfied to 0 for top and bottom face ($y = \pm d$), only if both odd functions of y and even functions of y go to 0 or vanish independently. How do you define this odd and even function? For example, sinh (sine hyperbolic) is an odd function: $\sinh(-x) = -\sinh(x)$. However, if you try that for cosine hyperbolic, $\cosh(-x) = \cosh(x)$; that is an even function. So, all the even functions and odd functions of y should independently go to 0.

Here, you can see a total of 6 terms are there. Out of 6, if you carefully check, 3 terms are even and 3 terms are odd. If you look at the even terms, this cosine hyperbolic is even term. Here, you can see, we have one more cosine hyperbolic that is also even term, and if you look at this particular term $D\beta y \sinh(\beta y)$, this total term, that is also even term because $\sinh(\beta y)$ itself is odd term and that is multiplied with y. So, this is one odd function, and y is another odd function. Product of these two odd functions will make the total function to be even. So, that is one even term.

All three even terms together should go to 0. Similarly, remaining term, this one, this one and this one, these 3 are odd terms. Summation of these 3 odd terms should go to 0 separately. Solving these two equations, we can express A and B in terms of D and C as $A = -D(\beta d \tanh \beta d + 1)$, and $B = -C(\beta D \coth \beta d + 1)$.



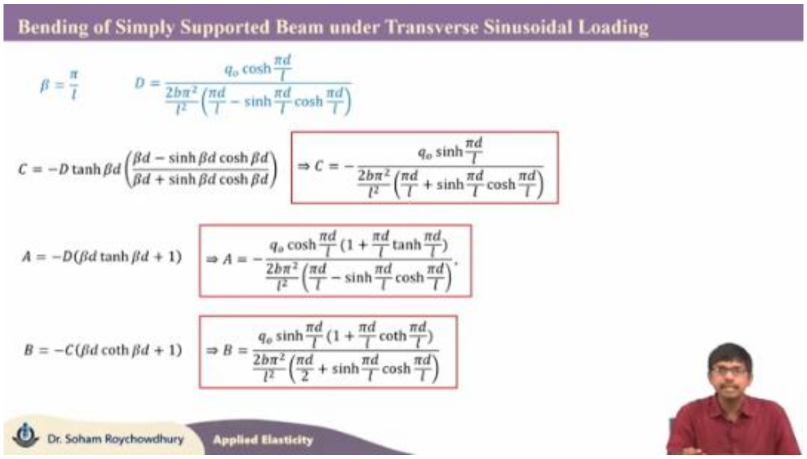
So, two of the constant we have replaced in terms of other two constants and this C and D are yet to be solved. So, I have written those A and B expressions in terms of C and D. Now, we are replacing those A and B in the expression of σ_{yy}. So, σ_{yy} stress is now written as function of C and D only; only two unknown constants at this stage. Using the fourth boundary condition, σ_{xx} for the bottom edge $(x, -d)$ is 0, i.e., bottom edge is free of any kind of surface fraction.

So, if you substitute σ_{yy} here, with $y = -d$, and set it to 0. Here, we will get one equation relating C and D. So, here, only D and C, two unknown constants, are present. Setting σ_{yy} to 0 at $y = -D$, we will relate C and D in this particular fashion. Coming to the third boundary condition on the top face, the normal traction σ_{yy} at $(x, +d)$ would be equal to $-\frac{q_0}{b}\sin\left(\frac{\pi x}{l}\right)$.

So, I am replacing σ_{yy} as $+d$ and also replacing this C, whatever we got from boundary condition 4. After replacing C in terms of D, σ_{yy} is now just a function of D, only one unknown constant. You can see the right-hand side is σ_{yy} at $(x, +d)$. It contains only one constant, which is D, and the left-hand side is $\frac{q_0}{b}\sin\left(\frac{\pi x}{l}\right)$. The minus sign has been canceled from both sides.

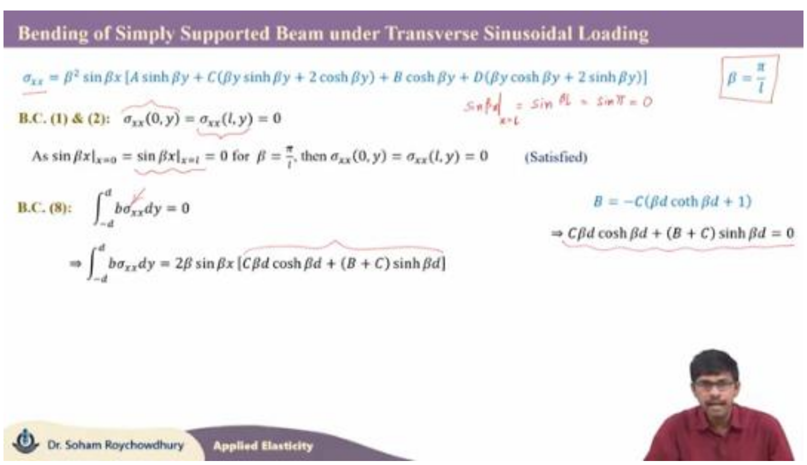
You can see, $\sin(\beta x)$ term is the only x-dependent part on the right-hand side, whereas on the left-hand side, the x-dependent part is $\sin\left(\frac{\pi x}{l}\right)$. The left-hand side constant is $\frac{q_0}{b}$, whereas the right-hand side constant is $2\beta^2$ times this big expression: D times this term within brackets - some numerator by denominator. This equation must be valid for all values of x. We must have $\sin\left(\frac{\pi x}{l}\right) = \sin(\beta x)$, or $\beta = \frac{\pi}{l}$, and D equal to this. The coefficient of the sine term should be the same, and the function within the sine term, $\left(\frac{\pi x}{l}\right)$ should be the same as (βx), thus $\beta = \frac{\pi}{l}$.

Putting that β here, as $\frac{\pi}{l}$, in all the terms in the expression of D, D would be $q_0 \cosh\left(\frac{\pi d}{l}\right)$ divided by $\frac{2b\pi^2}{l^2}\left(\frac{\pi d}{l} - \sinh\frac{\pi d}{l}\cosh\frac{\pi d}{l}\right)$. So, the expressions are quite big. We are getting this expression of D. As I had evaluated D in terms of external force q_0, it is not just a certain relation between the unknown constant. This D is completely evaluated in terms of external force q_0 and other parameters like d and l, the geometry.



You can replace this D back in these three expressions of A, B, C, and using $\beta = \frac{\pi}{l}$, you can write the expression of the remaining three unknown constants C, A, and B in terms of the externally applied sinusoidal load intensity q_0.

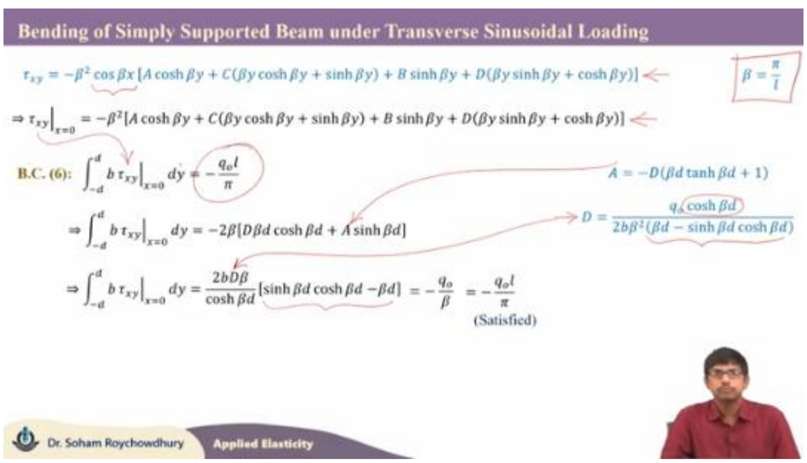
This completes the solution. We had obtained A, B, C, and D, with D being obtained first, and then using the interrelation between A, B, C, and D, we get all three other constants A, B, C in terms of the applied load intensity q_0, and those are obtained like this.



After obtaining these four constants A, B, C, and D, now we will check the remaining boundary conditions. σ_{xx} is written here. If you look at boundary conditions 1 and 2, those are σ_{xx} at $x = 0$ and $x = l$, that is, on both faces, the normal surface traction should vanish: $\sigma_{xx}(0, y)$, $\sigma_{xx}(l, y)$ should be 0. If you put $x = 0$ in this expression of σ_{xx}, the $\sin(\beta x)$ term will directly go to 0. Hence, for this $x = 0$, this particular term will go to zero; σ_{xx} is 0 as x goes to 0.

Coming to $x = l$ case, we know that $\beta = \frac{\pi}{l}$. So, $\sin(\beta x)$ at $x = l$ is nothing but $\sin(\beta l)$, which is equal to $\sin \pi$, and once again that term is 0. $\sin(\beta x)$ at $x = l$ is $\sin \pi$, which equals 0, and hence the second term σ_{xx} at (l, y) is 0. Both of these two boundary conditions are directly satisfied with $\beta = \frac{\pi}{l}$.

Coming to the next boundary condition, that is the case of zero axial force: $\int_{-d}^{+d} b\sigma_{xx} dy = 0$. Replacing the form of σ_{xx} in this particular equation and performing this integral over dy from $-d$ to $+d$, we will get this equation involving two constants, B and C. If you recall the relation between B and C, that was $B = -C(\beta d \coth \beta d + 1)$. If you write this $\coth \beta d$ as $\cosh \beta d$ divided by $\sinh \beta d$ and simplify it, this equation would be $C\beta d \cosh \beta d + (B + C)\sinh \beta d = 0$. If I compare this particular case, this particular term is exactly the same as the term present within this bracket of boundary condition 8. Hence, this term is equal to 0, so the boundary condition of zero axial force is also automatically satisfied.

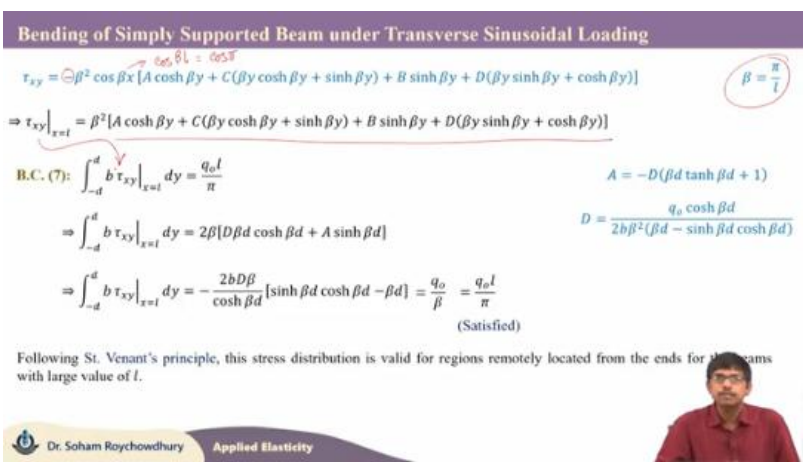


Coming to the last two remaining boundary conditions, which are defined over τ_{xy}, that is the vertical shear force at both the left and right edges. So, the τ_{xy} expression is given here with β being $\frac{\pi}{l}$. At $x = 0$, at the left edge, this $\cos(\beta x)$ term would go to 1, i.e., with $x = 0$, $\cot(\beta x) = 1$. τ_{xy} at $x = 0$ is given here. If you recall, boundary condition 6 was this: that is the reaction force at the left edge $x = 0$ edge. The integral from $-d$ to $+d$ of $b\tau_{xy}$ at $x = 0$ is equal to $-\frac{q_0 l}{\pi}$. τ_{xy} at $x = 0$ we have already evaluated; I will substitute that here, and then try to evaluate the left-hand side integral and verify whether it comes out to be $-\frac{q_0 l}{\pi}$ or not.

If you replace τ_{xy} at $x = 0$ on the left-hand side integral and integrate this, it would come out to be minus $-2\beta(D\beta d \cosh \beta d + A \sinh \beta d)$. Now, recall the relation between A and D; A was obtained to be $-D(\beta d \tanh \beta d + 1)$. Replacing that A here in terms of D, this integral can be written as a function of D only: $\frac{2bD\beta}{\cosh \beta d}(\sinh \beta d \cosh \beta d - \beta d)$.

Now, if you think about the expression of D we had obtained, the constant that was obtained like this, which was having a term in the numerator like this, which is almost the same as this term with just a negative sign. If I replace this in terms of D, substituting this D expression here, this term would get cancelled with the numerator factor of D. Also, this $\cosh(\beta d)$ term will also get cancelled. And with that, the right-hand side will be simplified to $-\frac{q_0}{\beta}$.

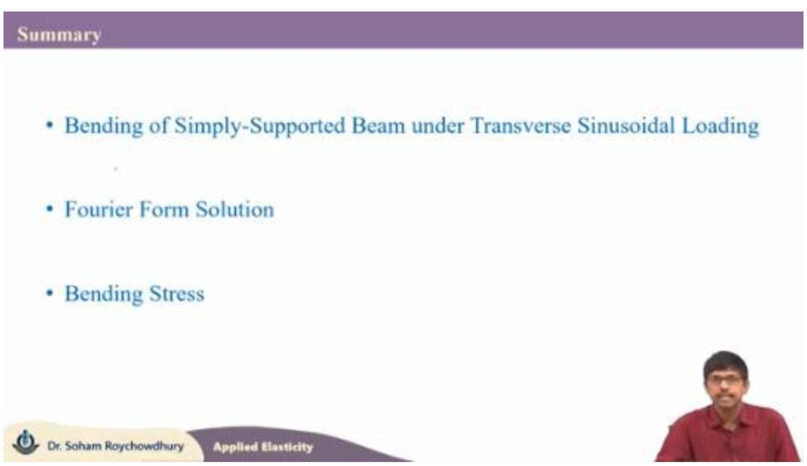
We know that $\beta = \frac{\pi}{l}$ for this problem. Substituting β with $\frac{\pi}{l}$, this would be $-\frac{q_0 l}{\pi}$, which was the same as the actual right-hand side given in boundary condition 6. Boundary condition 6 is also automatically satisfied with the obtained solution.



Coming to the last one, that is boundary condition 7. At the right edge, net vertical force is equal to $\frac{q_0 l}{\pi}$ with a positive sign. Integral $b\tau_{xy}$ at $x = l$ dy from $-d$ to $+d$ is $\frac{q_0 l}{\pi}$. This particular boundary condition is exactly the same as the previous one. Just two small differences: one is the previous one was at the left edge, $x = 0$, and this is at the right edge, $x = l$.

And for the last case, boundary condition 6, there was a minus sign on the right-hand side; here it is positive on the right-hand side, $+\frac{q_0 l}{\pi}$. So at $x = l$, if you replace and use this $\beta = \frac{\pi}{l}$, this $\cos(\beta x)$ would be $\cos(\beta l)$, which would be $\cos \pi$. That is -1, and if you replace that, this minus sign and that minus sign coming from $\cos \pi$ will cancel. So, τ_{xy} at $x = l$ will be β^2 multiplied by this big term within the bracket. Replacing these here in the integral, it is exactly the same as the previous one, just a minus sign is missing in the expression of τ.

Following a similar order of steps, writing A in terms of B first, and then finally putting the explicit expression of B in terms of q_0, you can obtain this left-hand side integral, $b\tau_{xy}$ at $x = l\ dy$ integral from $-d$ to $+d$, as $\frac{q_0}{\beta}$. Replacing β as $\frac{\pi}{l}$, you will get $\frac{q_0 l}{\pi}$. With this, the present Fourier form of solution satisfies all the boundary conditions exactly. And this state of stress, which is expressed by these four constants A, B, C, D, is valid for the regions remotely located from the two ends of the simply supported beam. Apart from those two edges, the stress field we obtained is going to give us the exact stress distribution using the Saint-Venant's principle for this particular Fourier form of solution.



In this particular lecture, we talked about the simply supported beam subjected to sinusoidal loading, and instead of a polynomial form of solution, we used the Fourier form of solution. Using that, we obtained the stress distribution. We got the expression of σ_{xx} (bending stress), σ_{yy}, and transverse shear stress τ_{xy} using the Fourier form of the stress function.

Thank you.