APPLIED ELASTICITY

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Lecture- 32

Welcome back to the course on applied elasticity. In this lecture, we are going to continue our discussion on the bending of beams, which we started in the last lecture. A beam element is a one-dimensional continuum with one length much larger compared to the other lengths, and it undergoes bending when subjected to either a bending moment or a transverse load, a transverse shear load. So, we are going to consider various bending of beam problems under different types of loading.

The one we have already discussed is the pure bending problem of a cantilever beam subjected to a pure bending moment. Now, in today's lecture, we are going to discuss the second problem, where a beam is subjected to a concentrated transverse load at a specific section, and we are going to solve this bending of beam problem. Now, considering this beam subjected to a shear load with the cantilever displacement boundary condition. So, we are considering a cantilever beam which is subjected to a shear load W. l is the length of the beam, b is the width of the beam, d is the depth of the beam, and W is the vertical shear load at the free end of the beam. So, considering this point O to be the origin, The x equals 0 edge, the left-hand side edge, which is the x equals 0 edge, is the fixed edge.

end where the vertical shear load W is acting. The plane of bending is the xy-plane. All the assumptions discussed for the Euler-Bernoulli beam theory, which we covered in the last lecture, are also valid for this problem. Now, at any cross-section, the bending moment $M(x) = W(l - x)$. So, let us consider a section here at a distance x from the fixed edge.

So, the bending moment acting at this section is $M(x)$, which can be written as $W(l - x)$. The distance of that section toward the free end equals $(l - x)$, which, when multiplied by W, gives the bending moment at any section of this cantilever beam at a distance x from the fixed end. Now, we are going to write the boundary conditions for this cantilever beam subjected to transverse shear load at the free end. So, for the first one, the first condition is the condition on the free edge. So, this free edge is where x equals l, and y can take any value. So, once again, we are considering a plane stress problem. The beam is thin; the thickness, depth, and width of the beam are taken to be much smaller compared to l. So, as for this particular phase, the right-hand side vertical face free edge is free from any kind of axial loading. No axial load or no axial stress σ_{xx} is present. So, $\sigma_{xx}(l, y) = 0$. That is, the free edge is free of any kind of axial stress distribution.

Now, coming to the fixed end on the left-hand side, the bending moment for this fixed edge is equal to Wl. So, here bending moment M is equal to Wl, and we already know, as we had discussed, that the expression of bending moment is nothing but the integral of $b\sigma_{xx}ydy$ from $-d/2$ to $d/2$, and that is equal to 0 at the fixed edge. So, at x equals to 0, $\int_{-d/2}^{d/2} b\sigma_{xx}ydy = Wl$. So, this is valid only for x equals to 0, not for all. Coming to the third condition, as no axial forces are existing on the body, the net axial force, that is, the integral of $\int_{-\frac{d}{2}}^{\frac{d}{2}} b\sigma_{xx}dy = 0$ for all values of x. None of the sections are subjected to any axial force. So, this condition, if satisfied, ensures that there is no axial force present in the system.

Now, the next one is vertical force balance. At any particular cross-section, if you are taking a section here and then try to draw the free body diagram. So, W is acting here. This distance is l minus x. We are taking a section at a distance x from the fixed edge. Now, this particular section must be having a load W acting on it for force balance.

So, considering the free body diagram of this particular portion of the beam, on the right free edge, downward W is acting. So, the left free edge must have this upward W, along with that, some bending moment M is also there. Mx is also there, which would be W times l minus x. Now, these W give rise to a shear stress distribution. Due to this W, a shear stress would be there. Now, if you recall, this particular plane is the negative y plane. So, the shear stress on the negative y plane is in this direction.

So, considering a small element, σ_{xx} is in this direction, and σ_{yy} is in this direction. Now, the direction of τ_{xy} is like this. So, this particular plane being the negative x plane, the direction of τ_{xy} here is downward. So, this is τ_{xy}. If you integrate the τ_{xy} over that entire area.

So, $\int_{-d/2}^{d/2} b\tau_{xy}dy$. This integral would give us the net downward force. This is giving us the net downward force at any cross section. W from the free body diagram is acting in the upward direction. So, thus to balance it, we are adding a minus sign. So, every time whenever you are writing this force balance or moment balance equation, you have to clearly look for the direction. This left-hand side $\int_{-d/2}^{d/2} \tau_{xy} dy$ is going to give you the net downward force because that is acting on the negative x side where τ_{xy} is downward and W at that particular section is upward. So, it should be equal to $-W$, then only this condition is balanced and finally, as the top face and bottom face: top face is , $y = +\frac{d}{2}$ and bottom face is $y = -\frac{d}{2}$. So, both of these two faces being free of any normal and shear tractions, σ_{yy} and τ_{yx} both would be 0 for the top and bottom place, which are defined by $x, \pm\frac{d}{2}$.

So, that is the last boundary condition. So, in this particular problem, we are having these five conditions to be satisfied by the chosen stress function. We are going to start our solution with the most general stress function, which is a combination of second, third, and fourth-degree polynomial. So, while discussing about the polynomial form of stress function, we have seen that the zero and first-degree polynomial stress functions cannot be used for solving any problem.

Whereas, second, third, and fourth-degree can be used, or even higher degrees can also be used. So, here we are choosing our stress function to be a linear combination of second-degree, third-degree, and fourth-degree polynomials. So, a_2, b_2, c_2 these are the three constants in the second-degree polynomial; a_3, b_3, c_3, d_3 are the constants in the third-degree polynomial; a_4 to e_4 are the constants in the fourth-degree polynomial. So, in total, we have 3 plus 4 plus 5—total 12 unknown constants associated with this stress function.

Now, the first task after the choice of stress function is to check whether it satisfies the biharmonic equation or not, $\nabla^4\phi = 0$ or not. Now, as you know, the second-order and third-order polynomials directly satisfy the biharmonic equation. So, thus only the fourth-order or fourth-degree polynomial will give some extra constraint from the biharmonic equation. So, if you put this chosen ϕ in the biharmonic equation $\nabla^4\phi = 0$, that will give a relation between 3 constants of the fourth-degree polynomial as $c_4 = -3(a_4 + e_4)$.

If you impose this condition, then the fourth-degree polynomial is also going to satisfy $\nabla^4\phi = 0$. The second and third degree would always satisfy the polynomial equation, the biharmonic equation in general. Now, imposing this $c_4 = -3(a_4 + e_4)$ and using the definition of the stress components in terms of the partial derivative of ϕ. We can obtain σ_{xx}, σ_{yy}, τ_{xy}, the in-plane stress components as this. So, these 3 equations are giving the in-plane stress components, and here you can see c_4 is not there. I had replaced c_4 with the help of this particular equation.

Now, we are going to start our solution with these stress components, and we will try to evaluate all these unknown constants—11 unknown constants. by using the boundary conditions. So, I have written sigma yy and tau xy here from the previous slide. Now, we are considering two boundary conditions: five, σ_{yy} equals zero on the top and bottom surface. τ_{xy} is also zero on the top and bottom surface.

So, σ_{yy} and τ_{xy} equal zero for $x, \pm\frac{d}{2}$. So, these two boundary conditions, which are written in boundary condition five, are considered simultaneously. So, we know these are the expressions of σ_{yy} and τ_{xy}. This should be valid for all values of x, but $y = \pm\frac{d}{2}$, and then it should go to zero. So, substituting that here, we are substituting $y = \pm\frac{d}{2}$ in both σ_{yy} expression and τ_{xy} expression, and with that, we got these two expressions.

Now, I have clubbed the terms of x^2, x, and constant term separately. As you know, x is not getting replaced with any specific value, unlike y, y is replaced with $\pm\frac{d}{2}$, but x remains as x, and hence, we will have the terms of x, x^2, and those independent of x.

So, the x^2 term is kept first. So, $12a_4x^2$ is the x^2 coefficient term. The x coefficient is $6\left\{a_3 + b_4\left(\pm\frac{d}{2}\right)\right\}$. And this is the last term, which is independent of x. Similarly, in the τ_{xy} case, the x^2 term is $-3b_4x^2$, then we have the x term, and then we have the x-independent term or constant term.

Now, these both equations can go to 0 only if the coefficient of x^2, the coefficient of x, and the x-independent term all three of them are going to 0. All three of them are individually 0; then only this equation can be true for all values of x. Thus, we are getting three equations from each of these two boundary conditions: 12 by 4 0, $a_3 \pm \frac{b_4 d}{2}$, and then this equals to 0. Similarly, from τ_{xy}, we are also getting 3 conditions: 1, 2, and 3.

And note that in these 4 conditions, plus-minus signs are included, meaning from each of this, we are having 2 equations: one with a plus sign and another one with a minus sign. So, using all these equations, we will try to solve some of the constants. Already, 1 constant b_4 came out to be 0. Also, as $12a_4 = 0$, from this we are going to get $a_4 = 0$. Now, substituting b_4 0 here in this equation, as b_4 is already 0, $6a_3$ should be 0, meaning $a_3 = 0$. Moving further, from this equation, this is having 2 terms: $2b_3 \pm 6(a_4 + e_4)d = 0$.

Now, this is basically giving 2 equations: one with a plus sign and another with a minus sign. Both can be satisfied only if the term before the plus minus is 0, and the term after the plus minus is 0; then only both the equations can be satisfied. So, both with plus and minus signs, this can go to 0 if $2b_3$ is 0 and $6(a_4 + e_4)d$ is also separately 0. Now, as we already know, a_4 is going to 0; thus, this expression would give the first case as $b_3 = 0$ and the second case as $e_4 = 0$, since a_4 is already 0. Similarly, if you go for this one.

The independent x-independent term for σ_{yy} would give us $a_2 = 0$, as all the rest of the constants are already 0, and the last one will give us two conditions: one is $c_3 = 0$, and another is $b_2 + \frac{3d_4d^2}{4} = 0$. So, from here, I think we have 7 constants already set to 0 and 1 constraint relating b_2 and d_4. Only from boundary condition 5. Now, we will proceed further.

So, these 7 constants are coming out to be 0. So, I have written all these 0 constants here, whatever is already evaluated, and the stress fields are also written. Now, coming to the axial total axial force to be 0 at any cross-section: there is no axial force for this problem of bending with transverse loading at the free edge of the cantilever beam, thus integral $\int_{-d/2}^{d/2} b\sigma_{xx}dy = 0$ at any particular x. Now, this particular form of equation can be satisfied. If σ_{xx} is an odd function of y because of the limit. The limit is from $-d/2$ to $+d/2$. We want integral $\int_{-d/2}^{d/2} \sigma_{xx}dy = 0$ within the symmetric limit from $-d/2$ to $+d/2$. That can be valid only if y is an odd function. σ_{xx} is an odd function of y.

So, looking at the general expression of σ_{xx} to ensure this to be an odd function of y, we must have c_2 to be 0, c_3 to be 0, a_4 to be 0, and e_4 to be 0. An odd function of y means we can only have terms with y, y^3, y^5, and so on. y^2 term should not be there, constant term should not be there, y^4 term should not be there. So, thus by setting the coefficient of y^2, y^4, and all those even power terms of y and setting the y-independent term, that is also an even term.

Setting that to 0, we would get these 4 constants $c_2 = c_3 = a_4 = e_4 = 0$. And thus, after setting all these constants to 0, you would be getting our stress field to be like this: $\sigma_{xx} = 6d_3y + 6d_4xy$, $\sigma_{yy} = 0$, $\tau_{xy} = -b_2 - 3d_4y^2$. We started with 12 constants and now, just after using 2 boundary conditions, we are left with only these 4 terms in the σ field. Now, using the second boundary condition, that is, bending moment at fixed end $x = 0$ is equals to Wl.

So, putting $x = 0$ in this σ_{xx} equation, we would only have $6d_3y$ left, and substituting that here because this is valid only at $x = 0$. So, σ_{xx} at $x = 0$ is $6d_3y$. Putting that here and integrating it, we can obtain the $\frac{2Wl}{bd^3}$ and defining I_{zz}, the second moment of area with respect to the Z-axis for the rectangular beam cross-section, as $\frac{bd^3}{12}$. d_3 can also be written as $\frac{Wl}{6I_{zz}}$. So, one of the constants now we have obtained in terms of W, the applied external shear loading. Now, moving forward to the first boundary condition: σ_{xx}

equals 0 at the free edge, that is, at $x = l$. If you substitute that, it would be $6d_3y + 6d_4ly = 0$. d_3 we had already obtained like this. So, if you substitute d_3 here, we can get d_4. So, d_4 is coming out to be $-\frac{2W}{bd^3}$ or, in terms of I, it would be $-\frac{W}{6I_{zz}}$. So, the second constant we are also getting now.

Now, coming to the fourth boundary condition, which is the vertical force balance, shear force balance in the y-direction. Integral of $b\tau_{xy}$ over the area $-d/2$ to $+d/2$ equals $-w$, where τ_{xy} is this. d_4 we had already evaluated; b_2 is unknown. So, putting τ_{xy} in this equation, then substituting d_4, we can get the expression of b_2. d_4 is written in terms of this $-\frac{2W}{bd^3}$. Putting that here, we can obtain b_2 as $\frac{3W}{2bd}$.

With this, we had obtained all the unknown constants and then putting them back in the stress field. Our σ_{xx} would be $\frac{Wly}{I_{zz}} - \frac{Wxy}{I_{zz}}$, and now using the bending moment at any section $M(x) = W(l - x)$ for this cantilever beam, we can write this $M(x) = W(l - x)$ as M, and thus σ_{xx}, the bending stress generated, will be $\frac{My}{I_{zz}}$, which is the well-known form of bending stress during bending of a cantilever beam with transverse shear load: $\frac{My}{I}$. The same form of σ_{xx} we are getting here. Coming to σ_{yy}, that equals 0, but τ_{xy} is non-zero here because this is a case of bending due to transverse shear load. Transverse shear load will directly give rise to transverse shear stress τ_{xy}, which is non-zero, putting the constant value b_2 and d_4 in that expression of τ_{xy}.

And then, writing it in terms of I_{zz}, τ_{xy} can be written as $\frac{W}{2EI_{zz}}\left\{y^2 - \left(\frac{d}{2}\right)^2\right\}$. So, here you can see this is clearly a parabolic distribution of τ_{xy} over y. So, along the depth, the shear stress is going to vary parabolically, whereas σ_{xx} is going to vary linearly over y along the depth. Now, after obtaining this stress field, the stress part is solved.

Now, we will proceed with the displacement. So, first we will get the strain fields for the given stresses using the constitutive equations, similar to the previous problem. The problem of pure bending is similar to that, and we are going to obtain the strain fields as $\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$ because ε_{yy} and ε_{zz} are both 0, and ε_{yy} and ε_{zz} would be $-\frac{\nu\sigma_{xx}}{E}$, where ν is the Poisson's ratio. So, ε_{xx} would be $\frac{My}{EI_{zz}}$, and ε_{yy} and ε_{zz}, the out-of-plane normal strains, would be $-\frac{\nu My}{EI_{zz}}$.

And ε_{xy}, the in-plane shear strain, would be $\frac{\tau_{xy}}{2G}$. ε_{xy} is the tensorial in-plane shear strain, and that would come out to be $\frac{(1+\nu)W}{2EI_{zz}}\left\{y^2-\left(\frac{d}{2}\right)^2\right\}$, and the other two out-of-plane, shear strains ε_{xz} and ε_{yz} are 0 because, recall the assumption: out-of-plane shear deformations are neglected. So, ε_{xz} and ε_{yz} must be 0.

Now, coming to the strain-displacement equation, we are going to replace these strain components in the strain-displacement equation. We have six strain-displacement equations, which are written like this. All three normal strains are non-zero, but only one shear strain, the in-plane shear strain, is non-zero. ε_{xz} and ε_{yz}, the out-of-plane shear strains, are 0. Now, starting with ε_{xx} is $\frac{\partial u}{\partial x}=\frac{My}{EI_{zz}}$. M can be written as $W(l-x)y$; we actually got this form of solution later and replaced M. So, $\varepsilon_{xx}=\frac{\partial u}{\partial x}$, that is $\frac{W(l-x)y}{EI_{zz}}$. Now, for finding u (the axial displacement), we are going to integrate this equation with respect to x,

and with that, we can get u to be $\frac{Wy}{EI_{zz}}\left(lx-\frac{x^2}{2}\right)+u_0$, which is a function of y. Earlier, we had $l-x$ integrated with respect to x, which would give $lx-\frac{x^2}{2}$. Now, we are taking this integration constant to be a function of y. Note that for this displacement formulation solution, we are going for an approximation. We are neglecting the out-of-plane displacement, meaning u is a function of x and y only (not z), v is a function of x and y only (not z), and w is 0. We are not considering it approximately 0; we are not considering the out-of-plane displacement of the body. For the pure bending, that was an exact solution; all 3 were considered to be non-zero. Now, here we are going for some approximation, setting w to 0 and considering u and v to be independent of z. These integration constants would only be functions of y. So, u_0

is a function of y because this is an integral with respect to x, and we are not considering these displacement components to be functions of the out-of-plane coordinate z. Now, after ε_{xx}, in a similar fashion, using ε_{yy} (this equation: $\frac{\partial v}{\partial y}=-\frac{\nu My}{EI_{zz}}$), and replacing M as $\nu W(l-x)$, we integrate this with respect to y to get v as $-\frac{\nu Wy^2}{2EI_{zz}}(l-x)$ plus this integration constant is function of x just $v_0(x)$ independent of z as this integral was with respect to y from $\frac{\partial v}{\partial y}$ we are getting v. So, these constant function coming due to integral should be function of x only that is taken to be $v_0(x)$. So, we got two in plane displacement fields u and v. Now, these two will be replaced back in this equation in the shear strain equation ε_{xy}.

So replacing both u and v on the left hand side of ε_{xy} tensorial shear strain expression we would be getting this big expression the left hand side is function of x and y right hand side is function of y now we will rearrange the terms and club them separately one set of term which is function of x another set of term which is function of y and right hand side is the term which is independent of x and y means constant term. So, these first term is function of x plus second term is function of y that is equals to constant and these should be true for all values of x and y.

That is possible only if both $f_1(x)$ and $f_2(y)$ are two constants. This should be one constant, and this should be another constant. The summation of those two will be this third constant, which is there on the right-hand side. For if that is valid, then only this equation is true for all values of x and y. So, $f_1(x)$ is equal to one constant C_1, and $f_2(y)$ is another constant C_2, where C_1 and C_2 are related through the right-hand side constant, which is $-\frac{(1+v)Wd^2}{4EI_{zz}}$. Now, using this f_1 and f_2, this part integrating this with respect to x, we can obtain v_0. Similarly, using $f_2(y)$, integrating this with respect to y, we can obtain u_0. So, using $f_1(x)$ equals to a constant C_1 integrating this with respect to x, we can obtain $v_0(x)$ to be this and during this integral with respect to x, another constant C_3 is coming.

Similarly, starting from $f_2(y) = C_2$, integrating it with respect to y, you can get u_0 as a function of y equals to $C_2 y + \frac{(2+v)Wy^3}{6EI_{zz}}$, where C_4 is another integral constant. So, note that we have 4 constants: C_1, C_2, C_3, C_4, but all 4 are not independent. C_1 and C_2 are related through this particular equation. So, overall displacement components were obtained like this and we had neglected the out-of-plane displacement component.

Assuming that the displacement is only confined to the mid-plane, that is, the plane of loading or plane of bending, which is nothing but the z equals to 0 plane. Now, we will try to find out the constants $C_1, C_2, C_3,$ and C_4 by using the boundary conditions. The left edge, x equals to 0 edge, is the cantilever end or built-in end. Now, applying the boundary condition at the built-in end. So, let us say this is the origin (0, 0, 0).

We must have u and v to be 0 at the origin. The third one, the z-coordinate, does not matter because we have already taken u and v to be independent of z. So, $u(0,0)$, $x = 0, y = 0$, at that point u should be 0, v should also be 0, and that would result in these two constants C_3 and C_4 going to 0. C_1 and C_2 are remaining. Now, if you consider the entire edge, the fixed edge, the axial displacement for all the points for the cantilever edge or built-in edge, that is, the x equals to 0 edge, for all values of y, u must be 0, so $u(0, y)$ should be 0 for this entire fixed edge. However, if you put it put x equals to 0 in this equation, you would be getting $C_2 y + \frac{(2+\nu)Wy^3}{6EI_{zz}} = 0$. Now, note that for all values of y, this cannot be 0.

Even if you force $C_2 = 0$, even after that, for all values of y, this cannot be set to 0, meaning this particular fixed edge. Cannot have 0 axial displacement from this solution of the elasticity theory. However, by the displacement boundary condition, it is enforced to have 0 displacement, meaning that particular cross-section is going to distort. The distortion of the beam cross-section is unavoidable because, along the entire built-in end, we cannot ensure that u equals 0. That results in the distortion of the beam cross-section at the origin. We are able to ensure that, but not for all values of y on the built-in end.

Now, we are still left with the constants C_1 and C_2. To evaluate those, we will enforce the slope condition: slope equals 0 at the built-in end. From the 0 slope condition, 0 slope is defined as $\left.\frac{\partial v}{\partial x}\right|_{(0,0)}$ is 0. So, from that 0 slope condition, putting the expression of v, obtaining $\frac{\partial v}{\partial x}$, substituting that here, and setting $0, 0, x\ 0, y\ 0$—that is, slope at the origin $(0, 0)$ is 0. At $x\ y\ 0$, $\frac{\partial v}{\partial x} = 0$, and from that, we would get C_1 to be 0. Now, C_1 and C_2 are related through this equation: $C_1 + C_2$.

They were not independent constants; they are related through this. C_1 being 0, C_2 should be $-\frac{(1+\nu)Wd^2}{4EI_{zz}}$. Now, with this, three constants came out to be 0: C_1, C_3, and C_4. C_2 is non-zero. If you replace them back in u and v, we will get the overall displacement field. Now, if you consider the deflection of the central line of the beam, that is this line. The neutral axis is defined by y equals 0.

So, $u(x, 0)$ — no axial displacement of the neutral axis. If you replace y to be 0 in the expression of u, you can verify the neutral axis change in length is 0. The neutral axis is not supposed to elongate; that theory is getting verified here. Now, $v(x, 0)$, the vertical deflection of the neutral axis, would come out to be $\frac{W}{EI_{zz}}\left(\frac{x^3}{6} - \frac{lx^2}{2}\right)$. Now, if you are considering the right endpoint of the neutral axis, that is $x = l$, y equals 0. So, $v(l, 0)$, the free-end transverse deflection of the beam, putting x equals L here in this equation, would come out to be $-\frac{Wl^3}{3EI_{zz}}$.

And that is identical to the cantilever beam deflection obtained from the elementary or simple Euler-Bernoulli beam theory. A cantilever beam subjected to a load W is supposed to have a downward deflection of $\frac{Wl^3}{3EI_{zz}}$ from the simple beam theory. Here also, we are getting the same thing from this elasticity approach. So, we are getting exactly the same displacement field at the free-edge midpoint of the transverse deflection to be $-\frac{Wl^3}{3EI_{zz}}$ in the direction of W, that is, along the negative y-direction.

However, this solution is approximate because distortion is unavoidable at the fixed edge of the beam. One of the boundary conditions we are unable to satisfy. So, in total, in this lecture, we discussed the bending of a cantilever beam subjected to a transverse shear load at the free end. We discussed the bending stresses, and the free-end deflection was obtained for such bending problems. Thank you.