APPLIED ELASTICITY

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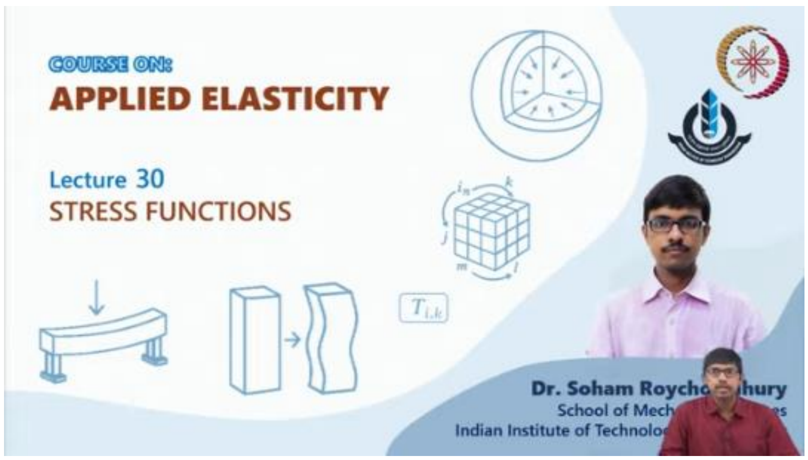
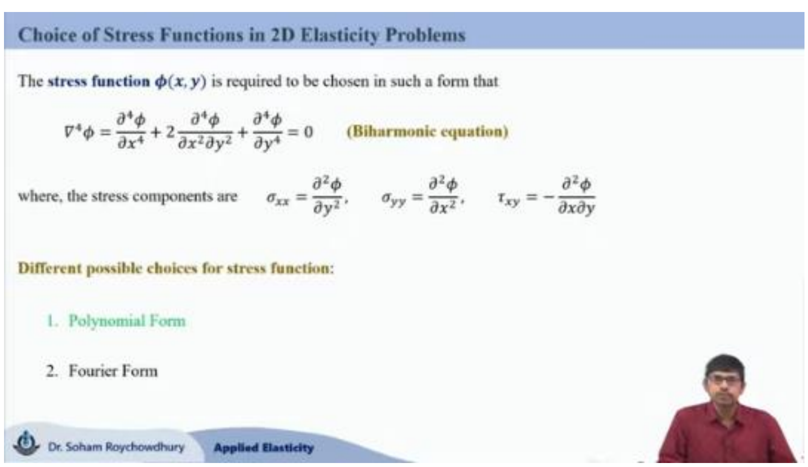
WEEK: 06
Lecture- 30



Welcome back to the course on applied elasticity. In today's lecture, we are going to talk about stress functions. We had already started our discussion on stress functions in the last lecture.

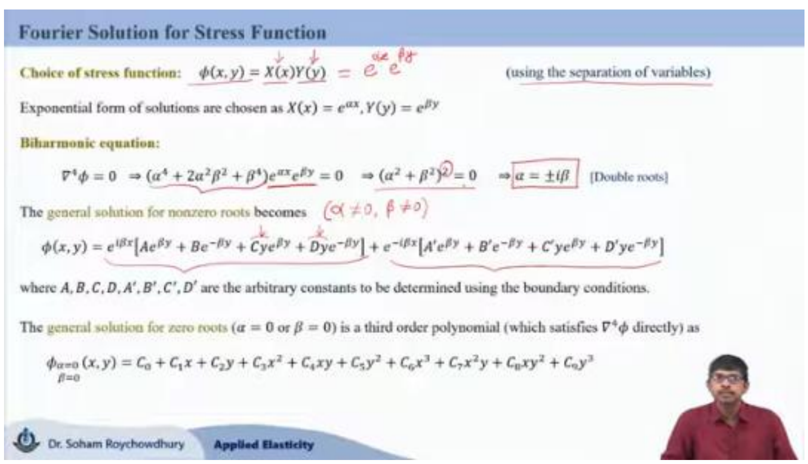


So, the stress function phi in the rectangular Cartesian coordinate system is required to be chosen in such a form that it satisfies the biharmonic equation.

So, $\nabla^4\phi = 0$. If this is satisfied by any chosen ϕ or stress function, which is a function of x and y, then we can use that particular function as a stress function for solving any elastic deformation problem. Now, this particular equation is valid only for two-dimensional elasticity problems. That means, either for plane stress problems or for plane strain problems, which we had discussed in previous lectures.

Now, in the rectangular Cartesian coordinate system, this biharmonic operator $\nabla^4\phi$, if we expand it, becomes $\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$. Now, in terms of this stress function ϕ, the three in-plane non-zero stress components can be defined as $\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$. These two are in-plane normal stresses and the in-plane shear stress $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$.

And if you replace these stress components in terms of the stress function in the equilibrium equation, that would be directly satisfied. Now, for the choice of ϕ, the stress function, we have two possible options. The first one is the polynomial form of solution for ϕ or power series solution that was discussed in the last lecture. Now, in today's lecture, first we are going to talk about the Fourier form solution of $\phi(x, y)$ or stress function. So, stress functions can be chosen either in polynomial form or in Fourier form.



Now we are going to talk about the Fourier form of solution for the stress function to solve any 2D elasticity problem. Now, as I told, there are two possible options for the choice of stress function. One is polynomial, another is Fourier. So, if you are going for the Fourier form of solution, the first assumption is we are assuming a separated solution. So, the ϕ stress function is a function of both x and y, two in-plane rectangular Cartesian coordinates.

Now, ϕ is chosen as a product of two different functions, capital X and capital Y. Now, capital X is a function of solely small x, a single Cartesian coordinate, small x. And capital Y is taken to be a function of another variable, small y. So, ϕ was a general function of both the Cartesian variables x and y, which is taken to be a product of two separate functions, capital X, which is dependent only on the small x variable.

Small x variable, and then capital Y, which is dependent only on the small y variable. So, this is the first function. This is the second function. The product of these two functions is going to give us the overall stress function. So, as we are taking two separate functions of x and y, we are calling it the separated solution or the method of separation of variables is being used here.

Now, exponential form of functions are chosen both for capital X and capital Y. In this particular manner, capital X is chosen to be $e^{\alpha x}$, and capital Y is chosen to be $e^{\beta y}$. So, first we are separating the stress function solution: one is a function of small x, another is a function of small y. Both these functions are chosen to be exponential functions as $e^{\alpha x}$ and $e^{\beta y}$.

So, if you replace them back. This phi would be e to the power alpha x times e to the power beta y. Now, this particular ϕ or the stress function must satisfy the biharmonic equation. So, $\nabla^4\phi = 0$. If you replace this form of phi in the biharmonic equation, the expression would look like this: $\alpha^4 + 2\alpha^2\beta^2 + \beta^4$.

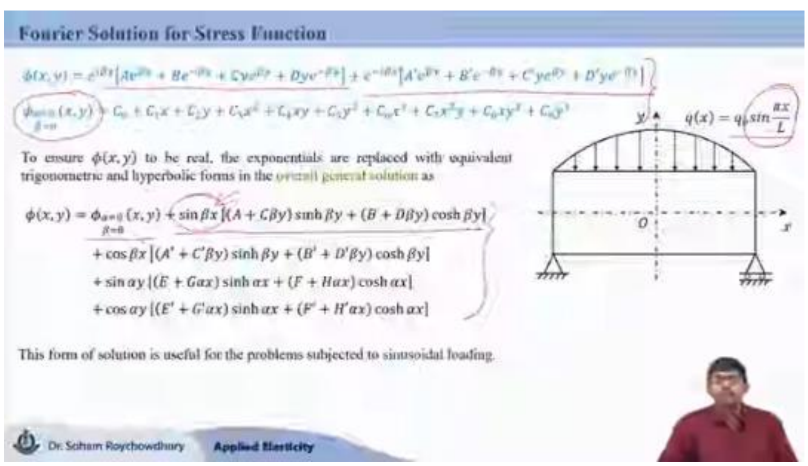
This entire thing multiplied with $e^{\alpha x}e^{\beta y} = 0$. Now, this particular expression being 0 for all values of x and y, this particular part $e^{\alpha x}e^{\beta y}$ cannot be 0 always. So, hence the coefficient, whatever is there within the bracket, that should be equal to 0, and then only this equation can go to 0.

Now, this coefficient $\alpha^4 + 2\alpha^2\beta^2 + \beta^4$ is nothing but $(\alpha^2 + \beta^2)^2$, and that must be equal to 0. So, from this, we would be getting a relation between α and β. And that relation would come out to be $\alpha = \pm i\beta$, and note that there exists a double root of α because of this square. So, as $(\alpha^2 + \beta^2)^2 = 0$, we would be having 4 solutions of α: 2 solutions are plus $i\beta$, another 2 solutions are $-i\beta$. So, plus $i\beta$ are repeated roots for this particular equation. Now, substituting this relation between α and β back in the expression of α. For non-zero values of α and β. So, α not equals to 0 and β not equals to 0.

For non-zero solution of α and β, if you replace this α equals to $\pm\beta$ with a double root case in the expression of α here, that $\phi(x, y)$ would be having two terms. One first term is $e^{i\beta x}$ multiplied with $Ae^{\beta y} + Be^{-\beta y} + Cye^{\beta y} + Dye^{-\beta y}$.

So, this entire set of term this is coming due to $\alpha = \pm i\beta$ and that is repeated twice. So, that is why this Cy term and Dy term these are the repeated root terms. if there was a single root then only first 2 A and B terms would be coming as there are repeated roots along with a Cy term is there along with b Dy term is also there similarly for $\alpha = \pm i\beta$ this similar second term the entire second term is there So, in total $\phi(x, y)$ is having 8 terms 4 involving $e^{i\beta x}$ another 4 involving $e^{-i\beta x}$ and in total $A,\ B,\ C,\ D,\ A',\ B',\ C',$ and D' total 8 arbitrary unknown constants are present which we need to solve by using the boundary conditions of the elasticity problem. So, this solution is the general solution for the non-zero roots α and β not equals to 0 with which the bi-harmonic condition can be satisfied by this chosen form of ϕ as $e^{\alpha x}\ e^{\beta y}$.

Now, if you are looking at the 0 roots with $\alpha = 0$ and $\beta = 0$, that may be another possible solution with which this particular equation would also be satisfied. So, with $\alpha = 0$ and $\beta = 0$, this should be a third-order polynomial function. And that can be written as $C_0 + C_1x + C_2y + C_3x^2 + C_4xy + C_5y^2 + C_6x^3 + C_7x^2y + C_8xy^2 + C_9y^3$. So, this is a general third-order polynomial containing all the zeroth-order, first-order, second-order, and third-order terms with a total of 10 unknown constants. So, if we combine both the non-zero root for α not equal to 0, β not equal to 0, and the zero root, the total general solution, the overall general solution, can be written as the summation of these two terms.

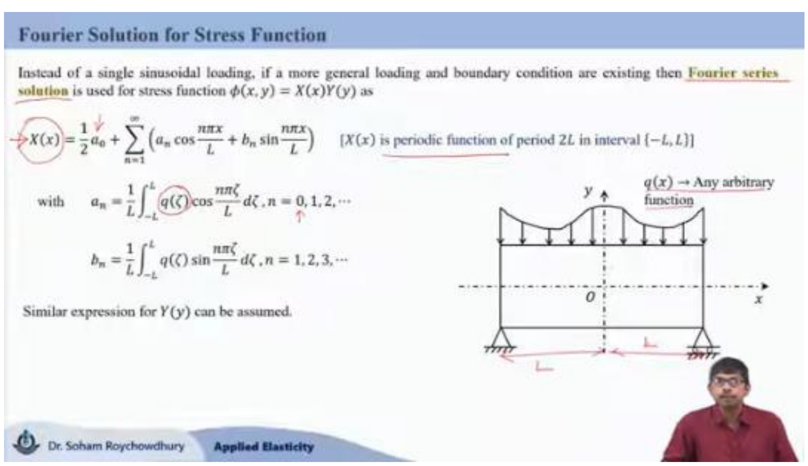


Now, here you can see we are having $e^{i\beta x}$, $e^{-i\beta x}$ terms. We also have $e^{i\beta x}$, $e^{-i\beta x}$ terms. Now, these terms can be rewritten in terms of sine and cosine. These trigonometric functions, as well as sine hyperbolic and cosine hyperbolic, can be expressed in terms of these two hyperbolic functions. If we rewrite that, the overall ϕ would be like this.

So, the first term of ϕ is the 0 solution, which is nothing but this—the general third-order polynomial or polynomial up to the third order. Then, these two terms for the non-zero solution are rewritten as this. We are having four terms here: $\sin\beta x$, $\cos\beta x$, $\sin\alpha y$, $\cos\alpha y$, and they are multiplied with respective factors. This also contains eight constants, which are required to be obtained with the help of boundary conditions. Now, this form of solution is useful if the sinusoidal loading is acting on the system.

So, if you consider this example of a simply supported beam, which is subjected to a transverse load of intensity $q(x) = q_o sin\frac{\pi x}{L}$, you can see this load acting here has a sinusoidal variation on the top surface. So, for solving such a problem, the polynomial form of stress functions would not be useful—it cannot be used to solve such problems. So, for this kind of problem, we can use this Fourier form of solution with the $\sin\beta x$ term being present, as the loading is of $sin\frac{\pi x}{L}$ type. This $\sin\beta x$ term should only be there; the other three should be neglected.

If you have a loading of $\cos\beta x$ type, then the cosine term should be chosen. So, based on the type of loading, only those set of constants can be chosen to be non-zero, and by setting the rest of the constants to 0, we can get a proper Fourier form of solution suitable for the problem. So, for solving this particular example, we should have only A$, B, C$, and D—these four constants—to be non-zero, which are associated with the $\sin\beta x$ term. That would be added with the 0 polynomial form solution, and that would give us the actual solution for this particular elasticity problem: the deformation of a beam subjected to transverse sinusoidal loading.



Now, instead of this simple sinusoidal loading, we may have some complex loading where the loading is not just a single sine. Along with the sign, there may be multiple other functions appearing. So, for such cases of a more general loading or more general boundary conditions, a Fourier series solution is used for the stress function ϕ. So, ϕ is written as capital X, which is a function of small x, and capital Y, which is a function of small y.

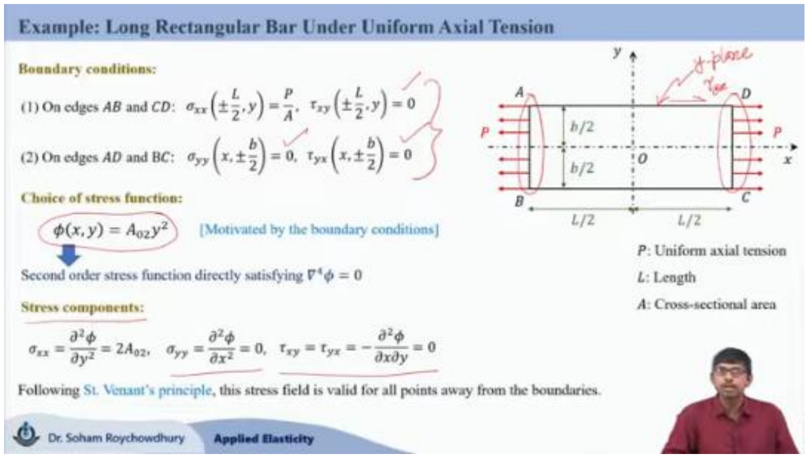
Now, capital X and capital Y are both chosen as Fourier series solutions. So, first we are writing capital X as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$. So, you can see this a_0, a_1 till a_∞, b_1, b_2 till b_∞; these are the constants for the Fourier cosine and sine series that we need to evaluate. This capital X can be written in terms of this Fourier series only if it is a periodic function of period $2L$.

over an interval of $-L$ to $+L$, and for such cases, the constants a_{n} and b_{n} can be obtained like this: a_{n}, the cosine function constants or coefficients, can be obtained as $\frac{1}{L}\int_{-L}^{L} q(\zeta)\cos\frac{n\pi\zeta}{L}d\zeta.\ n = 0,\ 1,\ 2,\ \cdots\infty$, and b_{n} is obtained as $\frac{1}{L}\int_{-L}^{L} q(\zeta)\sin\frac{n\pi\zeta}{L}d\zeta,\ n = 1,\ 2,\ 3, \cdots$ to ∞. Note that for a_{n}, we have n varying from 0 because we have this a_0 term. For b_{n}, it starts from 1; no b_0 term is present.

So, once we are given some arbitrary loading $q(x)$, we can use that q here, and with that, obtain the corresponding a_{n} and b_{n}, the coefficients of this Fourier series. Then, with respect to that, we can express this capital X. Similarly, in a similar fashion, the coefficients of sine and cosine of capital Y can be obtained, and a similar expression can be written. So, for any general periodic function if that is acting as loading on the system, then we can use this Fourier form of solution.

Let us once again consider a simply supported beam which is subjected to a general periodic transverse loading $q(x)$. So, this $q(x)$ is any arbitrary function acting on the top face, but it must be periodic. So, that constraint is there. So, let us say the total length is $2L$. So, this much is L, this much is also L, so the total length being $2L$, and $q(x)$ being periodic with period $2L$.

We can write this $q(x)$ with the help of a Fourier series, and then this particular choice of Fourier series form of the stress function can be used to solve this problem.



Now, we are going to take one example problem, which is a very simple example of a thin rectangular bar. It has quite a large length as compared to its width and thickness, and it is subjected to uniaxial tension. So, x is the axis along the length of the bar, and z is along the thickness, which is very small. We are considering the bar subjected to uniaxial tension of load P along the x-axis, along its length.

Now, we are interested in obtaining the deformation solution, the stress distribution, and the displacement fields generated within this particular bar subjected to uniaxial tension with the help of the stress function or elasticity approach. From undergraduate solid mechanics, the solution is well known. It should only have tensile stress of $\sigma_{xx} = \frac{P}{A}$, where A is the area of the cross-section, and we can also get the deflection.

Easily with the help of Young's modulus E. Now, here using the stress function approach, we would try to get the overall solution of this particular simple problem. Let us consider the boundary conditions: the length of the bar is taken to be L, the width of the bar is taken to be B, and the thickness is negligible along the z direction. So, as the thickness is negligible, this is basically a plane stress problem. The origin O is chosen at the center of the bar.

Thus, x varies between $-\frac{L}{2}$ to $+\frac{L}{2}$, and y varies between $-\frac{B}{2}$ to $+\frac{B}{2}$. So, that defines the domain of the problem. So, P is the uniform axial tension acting along the x direction, L is the length of the bar, and A is the cross-sectional area of the bar. Now, let us write the boundary conditions. So, there are four boundaries defined: AB, BC, CD, and AD. These are the four boundary edges.

First, consider the edges AB and CD, which are parallel to the y-axis or these are the x-plane and minus x-plane, the left-hand side and right-hand side boundaries. So, first, we are considering these two boundaries, AB and CD. Both of them are free of any kind of shear stresses. So, thus, τ_{xy} for those two edges So, the x value for AB is $-\frac{L}{2}$, and the x value for CD is $+\frac{L}{2}$.

So, these two planes AB and CD are defined with $x = \pm\frac{L}{2}$ with any value of y. y can be anything because for all these points on AB, y value is changing between $-\frac{B}{2}$ to $+\frac{B}{2}$. Now, shear stress is 0 for all the points for $\pm\frac{L}{2}, y$. Whereas, normal stress $\sigma_{xx} = \frac{P}{A}$ because P is the axial load applied for both the faces A, B and C, D. So, $\sigma_{xx}\left(\pm\frac{L}{2}, y\right) = \frac{P}{A}$ whereas, shear stress $\tau_{xy} = 0$ for those two edges.

Now considering other two edges AD and BC those are completely stress free. No external traction is acting on top and bottom faces AD and BC. Thus both normal stress σ_{yy} and shear stress τ_{yx} would be 0. Now how are we defining these two edges? here x is varying between $-\frac{L}{2}$ to $+\frac{L}{2}$ along this edge x is varying that is not constant.

However, y is constant as $+\frac{B}{2}$ for edge AD y is constant as -$\frac{B}{2}$ for edge BC. So, thus $\sigma_{yy}\left(x, \pm\frac{b}{2}\right) = 0$ means these 2 edges are free of any normal traction. $\tau_{yx}\left(x, \pm\frac{b}{2}\right) = 0$ means these 2 edges AB and BC are free of any shear traction. And note that here I am using τ_{yx} because these are y planes as these are y.

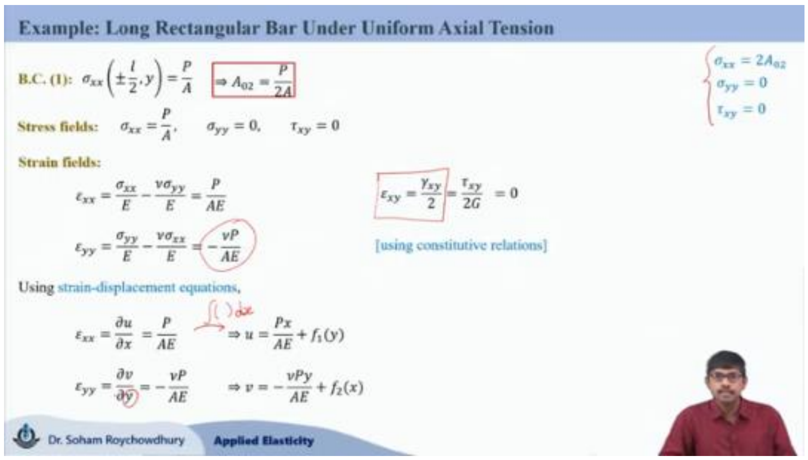
The first index should be y and the second index is x, which is the direction of shear stress along the x on y plane. So, that is equal to 0. However, we know from the equality of cross shear we can always write $\tau_{xy} = \tau_{yx}$. Now, moving forward to the choice of stress function here, motivated by this set of boundary. We should choose our stress function to be having only a single non-zero term as $A_{02}y^2$.

This is a polynomial form, second degree polynomial, and with only one non-zero term that is $A_{02}y^2$. Because if you go with this stress function, which obviously satisfies the biharmonic equation as it is second order, the stress component σ_{xx} can be obtained as $\frac{\partial^2\phi}{\partial y^2} = 2A_{02}$. Both σ_{yy} and τ_{xy} are 0 with these chosen form of ϕ. We are getting $\sigma_{yy} = 0$, $\tau_{xy} = 0$, and that would satisfy this set of boundary conditions automatically. We need to choose a stress function in such a way.

The boundary conditions, all the boundary conditions are satisfied. So, we know that we are expecting a constant stress along x direction, normal stress along x direction, and rest of the stresses should be 0. So, that is why this particular choice of some constant y square L stress function would give us that solution. So, you can take other terms like.

$A_{20}x^2$, but that constant will go to 0 as soon as you impose this boundary conditions of τ_{xy}, τ_{yx} and σ_{yy} to be 0.

And now, following the Saint-Venant principle, we know that this particular stress field, the stress components defined through these equations are valid away from the boundary. So, near this region, the effect of stress concentration may be present. So, apart from the region of loading, for all the far-field points, this particular stress component or stress field solution is valid.



Now, the only boundary condition left to be satisfied is this: $\sigma_{xx}\left(\pm\frac{l}{2}, y\right) = \frac{P}{A}$. Now, we had obtained this particular stress field where σ_{xx} is a constant $2A_{02}$, putting it here. A_{02} would come out to be $\frac{P}{2A}$. So, if you put this back in the stress field, our σ_{xx} would be $\frac{P}{A}$, $\sigma_{yy} = 0$, and $\tau_{xy} = 0$. Using this, we can obtain the strain fields. $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E}$, $\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E}$, and ε_{xy}, which equals $\frac{\gamma_{xy}}{2}$.

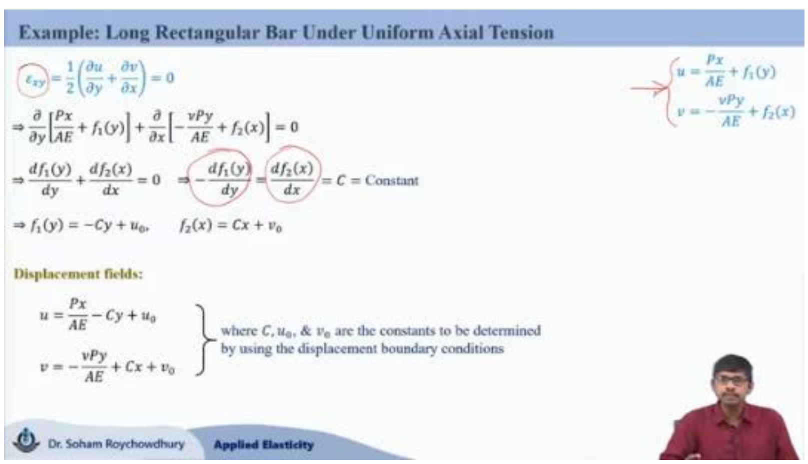
So, ε_{xy} is tensorial shear strain, and γ_{xy} is engineering shear strain. So, tensorial shear strain and engineering shear strains are related such that half of the engineering shear strain equals tensorial shear strain, and engineering shear strain equals the shear stress by G. So, $\varepsilon_{xy} = \frac{\tau_{xy}}{2G}$. Now, putting this $\sigma_{xx} = \frac{P}{A}$, $\sigma_{yy} = 0$, and $\tau_{xy} = 0$, we would get ε_{xx} to be $\frac{P}{AE}$, ε_{yy} to be $-\frac{\nu P}{AE}$, and ε_{xy} to be 0 as τ_{xy} is 0. Now, moving forward after obtaining the strain fields, using the strain-displacement equations, we would like to get the displacement fields.

Now, we know that ε_{xx}, that is the normal strain along the x-direction (longitudinal axis), is equal to $\frac{\partial u}{\partial x}$, and we had obtained ε_{xx} as $\frac{P}{AE}$. Similarly, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, which is obtained as $-\frac{\nu P}{AE}$. So, we are going to substitute it in the expression of strain-displacement relations.

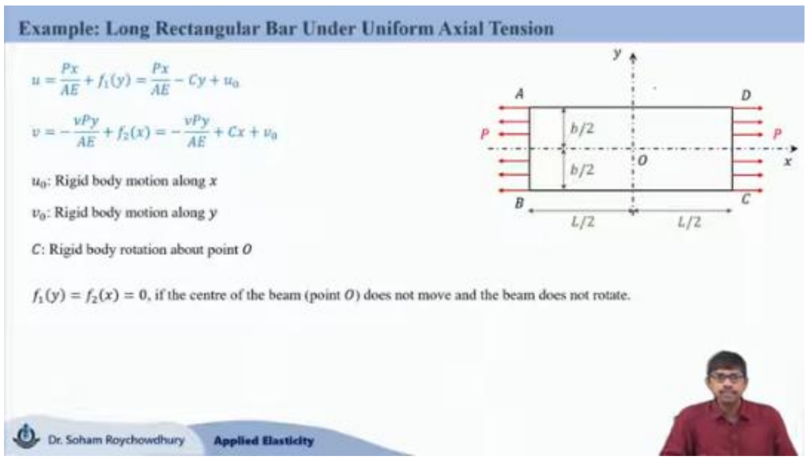
So, $\frac{\partial u}{\partial x} = \frac{P}{AE}$, and $\frac{\partial v}{\partial y} - \frac{\nu P}{AE}$. Now, integrating the first equation with respect to x, we can get the expression of u.

Similarly, integrating the second equation with respect to y, we can get the expression of v. So, if you integrate the first equation, $\frac{\partial u}{\partial x} = \frac{P}{AE}$, with respect to x, we would be getting $u = \frac{Px}{AE} + f_1(y)$. Now, what is this $f_1(y)$? $f_1(y)$ is the integration function. So, as we are going for the integral with respect to dx, we can have a constant present as an integration constant, or we can have any general function of y present as an integration function. Now, if you differentiate this du to get $\frac{du}{dx}$ or $\frac{\partial u}{\partial x}$, that would give us $\frac{P}{AE}$ irrespective of the form of $f_1(y)$. So, the general solution of u is taken to be $\frac{Px}{AE} + f_1(y)$. Similarly, integrating the next equation, $\frac{\partial v}{\partial y} - \frac{\nu P}{AE}$, with respect to y. This integral is with respect to y because we were having the derivative of v with respect to y.

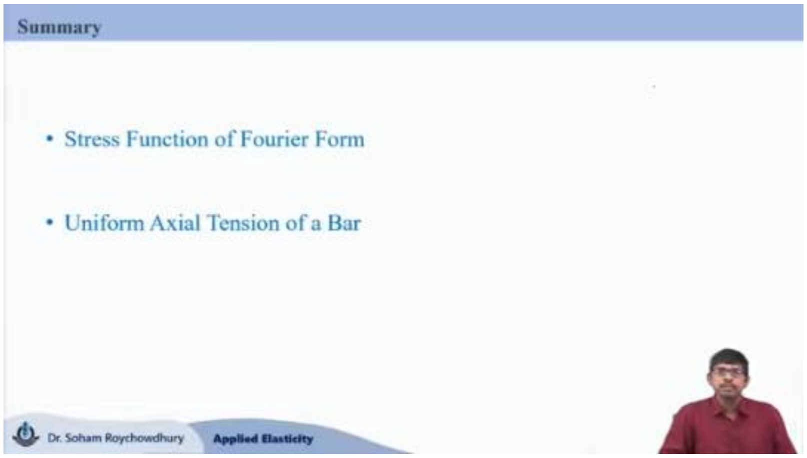
So, this would give $v = -\frac{\nu P y}{AE} + f_2(x)$. Here, the arbitrary function coming from integration is a function of x because the integral was with respect to y. So, we got these u and v expressions as $u = \frac{Px}{AE} + f_1(y)$ and $v = -\frac{\nu P y}{AE} + f_2(x)$, and we are left with a shear strain condition $\varepsilon_{x\gamma}$ should be 0, which is half of $(\partial u/\partial y + \partial v/\partial x)$.



So now, these obtained forms of u and v, which are derived from the normal strains, are substituted back into the expression for $\varepsilon_{x\gamma}$. If you simplify this by substituting u and v expressions here, it would look like this. So, $\partial/\partial y$ of $\frac{Px}{AE}$ term would go to 0. Similarly, $\partial/\partial x$ of $\frac{\nu P y}{AE}$ term would also go to 0. So, we would only be left with two terms involving f_1 and f_2 as $\frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} = 0$, which alternately we can write as minus $\frac{df_1(y)}{dy}$ equals to $\frac{df_2(x)}{dx}$ and this left hand side of this is only function of y whereas right hand side is only function of x and this is one function of x is equals to another function of y is valid for all values of x and y only if those two functions are constant. Otherwise, this being a function of y cannot be equal to another function of x for all values of x and y. This is possible only if both the functions are constant. Let us choose that constant to be C and hence, we can get $f_1(y) = -Cy + u_0$ and $f_2(x) = Cx + v_0$, where u_0 and v_0 are two arbitrary integration constants. So, after obtaining $f_1(y)$ and $f_2(x)$, if you replace them back in these expressions of u and v. The total displacement field is obtained as $u = \frac{Px}{AE} - Cy + u_0$ and $v = -\frac{\nu Py}{AE} + Cx + v_0$. So, these are the axial and transverse displacements of the bar subjected to uniaxial tension. Now, this involves three constants: C, u_0, and v_0.



These are required to be obtained with the help of displacement boundary conditions. So far, we have not considered any edge of the bar to be fixed or the center point to be fixed. Once you add such displacement boundary conditions, we can obtain the values of C, u_0, and v_0. Here, u_0 refers to the rigid body motion of the bar along the x-direction, v_0 refers to the rigid body motion of the bar along the y-direction, and C refers to the rotation of the bar about its center point O. These are the physical interpretations of these terms. If you restrict the bar from x-axis, y-axis motions, and rotation, then all those constants would be zero. So, both $f_1(y)$ and $f_2(x)$ would be zero if the center of the bar is not allowed to move in space and the bar is not allowed to rotate about its center. For such cases, u would only be $\frac{Px}{AE}$, and v would be $-\frac{\nu Py}{AE}$.



So, in today's lecture, we discussed the general Fourier form and Fourier series solution for stress functions and then solved an example problem of uniform axial tension of a long, thin bar using the stress function approach. Thank you.