

APPLIED ELASTICITY

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Week 7

Lecture 34: Bending of Beams IV



The slide features a purple background with white and blue text. On the left, it reads 'COURSE ON: APPLIED ELASTICITY' and 'Lecture 34 BENDING OF BEAMS IV'. Below this, there are three diagrams: a beam on two supports with a downward arrow, a rectangular beam element, and a curved beam element. A stress tensor $T_{i,k}$ is shown next to the curved element. In the center, there is a 3D grid with axes i, j, k and m . On the right, there is a portrait of Dr. Soham Roychowdhury, a circular logo of IIT Bhubaneswar, and the text 'Dr. Soham Roychowdhury School of Mechanical Sciences Indian Institute of Technology Bhubaneswar'.

Welcome back to the course on Applied Elasticity. In this lecture, we are going to continue our discussion on the bending of beam problems.

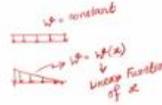
So, we had started our discussion on the bending of beams, where a beam is a one-dimensional continuum where the length of the element or the continuum is much larger compared to the other two dimensions, and it undergoes bending when subjected to either a bending moment or any kind of transverse shear loading.

Bending of Beams

Any beam (one dimension is longer than rest two) undergoes bending when it is subjected to bending moment or transverse shear load.

Bending of beam under different types of loading:

1. Beam under pure bending
2. Beam subjected to concentrated transverse load
3. Beam subjected to uniformly distributed load
4. Beam subjected to linearly varying distributed load
5. Beam subjected to sinusoidal transverse load



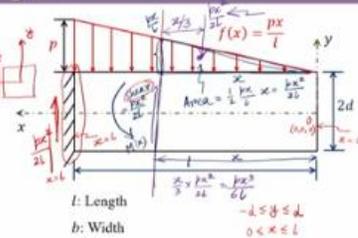
So, we are considering the bending of beam problems under different types of loading. Till now, we have solved three different problems: the first one is the bending of a cantilever beam subjected to pure bending, then the bending of a cantilever beam subjected to a concentrated shear load at the free end, and the third one was the bending of a simply supported beam subjected to a uniformly distributed load over the entire span.

Now, in today's lecture, we are going to consider a fourth problem where the beam is subjected to a linearly varying distributed load. The difference between the previous one and the present one is that for the previous problem, the load acting had a constant intensity over the entire span of the beam, whereas in this one, the intensity of the load varies linearly. At one end it is 0, and at another end it is maximum. This is the distribution w , which was constant force per unit length was constant for UDL, whereas for a linearly varying distributed load, this w is a function of x and that is a linear function.

Bending of Cantilever Beam with Linearly Varying Load

Boundary conditions:

- (1) Due to absence of any axial force at any x , $\int_{-d}^d b\sigma_{xx} dy = 0$
- (2) Bending moment at any x , $\int_{-d}^d b\sigma_{xx} y dy = \frac{px^3}{6l}$
- (3) Vertical shear force at fixed end ($x = l$), $\int_{-d}^d b\tau_{xy} dy = \frac{pl}{2}$
- (4) Along the bottom surface, $\sigma_{yy}(x, -d) = \tau_{xy}(x, -d) = 0$



- l : Length
- b : Width
- $2d$: Depth
- p : Maximum intensity of linearly varying distributed transverse load



So, let us start. We are going to consider a cantilever beam fixed at one end and free at another end, subjected to a linearly varying distributed load on the top face. Let us consider this beam where the left edge is fixed. The origin is somewhere here: 0, 0, 0. The length is taken to be l , and the total depth is taken to be $2d$. So, y varies between minus d to plus d , and x varies between 0 to l . $x = 0$ refers to the free edge;, whereas $x = l$ refers to the fixed edge on the left-hand side.

Coming to the loading, you can see that $f(x)$ is the linearly varying distributed load intensity, which equals $\frac{px}{l}$. So, at $x = 0$, which is at the right-hand side (the free edge), if you put $x = 0$, $f(0)$ will be 0. Thus, at the right-hand side, at this particular point $x = 0$, the value of the load intensity goes to 0. With $x = l$, for the left edge, which is the cantilever or the fixed edge, with $x = l$, $f(x)$ will go to $\frac{pl}{l}$, or a constant p . Thus, this p is the highest value - the maximum load intensity occurring for this linearly varying load at the left edge or at the built-in edge. So, l is the length, b is the width, $2d$ is the overall depth or thickness, and p is the maximum intensity of the linearly varying distributed transverse shear load, which occurs at the cantilever or built-in end or at the left edge.

Now, coming to the boundary conditions. Since there is no axial force present in the problem, we have the integral of $b\sigma_{xx}dy$ equals 0, where the integral is from $-d$ to $+d$. Note that the total depth is $2d$, so the range of y is from $-d$ to $+d$. So, y varies in this range - not $-\frac{d}{2}$ to $+\frac{d}{2}$ - the total depth is $2d$ here, and x varies between 0 to l .

Coming to the next boundary condition, which is the bending moment at any x . If you are considering any x , let us say somewhere here, what would be the bending moment? So, the left-hand side expression for bending moment, M , equals $\int_{-d}^{+d} b\sigma_{xx}ydy$. The right-hand side is the expression of the bending moment due to this linearly varying load intensity $f(x)$, which is obtained as $\frac{px^3}{6l}$.

How can that be obtained? If you are considering this particular section, the total load acting within that span is this; this much triangular load is acting. So, what is the total area within this? Here, this height is equal to $\frac{px}{l}$ and this bottom, base length is x , so total

area will be $\frac{1}{2} \frac{px}{l} x$, i.e., $\frac{px^2}{2l}$. So, the total shear load acting at this section, if you cut the section here at any x from left end, you will be getting the shear load to be equal to $\frac{px^2}{2l}$, which is the total downward load coming due to that portion of the triangle, i.e., that portion of the linearly varying load.

Coming to moment, at this section some moment will also be there $M(x)$, which is equal to moment of this linearly varying load about this particular surface that can be obtained by choosing the centroid. So, if you consider this triangle, the centroid of the triangle is somewhere here, through which this load $\frac{px^2}{2l}$ is passing, and the distance of this centroid should be at a distance of $\frac{x}{3}$ from the base of the triangle. This is well known from geometry. Now, the moment of this uniformly distributed load about these particular section would be $\frac{x}{3} \frac{px^2}{2l}$, which is $\frac{px^3}{6l}$, that is written here. Moment at any section x would be $\frac{px^3}{6l}$ and that is equal to $\int_{-d}^{+d} b\sigma_{xx}ydy$.

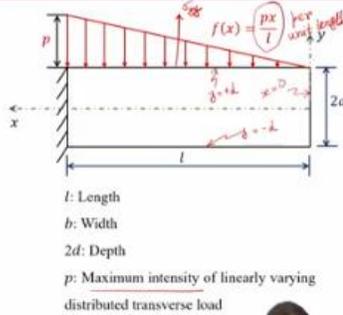
Coming to the vertical shear force. Vertical shear force, as I told, at any section is this: $\frac{px^2}{2l}$. Now, if you are considering the fixed edge, the left hand side edge, vertical shear force at that particular edge should be equal to $\frac{px^2}{2l}$ at $x = l$, because $x = l$ refers to the fixed edge, the left hand side face. Putting $x = l$ there, it would be $\frac{pl}{2}$. So, net vertical force on the left hand side that would be equal to $\int_{-d}^{+d} b\tau_{xy}dy$, and it should be $\frac{pl}{2}$, and this is positive; note that this should be positive.

Considering our element, positive x is on this side, so we should have τ_{xy} to be upward on that face. Here, the x -axis is towards the left (positive). So, τ_{xy} should be positive there, and this shear force is upward. This is the net upward shear force $\frac{px^2}{2l}$ at $x = l$, which has the same direction as τ_{xy} for the positive x -plane. Thus, $\int_{-d}^{+d} b\tau_{xy}dy = \frac{pl}{2}$. This is the third boundary condition.

Bending of Cantilever Beam with Linearly Varying Load

Boundary conditions:

- (1) Due to absence of any axial force at any x , $\int_{-d}^d b\sigma_{xx}dy = 0$
- (2) Bending moment at any x , $\int_{-d}^d b\sigma_{xx}ydy = \frac{px^3}{6l}$
- (3) Vertical shear force at fixed end ($x = l$), $\int_{-d}^d b\tau_{xy}dy = \frac{pl}{2}$
- (4) Along the bottom surface, $\sigma_{yy}(x, -d) = \tau_{xy}(x, -d) = 0$
- (5) Along the top surface, $\sigma_{yy}(x, d) = -\frac{px}{bl}$, $\tau_{xy}(x, d) = 0$
- (6) At the free end face, $\sigma_{xx}(0, y) = \tau_{xy}(0, y) = 0$



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Coming to the next one: along the bottom face, both σ_{yy} and τ_{xy} should equal 0. This bottom face is defined by $y = -d$ and is free of any normal or shear traction. So, both σ_{yy} and τ_{xy} should be 0 for the bottom face, whereas for the top face, which is defined by $y = +d$, τ_{xy} is 0, but σ_{yy} (the normal traction) is non-zero. The normal is acting like this for the top face (σ_{yy}), and it is non-zero because of $\frac{px}{l}$.

Now, once again, $\frac{px}{l}$ is the intensity, maximum load intensity for this linearly varying load. This is defined per unit length. Hence, if you divide this by b , we get the equivalent stress quantity - the force per unit area. So, σ_{yy} for the top plane ($y = +d$) for any value of x would equal $-\frac{px}{bl}$. The minus sign is because $\frac{px}{l}$ is acting downward, whereas σ_{yy} for the top plane is acting upward. That is why a minus sign is coming, and τ_{xy} is 0 for the top plane because no shear traction is there on the top face.

Coming to the free end face, that is this face - the right-hand side face - which is defined by $x = l$. σ_{xx} , that is the normal stress on the right-hand side free edge, is 0; τ_{xy} , that is the shear traction on the right-hand side face, is 0 because that is free of any kind of external tractions. These are the six boundary conditions which our solution should satisfy.

Bending of Cantilever Beam with Linearly Varying Load

Choice of stress function:

$$\phi(x, y) = A_1xy + A_2x^3 + A_3x^3y + A_4xy^3 + A_5(5x^3y^3 - 3xy^5) \leftarrow$$

Biharmonic equation:

$$\nabla^4 \phi = 0$$

$$\Rightarrow \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 + 360A_5xy - 360A_5xy = 0 \quad (\text{Satisfied})$$

Stress components:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6A_4xy + A_5(30x^3y - 60xy^3)$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 6A_2x + 6A_3xy + 30A_5xy^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -A_1 - 3A_3x^2 - 3A_4y^2 - 45A_5x^2y^2 + 15A_5y^4$$

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Now, the choice of stress function here is something like this. If you recall the previous two solutions, when we were solving the bending of a beam problem with a single transverse shear loading acting at a single section, for that, we were taking the polynomial up to the fourth degree. Then, when we were going for the uniformly distributed loading over the entire span, we had to take a polynomial up to the fifth degree. The order increased, or the degree of the polynomial increased by 1, when we increased the order of the load.

Here, we are going from a uniformly distributed load to a linearly varying load. Basically, the expression of the load, if you write mathematically, we are increasing the order of the load - meaning if you integrate the previous constant UDL load, we will be getting a linearly varying expression. By integrating the previous load, we are getting the linearly varying load; thus, the order of the polynomial required in the stress function should also increase by 1.

So, here, the highest order required to be considered in the polynomial is 6. The solution should be a combination of second, third, fourth, fifth, and sixth-degree polynomials. There will be a long expression of ϕ , which is a general expression of superposition of second to sixth-order or sixth-degree polynomials. Many of the constants would be there. Then, if you start solving it after putting the boundary conditions, the huge set of constants will be set to 0, as you had observed in the last 2 problems.

Here, I am only considering a solution with these 6 terms, which are going to result in non-zero constants. You can start with a general solution with, let us say, some 20 or 25

terms, and then after following a similar procedure as discussed in the previous two lectures, you will end up with many of the constants to be 0, and only these 6 non-zero constants will be left. I am directly starting with this solution for reducing the complexity or reducing the computational complexity, rather.

So, only non-zero terms present in ϕ are $A_1xy + A_2x^3 + A_3x^3y + A_4xy^3 + A_5(5x^3y^3 - 3xy^5)$. You can note that these are the sixth-order terms. This must satisfy the biharmonic equation. As I had already told, I had set the 0 constants out, and thus this particular choice of ϕ satisfies the biharmonic equation automatically. Whatever conditions are required to be imposed to satisfy $\nabla^4\phi = 0$, that I have already incorporated, and thus, this chosen form of ϕ stress function satisfies the biharmonic equation directly. From this, we can obtain the stress component σ_{xx} as $\frac{\partial^2\phi}{\partial y^2}$, σ_{yy} as $\frac{\partial^2\phi}{\partial x^2}$, and τ_{xy} as $-\frac{\partial^2\phi}{\partial x\partial y}$, as this.

These would be the three equations for the three plane stress components for the present problem with this chosen form of ϕ . We need to see whether this satisfies all the boundary conditions or not. If yes, what should the values be of these constants A_1 to A_5 , so that we can obtain the overall stress distribution for the bending of a cantilever beam subjected to a uniformly varying load.

Bending of Cantilever Beam with Linearly Varying Load

B.C. (4) and (5): $\tau_{xy}|_{y=\pm d} = 0$

$$\Rightarrow -A_1 - 3A_3x^2 - 3A_4d^2 - 45A_5d^2x^2 + 15A_5d^4 = 0$$

$$\rightarrow A_1 + 3A_4d^2 - 15A_5d^4 = 0$$

and, $3A_3 + 45A_5d^2 = 0 \Rightarrow A_3 = -15A_5d^2$

B.C. (4): $\sigma_{yy}|_{y=-d} = 0 \Rightarrow (6A_2 - 6A_3d - 30A_5d^3)x = 0$

$$\Rightarrow A_2 = -10A_5d^3$$

B.C. (5): $\sigma_{yy}|_{y=d} = -\frac{px}{lb} \Rightarrow (6A_2 + 6A_3d + 30A_5d^3)x = -\frac{px}{lb} \Rightarrow A_5 = \frac{p}{120lbd^3}$

$\therefore A_2 = -\frac{p}{12lb}$ and, $A_3 = -\frac{p}{8lbd}$

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Now, starting from boundary conditions 4 and 5, that is, shear traction for the top and bottom faces are 0; τ_{xy} is 0 for $y = \pm d$. This is the expression of τ_{xy} . If you replace y

with $\pm d$ and x remains as x , it should be satisfied for all values of x . So, we will get this equation. Here, you can see there are 3 terms - these 3, which are constant terms independent of x - whereas 2 terms are there: this one and this one, which are dependent on x^2 . This equation would be 0 only if the constant term and the coefficient of x^2 are set to 0 independently.

Thus, $-A_1 - 3A_4d^2 + 15A_5d^4 = 0$. The minus sign is canceled from all the terms, and we get the first equation, that is, setting the constant term to 0. And setting the coefficient of x^2 to 0, we will get this second equation: $3A_3 + 45A_5d^2 = 0$. From here, we get a relation between A_3 and A_5 as $A_3 = -15A_5d^2$.

Going to the remaining part of boundary condition 4, σ_{yy} at $y = -d$ is 0; the bottom face is free of any kind of normal traction. Putting the expression of σ_{yy} , this expression with $y = -d$, we will get this equation. This entire term, $(6A_2 - 6A_3d - 30A_5d^3)x = 0$. This can be satisfied for all values of x if this coefficient of x equals 0. We know A_3 and A_5 are already related. So, here, this 3 and 5 relation is substituted, and A_2 is written in terms of A_5 as $-10A_5d^3 = A_2$.

Going back to the remaining part of boundary condition 5, that is, normal traction on the top face σ_{yy} for $y = +d$ is $-\frac{px}{bl}$ due to linearly varying load. So, once again, in the expression of σ_{yy} , substituting y with $+d$, we would be getting this equation: $(6A_2 + 6A_3d + 30A_5d^3)x = -\frac{px}{b}$. Here, we can cancel x from both sides. A_3 and A_2 , both are known in terms of A_5 ; we can replace them. Then, this equation would give us the expression of A_5 directly in terms of p as $A_5 = \frac{p}{120lbd^3}$.

So, we got one of the constants. Substituting this A_5 back here, we get A_2 to be $-\frac{p}{12bl}$, and substituting this A_5 back here, we get A_3 to be $-\frac{p}{8lbd}$. So, three constants have been obtained: A_2 , A_3 , and A_5 . After this, we are going to check the remaining boundary conditions on σ_{xx} . σ_{xx} is written here, which we had obtained, and then the constant A_5 in σ_{xx} is already obtained as $\frac{p}{120lbd^3}$. A_4 is the remaining constant to be evaluated.

Bending of Cantilever Beam with Linearly Varying Load

B.C. (1): $\int_{-d}^{+d} b\sigma_{xx}dy = 0$

$\Rightarrow \int_{-d}^{+d} (6A_4xy + 30A_5x^2y - 60A_5xy^3)bdy = 0$ (Satisfied as σ_{xx} is an odd function of y)

$\sigma_{xx} = 6A_4xy + A_5(30x^2y - 60xy^3)$
 $A_5 = \frac{p}{120lbd^3}$

B.C. (2): $\int_{-d}^{+d} by\sigma_{xx}dy = \frac{px^3}{6l}$

$\Rightarrow \int_{-d}^{+d} (6A_4xy^2 + 30A_5x^3y^2 - 60A_5xy^4)bdy = \frac{px^3}{6l}$

$\Rightarrow 20A_5bd^3x^3 + (4A_4bd^3 - 24A_5bd^5)x = \frac{px^3}{6l}$

$\Rightarrow 20A_5bd^3 = \frac{p}{6l} \Rightarrow A_5 = \frac{p}{120lbd^3}$

and, $4A_4bd^3 - 24A_5bd^5 = 0 \Rightarrow A_4 = \frac{p}{20lbd}$



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Using the first boundary condition, that is, the condition of zero axial force: $\int_{-d}^{+d} b\sigma_{xx}dy = 0$. Replacing this form of σ_{xx} in this particular integral equation, we will get this term. Here, this equation is automatically satisfied because σ_{xx} is an odd function of y . You can see only y term or y^3 terms are present in the expression of this σ_{xx} , and y integral is from $-d$ to $+d$. Thus, for σ_{xx} being an odd function of y , this equation will be automatically satisfied and will go to 0. So, this equation is satisfied, boundary condition 1 is satisfied, and we cannot find A_4 using this particular condition.

Let us proceed to the second boundary condition, which is the moment balance equation. The moment at any particular cross-section is written as $\frac{px^3}{6l}$, and in terms of σ_{xx} , that is $\int_{-d}^{+d} by\sigma_{xx}dy$. Here, if I am replacing this form of sigma in this particular equation, then, it would be something like this.

Integrating the left-hand side with respect to y from $-d$ to $+d$, we will get these two sets of terms. The first term you can see contains x^3 , some constant $20A_5bd^3$ times x^3 , and the second term contains x , which has a coefficient $4A_4bd^3 - 24A_5bd^5$. Comparing the coefficients of x and x^3 on both sides: on the right-hand side, the x^3 coefficient is $\frac{p}{6l}$, which should be equal to the x^3 coefficient on the left-hand side. Similarly, as there is no term on the right-hand side with x , the left-hand side coefficient of x should be 0.

So, we get these two conditions, from which, from the first one, A_5 can be obtained as $\frac{p}{120lbd^3}$. Using that A_5 here in the second condition, we get A_4 to be $\frac{p}{20lbd}$. So, we get both

the constants A_4 and A_5 associated with σ_{xx} here. Now, if you check we had already obtained A_5 from one of the previous boundary condition and comparing them, you can verify that this A_5 form is same from this boundary condition 2 and obtained from the previous boundary condition. As we are getting the same constant from different boundary condition means the chosen stress field is valid. This is compatible with all the boundary conditions.

Bending of Cantilever Beam with Linearly Varying Load

$$\tau_{xy} = -A_1 - 3A_3x^2 - 3A_4y^2 - 45A_5x^2y^2 + 15A_3y^4$$

$$\tau_{xy} = -A_1 + \frac{3px^2}{8bd} - \frac{3py^2}{20bd} - \frac{3px^2y^2}{8bd^3} + \frac{py^4}{8bd^3}$$

B.C. (3): $\int_{-d}^d b \tau_{xy} \Big|_{x=1} dy = \frac{pl}{2}$

$$\Rightarrow \int_{-d}^d \left(-A_1b + \frac{3pl}{8d} - \frac{3py^2}{20d} - \frac{3ply^2}{8d^3} + \frac{py^4}{8d^3} \right) dy = \frac{pl}{2}$$

$$\Rightarrow -2A_1bd + \frac{3pl}{4} - \frac{pd^2}{10l} - \frac{pl}{4} + \frac{pd^2}{20l} = \frac{pl}{2}$$

$$\Rightarrow A_1 = -\frac{pd}{40bl}$$

$$\begin{cases} A_3 = -\frac{p}{8bd} \\ A_4 = \frac{p}{20bd} \\ A_5 = \frac{p}{120bd^3} \end{cases}$$



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Now, we are only left with a single unknown that is, I guess, A_1 . Using the expression of τ_{xy} , this is τ_{xy} where apart from A_1 , rest of the constant A_3 , A_4 , A_5 are known. Replacing that in the expression of τ_{xy} , we would be getting τ_{xy} to be this. First term, minus A_1 , is unknown, rest 4 terms are known in terms of x and y functions.

Now, the remaining boundary conditions we need to satisfy $b\tau_{xy}$ at $x = 1$ integrated over $-d$ to $+d$ is equal to $\frac{pl}{2}$, that is at the cantilever end total shear force or upward force is $\frac{pl}{2}$. Putting this τ_{xy} here, with $x = 1$ if you integrate it, and equate that to $\frac{pl}{2}$, the left hand side is having only one unknown constant: A_1 . Solving this, we get A_1 to be $-\frac{pd}{40bl}$. So, by using the third boundary condition that is balance of the vertical shear force at the cantilever end, net y -axis force balance, will give us this last remaining constant A_1 as $-\frac{pd}{40bl}$. Now, all the constants are known. Substituting this A_1 back here, we get the complete expression of τ_{xy} .

Bending of Cantilever Beam with Linearly Varying Load

B.C. (7): $\sigma_{xx}(0, y) = \tau_{xy}(0, y) = 0$

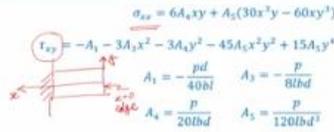
$$\sigma_{xx} \Big|_{x=0} = 6A_4xy + A_5(30x^3y - 60xy^3) \Big|_{x=0}$$

$$\Rightarrow \sigma_{xx} \Big|_{x=0} = 0 \quad (\text{Satisfied})$$

$$\tau_{xy} \Big|_{x=0} = (-A_1 - 3A_3x^2 - 3A_4y^2 - 45A_5x^2y^2 + 15A_5y^4) \Big|_{x=0}$$

$$\Rightarrow -A_1 - 3A_4y^2 + 15A_5y^4 = 0$$

This condition cannot be satisfied for all values of y



$$\begin{aligned} \sigma_{xx} &= 6A_4xy + A_5(30x^3y - 60xy^3) \\ \tau_{xy} &= -A_1 - 3A_3x^2 - 3A_4y^2 - 45A_5x^2y^2 + 15A_5y^4 \\ A_1 &= -\frac{pl}{40bl} & A_3 &= -\frac{p}{8bd} \\ A_4 &= \frac{p}{20bd} & A_5 &= \frac{p}{120bd^3} \end{aligned}$$



So, I have written the expression of σ_{xx} , τ_{xy} , and all the constants which are involved. Now, the remaining boundary condition 7 was σ_{xx} and τ_{xy} to be 0 at $x = 0$, that is at the free edge. The free edge of the cantilever beam is free from any normal traction and that is also free from any shear traction on the right hand side.

This particular edge, this edge is the $x = 0$ edge. This edge should have $\sigma_{xx} = 0$, and $\tau_{xy} = 0$. If you put $x = 0$ in the σ_{xx} expression, it will directly go to 0. This equation is automatically satisfied. Coming to τ_{xy} at $x = 0$, in τ_{xy} , if you replace $x = 0$, this term and this term would cancel. But three other terms would remain where one term is constant, one term is a function of y^2 , and another term is a function of y^4 . And this A_1 , A_4 , A_5 are already obtained as non-zero constants.

So, for all values of y , it is impossible to force τ_{xy} to be 0. It is not possible to satisfy this particular equation for all values of y . Thus, at the free edge, one of the surface tractions, that is, the shear traction boundary condition on the right face, is not getting satisfied at $x=0$. I think all the boundary conditions are satisfied.

Bending of Cantilever Beam with Linearly Varying Load

$$\sigma_{xx} = 6A_1xy + A_1(30x^2y - 60xy^3)$$

$$\sigma_{yy} = 6A_2x + 6A_3xy + 30A_4xy^2$$

$$\tau_{xy} = -A_1 - 3A_3x^2 - 3A_4y^2 - 45A_1x^2y^2 + 15A_2y^4$$

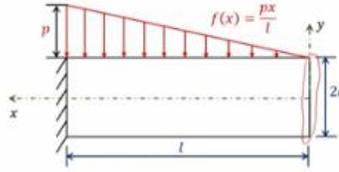
$$A_1 = -\frac{pd}{40bl} \quad A_2 = -\frac{p}{12lb} \quad A_3 = -\frac{p}{8lbd} \quad A_4 = \frac{p}{20lbd} \quad A_5 = \frac{p}{120lbd^3}$$

Stress components:

$$\sigma_{xx} = \frac{3pxy^2}{10lbd} + \frac{pxy}{4lbd^3}(x^2 - 2y^2)$$

$$\sigma_{yy} = -\frac{px}{2lb} - \frac{3pxy}{4lbd} + \frac{pxy^3}{4lbd^3}$$

$$\tau_{xy} = \frac{pd}{40bl} + \frac{3px^2}{8lbd} - \frac{3py^2}{20lbd} + \frac{py^2}{8lbd^3}(y^2 - 3x^2)$$



Now we are replacing this A_1, A_2, A_3, A_4, A_5 , all five constants which we had obtained in terms of the load intensity p . If I replace all of them in σ_{xx}, σ_{yy} , and τ_{xy} , the expressions of the stress field would be obtained like this: $\sigma_{xx} = \frac{3pxy}{10lbd} + \frac{pxy}{4lbd^3}(x^2 - 2y^2)$. Here, you can see we have the cubic order variation of σ_{xx} in terms of x . You can see one x is present here which will be multiplied with this x^2 .

So, the σ_{xx} is proportional to x^3 from the first term, it is proportional to x , and it is also proportional to y for the first term. Here, it is proportional to y^3 from this term. We cannot just say σ_{xx} is $\frac{My}{I}$; that particular assumption is not valid here. σ_{xx} is proportional to y^3 as well for this particular type of linearly varying load. Then, similarly, substituting the constant in σ_{yy} and then in τ_{xy} , you will get this equation.

These three expressions of σ_{xx}, σ_{yy} , and τ_{xy} give us the stress field - the stress components generated for the cantilever beam subjected to a linearly varying load. Note that we are able to satisfy all the boundary conditions except one, which is on the free edge. τ_{xy} was not - we were not able to force that to 0. Apart from that, all the rest of the conditions are satisfied. Following Saint-Venant's principle, these results are applicable for the region apart from the two edges, $x = 0$ and $x = l$.

Summary

- Cantilever Beam Subjected to Linearly Varying Distributed Load
- Bending Stress

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In total, in this lecture, we discussed the bending stresses for a cantilever beam subjected to a linearly varying distributed load.

Thank you.