APPLIED ELASTICITY

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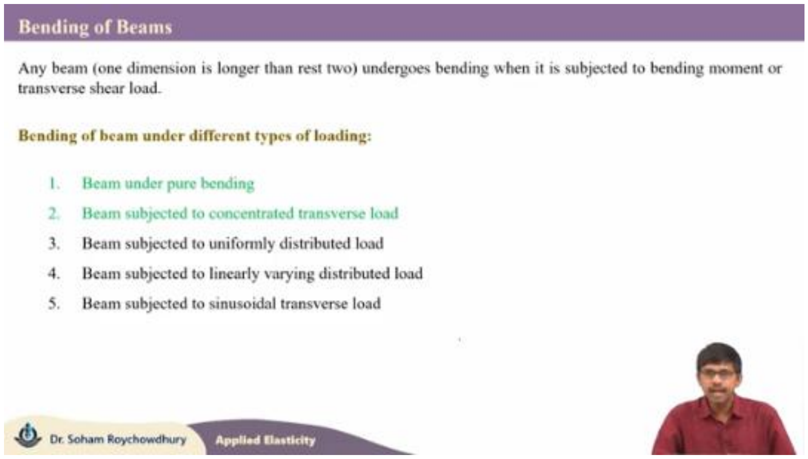
WEEK: 07
Lecture- 33

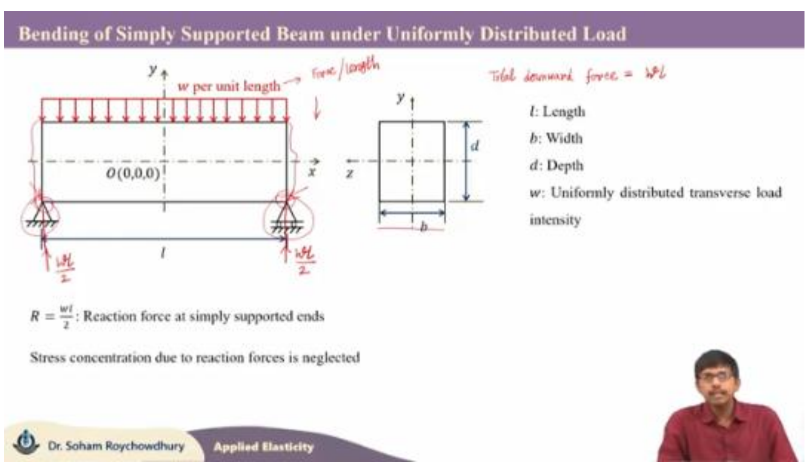
Welcome back to the course on applied elasticity. In today's lecture, we are going to continue our discussion on the bending of beams. We started this particular topic and were discussing the bending of beam problems when the beams are subjected to different types of loading. So, beams are basically one-dimensional continua where one of the dimensions, that is the length of the beam, is much larger compared to the other two dimensions: width and thickness.

Now, when such elements or beam elements are subjected to pure bending moment or any kind of transverse shear loading, they undergo bending. Now, we are going to consider different types of bending of beam problems when the beam is subjected to different types of loading.



So, we have already solved two bending problems. The first one was the bending of the beam subjected to pure bending moment, and the next one was the bending of the beam subjected to a concentrated transverse load at a particular point or section.

Now, the next topic which we are going to discuss in this particular lecture is the bending of a beam subjected to uniformly distributed loading. So, the external transverse shear load is not going to act at a particular plane; it is acting over a finite span of the beam. We are considering the intensity of that distributed load to be constant, and thus this is called uniformly distributed load or UDL. Now, coming to the bending of a simply supported beam under UDL, uniformly distributed loading.



So, first let us consider a beam with a rectangular cross section. The length of the beam is taken to be l, b is the width of the beam, and d is the depth of the beam. We are considering the length of the beam being aligned along the x-axis, and the beam is subjected to a distributed load of constant intensity W. So, W is the uniformly distributed transverse load intensity per unit length of the beam, and we are considering the load acting on the xy plane.

So, the mid-plane xy plane is the plane of loading, which is also the plane of bending. By the assumption of Euler-Bernoulli beam theory, we are considering the bending of the beam under the Euler-Bernoulli beam theory, and the beam is simply supported, so you can see At both edges, the simple supports or pinned or hinged supports are shown.

So, we are allowing rotation at the hinged or pinned edges. However, the transverse displacement is restricted. So, those two endpoints cannot move in the y-direction. However, the slopes are allowed to be non-zero. So, for such simply supported beams, the bending moment at both ends—at both hinge joints—must be zero.

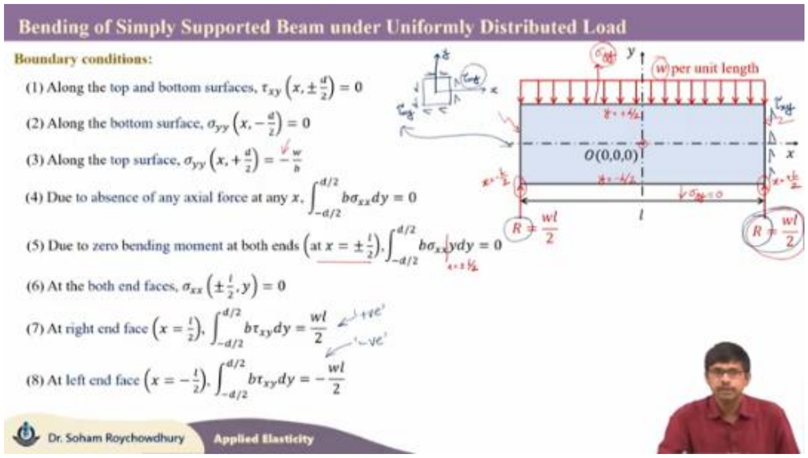
And this W per unit length is the force intensity. So, the unit of this is force unit divided by length unit, that is Newton per meter or Newton per millimeter, something like that. So, you should not get confused between W and the stress. W does not have the unit of stress, which is force per unit area. W is force per unit length along the x-axis.

So, if you divide W by this b, then you will get the stress on that particular phase, which would be W by b, or force per unit area. Now, the reaction forces will be generated at both hinge points, on the left hinge as well as on the right hinge. At both of these two points, the vertical reaction forces will be generated to balance the overall downward transverse external load acting on the system. So, what is the total downward force?

The total downward force for this problem is equal to W times l because we have W as the constant intensity multiplied by l that is, the total length of the beam will give us the net vertically downward force acting, and this should be balanced by the support reactions at both hinge points.

Now, due to the symmetry of the problem, the problem is symmetric about the y-axis. So, at both ends, we are supposed to have $\frac{wl}{2}$ and $\frac{wl}{2}$ amount of support reactions acting upward. Now, these $\frac{wl}{2}$ at both the right and left faces would result or would give rise to shear distribution on these edges. So, τ_{xy} would be non-zero, and the integral of τ_{xy} should be equal to $\frac{wl}{2}$ for both the right and left edges of the beam. That is, at x equals to $+l$ or $+l/2$ and $-l/2$.

And we are going to neglect the effect of stress concentration due to reaction forces near the hinge region. So, we are going to neglect the effect of stress concentration, and we will say that the obtained solution, the stress field, would be valid at the far field following the Saint-Venant principle.



Now, moving forward. Let us try to write down all the boundary conditions. So, here in the free body diagram, I had removed the hinge or pin supports and replaced them with $R = \frac{wl}{2}$. So, at both the hinge support points, a vertically upward force $\frac{wl}{2}$ is acting in total, which will balance the total wl amount of downward load acting on the top face of the beam due to UDL, uniformly distributed loading. Now, coming to the boundary condition, first let us consider the top face and the bottom face. Now, the top face is given by $y = \frac{d}{2}$ because total depth is d considering the origin O to be at the midpoint of the beam the top plane is defined with $y = +d/2$ and bottom plane is defined with $y = -d/2$. Now, if you carefully look at the given loading on the top plane only downward force is acting W per unit length which will give rise to normal stress on the top face, but this W cannot result any shear stress on the top face.

Thus, shear stress on the top face $\tau_{xy}\left(y = +\frac{d}{2}\right) = 0$. Similarly, the bottom plane is free of any kind of external loading. So, Both shear as well as normal stress on the bottom plane at $(y = +\frac{d}{2}) = 0$. So in the first boundary condition we are writing the shear stress τ_{xy} for top and bottom, both the planes to be 0. So, $\tau_{xy}\left(x, \pm\frac{d}{2}\right) = 0$ at any point on the top and bottom plane at any x with $y = +\frac{d}{2}$ or $y = -\frac{d}{2}$, we are not going to have any shear surface traction. So, $\tau_{xy} = 0$ for top and bottom plane. Now coming to the normal surface tractions first for the bottom plane bottom plane is having 0 surface traction no external forces are there on the bottom plane thus the normal traction σ_{yy} so this is y plane so normal traction acting here should be σ_{yy}. And for the bottom plane that must be 0 because no external force is acting there. So, σ_{yy} for all values of x with $y = -\frac{d}{2}$ that is at the bottom plane should be equals to 0.

However, on the top plane this is not equals to 0 because on the top plane if you draw that σ_{yy}. So, top plane means positive y plane because the unit normal for the top plane is along the positive y axis. So, for the positive y-axis, the positive y-plane along the positive y-axis, we are going to have σ_{yy}. Now, this is nonzero due to the presence of this w. Now, as I had already mentioned, w cannot be directly equal to σ_{yy} because of the mismatch in dimension.

w is force per unit length, whereas stress is force per unit area. So, w is defined per unit length along the x-axis, that is along l. If you divide that by b, the width along the z-axis, then $\frac{w}{b}$ would be equivalent to stress on that particular top face, and as w is acting in the downward direction. However, σ_{yy} for the top plane, by convention, is in the upward direction; hence, we are adding this negative sign.

So, σ_{yy} for the top plane $\left(x, +\frac{d}{2}\right) = -\frac{w}{b}$. Coming to the next one, this is 0 axial force throughout the beam at any x of the beam at any cross-section integral $\int_{-d/2}^{d/2} b\sigma_{xx} dy = 0$ because at no cross-section is any axial force acting. This is the same boundary condition as in the previous problems.

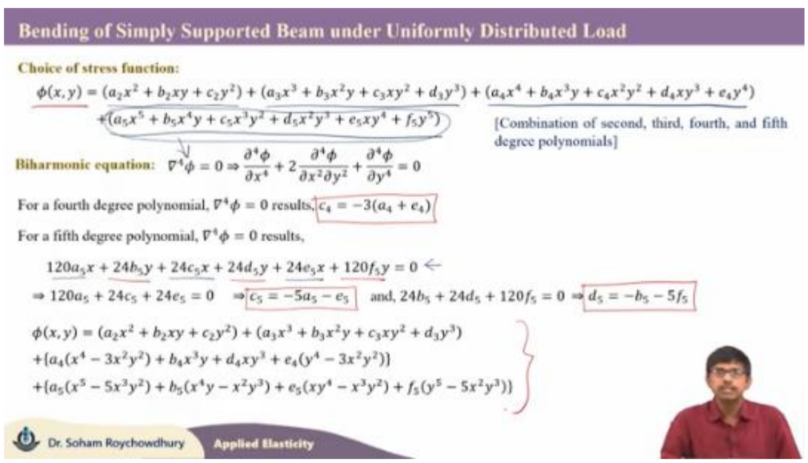
Coming to the bending moment at both the edges, the pinned edges. So, at the left-hand side edge and the right-hand side edge. The conditions, the boundary conditions were pin-pinned or simply supported ends, meaning at $x = -\frac{l}{2}, +\frac{l}{2}$ refers to the right edge and $-\frac{l}{2}$ refers to the left edge. At $x = \pm\frac{l}{2}$, we must have the bending moment to be 0 for the simply supported beam.

And the general expression of bending moment is $\int_{-d/2}^{d/2} b\sigma_{xx} y dy = 0$. But this is valid only for $x = \pm\frac{l}{2}$, not in general. Coming to the next one, these side faces, this face and this face should be free of any normal stress $\sigma_{xx} = 0$ for $x = \pm\frac{l}{2}$ for any value of y because no normal stresses no axial forces are acting on those two side planes. Coming to the vertical force balance or modeling of this $R = \frac{wl}{2}$ for the right face and then for the left face. So, for the right face here, this $R = \frac{wl}{2}$ is basically giving rise to some kind of τ_{xy}. So, you can think of this τ_{xy} as nothing but the resultant of this τ_{xy}. So, on the right face, which is the positive x plane, τ_{xy} will be acting. Positive τ_{xy} will be acting in the upward direction. So, if you integrate that τ_{xy} over the total area, that should give rise to this total.

Reaction at that particular right hinge, which is $\frac{wl}{2}$. So, integral $\int_{-d/2}^{d/2} b\tau_{xy} dy = \frac{wl}{2}$ on the right face. Similarly, for the left face at $x = -\frac{l}{2}$, $\int_{-d/2}^{d/2} b\tau_{xy} dy = -\frac{wl}{2}$. Now, why this minus? Because by convention, if you consider a block like this with this being the x-axis and this being the y-axis, by the sign convention, τ_{xy} directions are like this.

So, for the positive x-axis on the positive x plane on the right-hand side, τ_{xy} is upward. Integral of that is $\frac{wl}{2}$, R is also upwards. So, as the direction of this τ_{xy} and this R are the same, we had used a positive sign here. Whereas, for the left-hand side face, this particular face, these two are related. Here, net τ_{xy} is downward, and this R is upward.

So, because of this mismatch in direction, we are adding a negative sign to match it with the shear stress direction convention. So, the integral of $\int_{-d/2}^{d/2} b\tau_{xy}dy = -\frac{wl}{2}$ for the left-hand side face. So, in total, we have these eight boundary conditions for the simply supported beam under uniformly distributed loading.



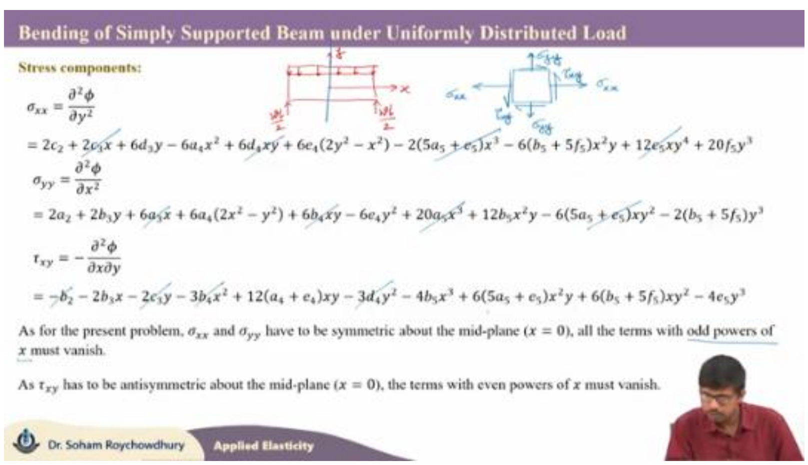
Now, coming to the choice of stress function. For the previous problem, we considered a stress function that was a combination of second-, third-, and fourth-degree polynomials.

That was done for the case of bending a beam with a single concentrated shear loading. Now, as you are considering the uniformly distributed load, the fourth-degree polynomial will not be sufficient to represent the stress distribution. Thus, we need to add or increase the degree of the polynomial to fifth. So, the general stress function we are considering is a combination of second-, third-, fourth-, and fifth-degree polynomials.

So, the first term involving a_2, b_2, c_2 is a second-degree polynomial, then this is a third-degree polynomial, this is a fourth-degree polynomial, and this is a fifth-degree polynomial involving respective unknown constants. Now, this choice of ϕ must satisfy the biharmonic condition. The second and third degree polynomial functions of directly satisfies the bi-harmonic equation as we had already discussed. Now fourth and fifth degree polynomial will give rise to some additional constraints, Some relation between their constants so that they can satisfy the bi-harmonic equation. So for the fourth-degree polynomial, the constraint we will get to satisfy the biharmonic equation is $c_4 = -3(a_4 + e_4)$. This was the same as the constraint obtained for the bending of a beam with a single concentrated shear loading. Now coming to the fifth-degree polynomial, if you simply use this fifth-degree polynomial in the biharmonic equation, you will get this equation.

Now here you can see there are a few terms involving x, there are a few terms involving y, and this must be 0 for all values of x and y. The coefficient of x and the coefficient of y must vanish independently; separately, the coefficient of x and the coefficient of y should be 0. This would give rise to these two conditions: $c_5 = -5a_5 - e_5$, and $d_5 = -b_5 - 5f_5$. So, we have these three conditions.

If we impose these three on the chosen stress function, which is a combination of second, third, fourth, and fifth-degree polynomials, we would be able to satisfy the biharmonic equation. So, imposing these three conditions in the stress function, this would be the final form of the stress function: a combination of these four polynomials of different degrees or different orders, and this satisfies the bi-harmonic equation directly.



Now, from this, we are going to write the stress components as $\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$, which is given by this equation.

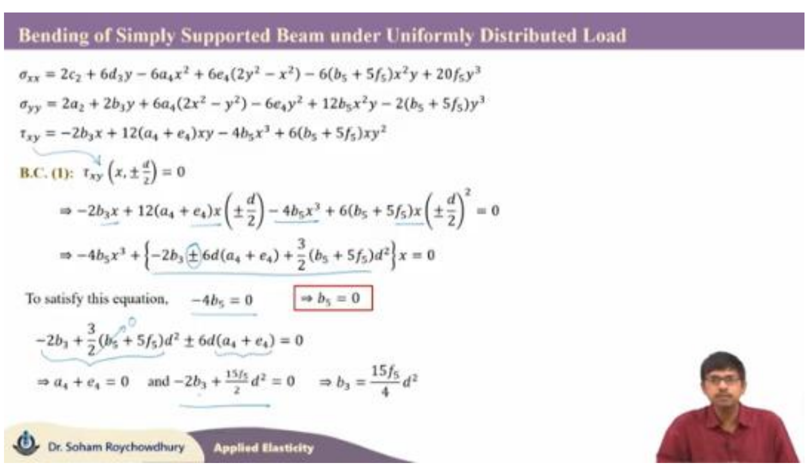
Then, $\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$ is given by this equation. And $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$, given by this equation. So, so many terms are there involving many non-zero constants. Now, we are going to invoke the symmetry principle. If you recall the problem, the problem was like this.

This was the x-axis, this was the y-axis, the support reactions were $wl/2$, and the loading was uniformly distributed loading throughout the length. Now, if you consider the vertical mid-plane about the mid-vertical plane, if you consider this is the plane of symmetry for the problem, so σ_{xx} and σ_{yy} variation must be symmetric about the vertical mid-plane, that is, the x equals to 0 plane. Because if you simply consider one element, this is σ_{xx}, and this is σ_{yy}.

So, both of them are simply preserving their symmetry about the respective mid-plane. So, that can be ensured only if both σ_{xx} and σ_{yy} are symmetric about the mid-vertical plane or x equals to 0 plane, and to ensure that all the odd power terms of x must vanish because they will break the symmetry. So, whatever odd power terms of x we have in the expression of σ_{xx} and σ_{yy}, they should be taken to 0.

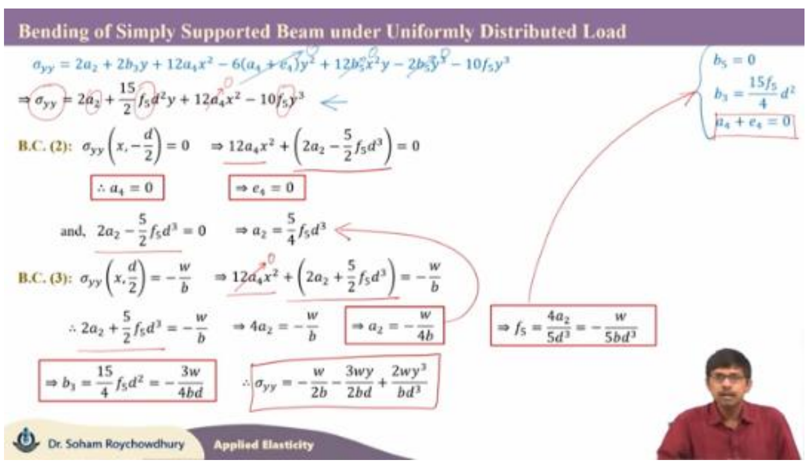
So, if you look at that, this is the odd power term $2c_3x$; x power is 1, odd power. This is independent of x. Now, $-6a_4x^2$—this is even power of x—can remain. Now, $6d_4xy$—once again, odd power—so should go to 0. Similarly, this x cube term should go to 0, and $12e_5xy^4$ term should go to 0. And in the same fashion, for σ_{yy}, we will also force the odd power terms of x to 0.

Now, coming to τ_{xy}. That is not symmetric; it is simply anti-symmetric about the mid-plane. Why? Because if you draw τ_{xy} on the right face, that is upward, whereas tau xy on the left face is downward by convention. We must have τ_{xy} to be anti-symmetric, and that can be ensured by dropping the even power terms of x from the expression of τ_{xy}. Even power means x square or x to the power 4, or even x to the power 0 terms—meaning constant terms or independent of x terms.



So, keeping that in mind, this set of terms should vanish. So, only x or x cube terms will remain; all rest—x square term, x 4 term, or x independent terms—would go to 0. So, after enforcing this symmetry condition, we can immediately reduce the stresses to this form; many of the terms are set to 0. Now, you are going to invoke the boundary condition $\tau_{xy}\left(x, \pm\frac{d}{2}\right)$—that is, on top and bottom face, we have 0 shear stress. Now, substituting this τ_{xy} here with $\pm\frac{d}{2}$, we would be getting this equation, and then you can see, then you can see we have some x cube term, we have some x term and if you couple them separately, so $-4b_5x^3$ plus this big term into x. and this term is having a plus minus sign as well so this should be satisfied this equation should be satisfied only if we force the coefficient of x cube and x to be separately 0 and then only this equation can be satisfied for all values of x that means we must have b_5 to be 0 from the first equation and from the second equation Both the terms before and after the plus minus sign should be individually 0. So, $a_4 + e_4 = 0$ and these total terms should also be 0. Now, already we had n 4 b_5 to be 0. So, this is 0.

So, second equation would be this. So, from here we are getting three condition $b_5 = 0$ $a_4 + e_4 = 0$ and the relation between b_3 and f_5 from the boundary condition 1. which is $b_3 = \frac{15f_5}{4}d^2$.

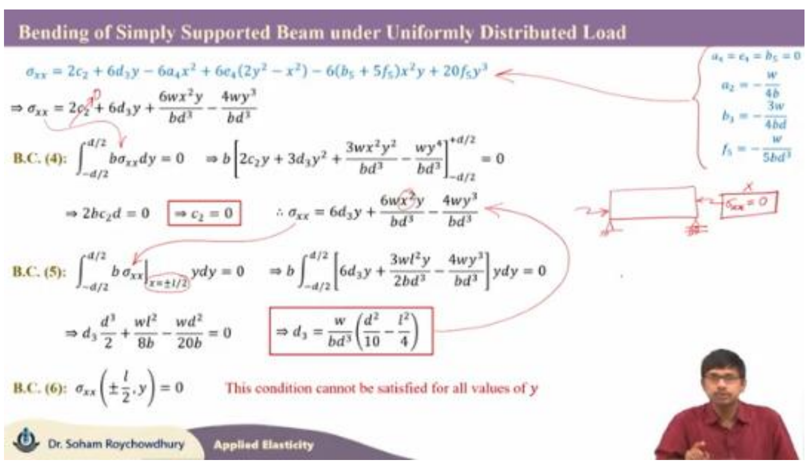


Now, moving forward to the next boundary condition on σ_{yy}. So, after imposing already obtained conditions for b_5 b_3 and a_4 e_4 the σ_{yy} equation reduces to this.

This a_4 e_4 term goes to 0, and b_5 terms also go to 0. Now, Using the second boundary condition, that is 0 normal traction on the bottom phase $x, -\frac{d}{2}$, σ_{yy} by 0. If you substitute that, we would be getting this equation where one term has x square, another is a constant term. This is 0 for all values of x if both terms are set to 0 individually, resulting in a_4 equals to 0 and $2a_2$ equals to $\frac{5}{2}f_5d^3$. Now, already we had a_4 plus e_4 is 0. If we now have a_4 as 0, that implies e_4 is also 0. So, both a_4 and e_4 are individually 0.

Similarly, the second one will give a relation between a_2 and f_5. Now, moving to the third boundary condition, that is normal traction boundary condition on top phase, that is $+\frac{d}{2}$. So, $\sigma_{yy}\left(x,\frac{d}{2}\right) = -\frac{w}{b}$ due to the uniformly distributed downward load. Now, substituting σ_{yy} expression, this expression here, we would be getting this.

Once again, the coefficient of x square and the constant, two terms are there on the left hand side and that is $= -\frac{w}{b}$. a_4 is already 0. So, this term is set to 0. now the second term $2a_2 + \frac{5}{2}f_5 d^3$ should be $= -\frac{w}{b}$. now using this condition and this condition we can write this $\frac{5}{2}f_5 d^3$ as $2a_2$ thus a_2 can be obtained as $-\frac{w}{4b}$ we are getting 1 nonzero constant $a_2 = -\frac{w}{4b}$ now substituting a_2 back in this equation we can obtain $f_5 = -\frac{w}{5bd^3}$ and then substituting these f_5 back in the second b_3 equation, we can obtain $b_3 = -\frac{3w}{4bd}$. So, 3 of the non-zero constants we had evaluated, replacing this a_2 f_5 and b_3, so f_5. a_2 and a_4 is 0. These, if you replace in the expression of σ_{yy}, σ_{yy} is obtained as this. So, we are able to get one of the non-zero stress components sigma yy as this.



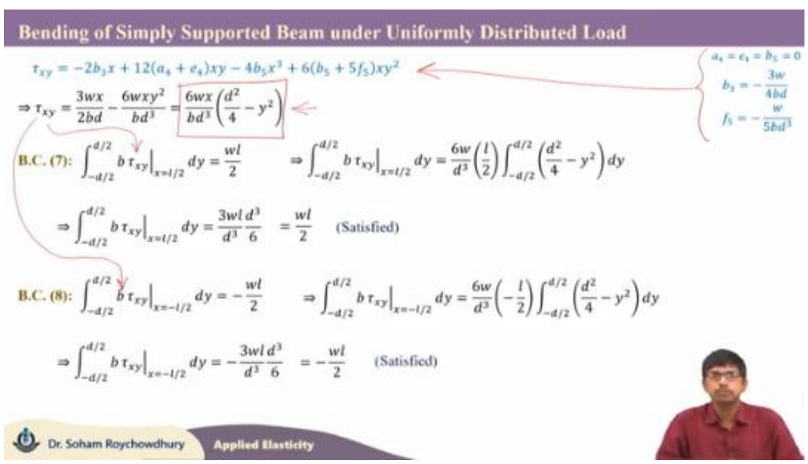
All the non-zero constants are now evaluated. Now, we will proceed to get σ_{xx} and τ_{xy} by using the other boundary conditions. So, σ_{xx} is equal to this, and these are the constants which we had already obtained. $\emptyset_3$ of them are 0: a_4, e_4, b_5, and the rest 3 are non-zero: a_2, b_3, and f_5, which are now known. So, if I substitute all these constants here, 0 and non-zero, then σ_{xx} is like this, having 4 terms with 2 unknown constants c_2 and b_3 plus 2 known terms. Now, using boundary condition 4, that is 0 axial force at any x, $\int_{-d/2}^{d/2} b\sigma_{xx}dy = 0$. Now, if I replace this σ here and try to check, that will give the value of only one non-zero unknown constant c_2. So, if you evaluate this, we will get $c_2 = 0$.

So, this constant of σ_{xx} is once again going to 0. So, σ_{xx} can be written like this where d_3 is the still unknown constant left to be evaluated. And that we can obtain by using the fifth boundary condition, that is 0 bending moment at both the edges. At both left and right simply supported ends, that is pinned ends, we are going to have 0 bending moment. So, $\int_{-d/2}^{d/2} b\sigma_{xx}|_{x=\pm l/2} y dy$, this refers to the right and left hinge points, integrated from $-d/2$ to $d/2$ equals 0.

Now, replacing these $\sigma_{xx}|_{x=\pm l/2}$. If you evaluate this integral, that will give us the value of $d_3 = \frac{w}{bd^3}\left(\frac{d^2}{10} - \frac{l^2}{4}\right)$. So, once this d_3 is obtained, you can replace this d_3 back in the σ_{xx} and get the complete expression of σ_{xx}. Now, coming to the sixth boundary condition, that is, The normal traction for right and left faces is 0.

σ_{xx} should be 0 at all values of y for $x = \pm l/2$. Now, this expression of σ_{xx}, if you substitute any value of y, you can see it is not possible to satisfy this particular condition. So, these right and left end faces are Here, sigma xx is supposed to be 0. This condition we are unable to satisfy.

Same here for this left-hand side face. So, this will cause the distortion of the face that is at the support point. So once again, we are going to have a solution which will have some approximation near the support point where one of the boundary conditions is not being satisfied.

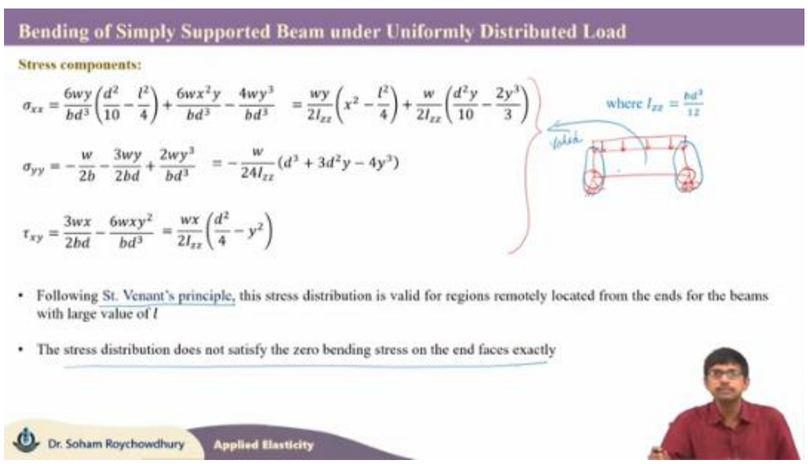


Now, finally, coming to τ_{xy}. τ_{xy}, once we replace or substitute all these zero and non-zero constants in the τ_{xy}, we will get this equation of in-plane shear stress τ_{xy}.

Now, we can see all the constants are now evaluated. $\tau_{xy} = \frac{3wx}{2bd} - \frac{6wxy^2}{bd^3} = \frac{6wx}{bd^3}\left(\frac{d^2}{4} - y^2\right)$. Once again, we have a parabolic variation of the transverse shear. But we need to see if the seventh and eighth boundary conditions are satisfied with this τ_{xy} or not because it is essential to satisfy this boundary condition.

These are coming from the vertical force balance. If these are not satisfied, our free body diagram is wrong. This is not just causing some distortion; it will cause the failure of the vertical force balance. So, we are bound to satisfy—our solution is bound to satisfy—this condition. So, $x = l/2$ on the right face, $\int_{-d/2}^{d/2} b\tau_{xy}\big|_{x=l/2} dy = \frac{wl}{2}$. Now, substituting τ_{xy} here with $x = l/2$, if you integrate it, you would get it to be the same as $\frac{wl}{2}$.

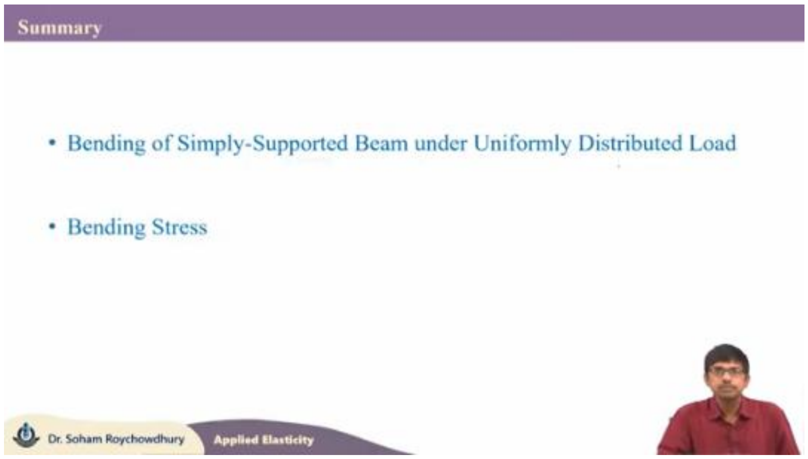
So, with these two obtained forms of τ_{xy}, this particular boundary condition 7 is automatically satisfied, which can be verified by substituting τ_{xy} in boundary condition 7 and x with plus $l/2$. You will get the left-hand side integral to come out as $wl/2$, as shown here. Similarly, if you go for boundary condition 8, that is, the force balance at the left-hand side hinge, $\int_{-d/2}^{d/2} b\tau_{xy}\big|_{x=-l/2} dy = -\frac{wl}{2}$. So, following a similar procedure, substituting $\tau_{xy}\big|_{x=-l/2}$, This integral will come out to be $-\frac{wl}{2}$. So, both boundary conditions 7 and 8 are automatically satisfied with the help of this particular τ_{xy} form.



Now, the overall stress components, if we write, they would be like this. This is σ_{xx}, then σ_{yy} and then τ_{xy}. These can be written in terms of I_{zz} as this. So, this is the total stress field present for the simply supported beam subjected to uniformly distributed loading. Note that these stress fields are valid only for the region which are remotely located from the two hinge points.

So, from the simple support beam around this two end hinge. these solutions are not valid for all the points away from those two hinge points apart from these and this region the solution these three equations are valid also note that the stress distribution is not satisfying the zero bending stress at the end faces exactly. Now following the Saint Venant's principle already we are excluding the solution near these end faces we are saying that these is valid for the far field. not near the end.

So, thus this violation of the of this last boundary condition is not going to affect our result because we had already imposed Saint Venant's principle and saying that solution is valid for the entire span of the beam just excluding the end portions where the hinges or reaction forces are there that will cause some stress concentration as well as distortion of the of the section at the end phases.



So, in total we had discussed the bending of a simple supported beam with the help of stress function approach when it is subjected to uniformly distributed load. In this particular approach, in this particular lecture, we obtained the expressions of all three in-plane stresses: σ_{xx}, σ_{yy} and τ_{xy} for this particular problem. Thank you.