APPLIED ELASTICITY

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WEEK: 01
Lecture- 02

Welcome back to the course of applied elasticity. So in the previous two lectures, we were talking about the tensor algebra. We will continue with the same topic in this third lecture as well. So earlier we had talked about the vectors and second order tensors, and their components.

First, we are going to discuss about the laws of transformation for the Cartesian components of vectors as well as second order tensors. So, here we are considering $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ as one set of base vectors. $\tilde{e}_1', \tilde{e}_2', \tilde{e}_3'$ to be another set of transformed base vectors or bases for the same vector space with the same origin. So, $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is one set of unit base vectors, $\tilde{e}_1', \tilde{e}_2', \tilde{e}_3'$ is another set of unit vectors for two different rectangular Cartesian coordinate systems. But both of them both these bases are being used to describe the same vector space. now $\tilde{e}_i$

can be coincided with $\tilde{e}_i'$ vector through a rigid body rotation if both the base vectors are same handed means both are right hand triad in that case, just by simple rotation of $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ by some fixed angle we can coincide those with $\tilde{e}_1', \tilde{e}_2', \tilde{e}_3'$ respectively. However, if these two chosen sets, one is left hand triad, another is right hand triad, they are different handed coordinate systems.

In that case, after this rotation, a reflection is also required. Then only $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ can coincide with the $\tilde{e}_1', \tilde{e}_2',$ and $\tilde{e}_3'$ frame. Now, as we have discussed about the orthogonal tensors. Now, orthogonal tensors were shown to have a determinant value of plus or minus 1, and we also discussed that the orthogonal tensor $\tilde{\tilde{Q}}$ has a determinant equal to 1

when it is a rotation tensor, and when it is a reflection tensor, then its determinant equals minus 1. Thus, with the help of an orthogonal tensor $\tilde{\tilde{Q}}$, $\tilde{e}_i$ can be

transformed to $\tilde{e}_i'$. So, two sets of unit vectors can be related with the help of an orthogonal tensor and that is done through this particular transformation law.

$\tilde{e}_i'$ is equal to $\tilde{\tilde{Q}}\tilde{e}_i$. So, $\tilde{\tilde{Q}}$ is an orthogonal tensor, $\tilde{e}_i$ and $\tilde{e}_i'$ are two sets of base vectors. for the same vector space. So, $\tilde{e}_i'$ we can write as $Q_{ji}\tilde{e}_j$ where Q_{ji} are the components of the tensor $\tilde{\tilde{Q}}$.

If we explicitly expand these equations for free index i varying from 1 to 3 $\tilde{e}_1'$ prime is $Q_{11}\tilde{e}_1 + Q_{21}\tilde{e}_2 + Q_{31}\tilde{e}_3$. Similarly, $\tilde{e}_2'$ and $\tilde{e}_3'$ can also be expanded and condensed form of all 3 equations is defined through this $\tilde{e}_i'$ is equals to $Q_{ji}\tilde{e}_j$ where Q_{ij} the component of the orthogonal tensor is defined as $\tilde{e}_i . \tilde{\tilde{Q}}$

acting over $\tilde{e}_j'$ and physically this quantity defines the cosine of the angle between $\tilde{e}_i$ and $\tilde{e}_j'$. So, cosine of the angle between $\tilde{e}_i$ and $\tilde{e}_j'$ defines the Q_{ij} component of the orthogonal tensor which is $\tilde{e}_i . \tilde{\tilde{Q}}$ acting over $\tilde{e}_j'$. And in the matrix form Q can be written like this as shown here Q_{11} Q_{12} Q_{13} Q_{21} Q_{22} Q_{23} Q_{31} Q_{32} Q_{33}. to another base vector ej prime.

Now, with the help of this transformation matrix or transformation tensor, we can transform any vector component from one base vector system to another base vector system. Similarly, tensor component from one base vector choice to another base vector choice. So, we will first discussing about transformation of vector components. This will be followed by the transformation of tensor components.

So, consider any vector $\tilde{a}$ which is described in e_i frame as $a_i\tilde{e}_i$ same vector is described in $\tilde{e}_i'$ frame as $a_i'\tilde{e}_i'$. So, a_i are the components of the vector $\tilde{a}$ with respect to $\tilde{e}_i$ unit vectors a_i' are the components of same vector $\tilde{a}$ with respect to second choice of base vector $\tilde{e}_i'$.

now we know that a_i' can be written by taking the dot product of the vector a_i' with the corresponding base vector $\tilde{e}_i'$ so $\tilde{a}.\tilde{e}_i'$ will give us the corresponding component a_i' prime in the prime base vector frame. Now $\tilde{e}_i'$ is written as $Q_{ji}\tilde{e}_j$ by definition of this orthogonal tensor transformation as discussed in the previous slide

$\tilde{e}_i'$ can be written as $Q_{ji}\tilde{e}_j$. so that is what we have written and thus a_i' becomes $\tilde{a}$ dot $Q_{ji}a_j$. Now taking the scalar Q_{ji} out and then this Q_{ji} is followed by $\tilde{a}$ dot $\tilde{e}_j$.

Now, $\tilde{a}$ dot $\tilde{e}_j$ will give us the jth component of vector $\tilde{a}$ in the $\tilde{e}_j$ frame. So, this would be $Q_{ji}a_i$.

So, thus a_i' is equals to $Q_{ji}a_j$. This is the transformation of vector component from $\tilde{e}_i$ frame. to $\tilde{e}_j'$ frame with the help of this equation we can transform the vector components. Now, if you write it in terms of the definition of orthogonal transformation tensor, this matrix is nothing but $[Q]^T$. So, $\tilde{a}'$ vector is equals to $[Q]^T\{a\}$ or a is equals to Q times $\tilde{a}'$.

So, a_i is equals to $Q_{ij}a_j'$. With the help of this transformation, we can transform a vector component from $\tilde{e}_i$ frame to $\tilde{e}_j'$ frame. Now, we will proceed for the transformation of second order tensor. So, we are considering a second order tensor $\tilde{\tilde{T}}$ whose components in ij frame $\tilde{e}_i\tilde{e}_j$ frame can be written as T_{ij} equals to $\tilde{e}_i.(\tilde{\tilde{T}}\tilde{e}_j)$. This is by the definition of tensor components.

$\tilde{e}_i$ and $\tilde{e}_j$ are two base vector in the original frame of reference. T_{ij} the tensor component is equals to $\tilde{e}_i.(\tilde{\tilde{T}}\tilde{e}_j)$. And in the transformed frame where the base vectors are $\tilde{e}_i'$ and $\tilde{e}_j'$ in that frame the transformed tensor components T_{ij}' can be written as $\tilde{e}_i'$ dot $\tilde{\tilde{T}}\tilde{e}_j'$.

Now, our objective is to relate T_{ij} with T_{ij}', the transformed tensor component with the original tensor component. Now, ei is related to $\tilde{e}_i'$ with the help of this particular transformation law, with the help of the orthogonal transformation tensor Q as $\tilde{e}_i'$ equals $Q_{ki}\tilde{e}_k$. Now, in the expression of T_{ij}', we will replace $\tilde{e}_i'$ as $Q_{ki}\tilde{e}_k$ and $\tilde{e}_j'$ prime as $Q_{lj}\tilde{e}_l$.

So, this part $Q_{ki}\tilde{e}_k'$ was earlier the $\tilde{e}_i'$, and this part $Q_{lj}\tilde{e}_l$ was earlier the $\tilde{e}_j'$. With the help of the transformation tensor Q, the orthogonal tensor Q, we have expressed both $\tilde{e}_i'$ and $\tilde{e}_j'$ like this. Now here, Q_{ki} and Q_{lj} are both scalars, so we can take those out: T_{ij}' is $Q_{ki}Q_{lj}\tilde{e}_k$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_l$. Now, by definition of the tensor component T in the $\tilde{e}_i$, this $\tilde{e}_k$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_l$ is the tensor component T_{kl}. So, thus in total, the right-hand side becomes $Q_{ki}Q_{lj}T_{kl}$. So, these are the three terms.

So, T_{ij}' is equal to $Q_{ki}Q_{lj}T_{kl}$ or If I write T_{ij} on the left-hand side, then this equation can be written as T_{ij} equals $Q_{ik}Q_{jl}T_{kl}'$. This is the equation which transforms one tensor component into a tensor component in another base

vector frame. So, T' equals $[Q]^T\, T$ times Q. This is called the transformation law for a second-order tensor. So, if you compare this with the first-order tensor

Here in the transformation law of a second-order tensor, you can see the Q term is appearing twice: $Q_{ik}Q_{jl}$. Whereas, for the transformation law of vectors, this was appearing once. The Q term was appearing once here, but this Q term is appearing twice for the case of transformation of a second-order tensor. So, as we go for higher-order tensors, the number of Q terms in the transformation will increase.

For the third-order tensor component transformation, there will be three orthogonal tensor terms present. For the fourth-order tensor transformation, four orthogonal tensor terms would be present, and so on. Thus, for any nth-order tensor, let us consider an nth-order tensor, capital A. So, capital ais an nth-order tensor whose transformation we are going to consider between the $\tilde{e}_i$ frame and the $\tilde{e}_i'$ frame, and in that transformation, capital N number of Q terms will be present, where Q is an orthogonal tensor.

So, this right-hand side A_{mno} like that, capital N number of indices are also present here that is the component of the nth-order tensor A in the $\tilde{e}_i$ frame and the left-hand side, A'_{ijk} here also, this is having capital N number of subscripts because that is an nth-order tensor. And the left-hand side, A'_{ijk}, which is continuing till n number of terms

This is the component of the tensor A in the transformed frame, the $\tilde{e}_i'$ frame. Now, we will be solving a few example problems dealing with the simplification of tensorial notations. So, the first problem is this: This particular expression is given: $\tilde{a}.\left(\tilde{b} \times \tilde{c}\right)$ equals $\left(\tilde{a} \times \tilde{b}\right).\tilde{c}$. And we need to prove this particular identity with the help of indicial notation.

So, we will take both the left-hand side and right-hand side of this expression separately. First, taking the left-hand side, $\tilde{a}.\left(\tilde{b} \times \tilde{c}\right)$, $\tilde{b} \times \tilde{c}$ is a vector quantity. So, $\tilde{a}.\left(\tilde{b} \times \tilde{c}\right)$ can be written as $a_i(\tilde{b} \times \tilde{c})_i$. And this $(\tilde{b} \times \tilde{c})_i$ can further be simplified with the help of the permutation symbol e_{ijk}. So, if we do so, $(\tilde{b} \times \tilde{c})_i$ can be written as $a_i e_{ijk} b_j c_k$.

So, in total, this expression becomes: the left-hand side expression is $e_{ijk} a_i b_j c_k$. Now, coming to the right-hand side, $\left(\tilde{a} \times \tilde{b}\right).\tilde{c}$. Here also, we can write this as a

dot product of two vectors: the first vector is $\tilde{a} \times \tilde{b}$, and the second vector is $\tilde{c}$. So here, it would be $(\tilde{a} \times \tilde{b})_k c_k$. This dummy index I have chosen as k, so that we can conveniently match the left-hand side with the right-hand side.

If you look at the expression, the simplified expression of the left-hand side, the index associated with c is k. So, that is why, while expanding the right-hand side equation, this dummy index I have chosen to be k, which will conveniently help us to equate the left-hand side with the right-hand side. So, $(\tilde{a} \times \tilde{b})_k c_k$. Now, $(\tilde{a} \times \tilde{b})_k$ can be written in terms of the permutation symbol as $e_{kmn} a_m b_n$, and c_k is also there from the second term.

Now, this e_{kmn} is the same as e_{mnk}. So, m, and n, these three are in cyclic order. So, if you start from k, then go to m, then go to n, that is equal to e_{kmn}, which is equal to 1. Now, instead of starting from k, if you start from m, then, in the same cyclic order—that is, in the clockwise rotation—if you go to the next quantity N and then the next quantity k, this will also be 1 because both of them are in cyclic order.

So, e_{kmn} is the same as e_{mnk}, and both of them equal 1. So, this equation is written in this particular form: $e_{mnk} a_m b_n c_k$. Now, you can compare both of them and just check that i is equivalent to m, j is equivalent to n. These are all dummy indices. Names are in our hands. If you replace m with i and n with j in the second equation, the second equation will be identical to the first one.

$e_{ijk} a_i b_j c_k$ and thus this particular given identity can be proved with the help of this tensorial or indicial manipulation. So, for such cases, we have to carefully choose the indices. We need to remember that no index can be repeated more than twice. And sometimes we need to change the name of some of the indices to match it with the other side of the equation. Now, let us proceed to the next example.

Here, these two terms are given which we need to simplify. The first one is $e_{1jk} \alpha_2 T_{kj}$. Now, $e_{1jk} \alpha_2 T_{kj}$, if we simplify j and k, these two can only take values of 2 and 3 for which the permutation symbol value would be non-zero. So, $e_{123} \alpha_2 T_{32}$ plus $e_{132} \alpha_2 T_{23}$. Now, e_{123} and e_{132} are opposite to each other: e_{123} is equal to 1, and e_{132} is equal to minus 1. So, substituting these values of the permutation symbol and taking α_2 common, the first equation can be condensed

to the form α_2 times T_{32} minus T_{23}. In the same fashion, if you simplify the second one,

$e_{1jk}\delta_{3j}v_k$. So, first we will simplify this part: δ_{3j} multiplied with the permutation symbol. So, when we are multiplying δ, the δ, with any other thing, the two subscripts of the δ are matched with each other. So, j is replaced with 3 here. So, this becomes the permutation symbol e_{13k}. j is replaced with 3 as it was multiplied with the Kronecker δ.

Now, here e_{13k} is non-zero only with a single value of k equals to 2. So, thus this equation becomes $e_{132}v_2$, and e_{132} being minus 1, this is equal to minus v_2. So, for the case of the permutation symbol, while expanding it, you just check which terms are non-zero. Now, here if you go for e_{131} or e_{133}, all these terms are 0 by definition of the permutation symbol. So, while expanding this summation convention, no need to write those expressions.

Only non-zero terms, which are having either a value of 1 or minus 1, those should be considered. Now, going to the next example. The matrix form of a second-order tensor S is given as 1 0 2, 0 1 2, 3 0 3. And a_j, the components of a vector a, are given as 1, 2, and 3. Now, you need to obtain all these quantities: $S_{ij}S_{ij}$, $S_{ij}S_{ji}$, $S_{ij}a_ia_j$, $S_{ji}a_ia_j$. Now, taking the first one, $S_{ij}S_{ij}$.

This is product of two tensors. But we know that product of two tensor TS_{ij} can be written as $T_{ik}S_{kj}$. So, second index of the first tensor should be same as first index of the second tensor. Here in the given form that is not so, they arenot same. The second index of first tensor is j.

First index of second tensor is i. Now to make them equal, if we are able to make them same then only the product of these two tensors can be taken. To make them equal we are writing the second term S_{ij} as $S_{ji}{}^T$ by taking the transpose sign the two subscripts are flipped j is taken first i is taken back

So, now $S_{ij}S_{ji}{}^T$ here it is possible to take the product of these two tensor S and S^T transpose. So, this is basically $(SS^T)_{ii}$ the first and last index would be appearing here. Now, from the given matrix S, you can easily find out S^T and take the product of those two matrices. If you do so, you can verify this. This particular matrix S, S^T will come, which is product of given S matrix and its transpose.

Now, we are having i, i component of this. Now, what is meant by i, i? Its same index is repeated twice. So, this summation is over ii and both the indices being same, this refers to the diagonal term, summation of the diagonal term.

That is 1-1 term, 2-2 term and 3-3 term. So, this is basically summation of all three diagonal terms of this matrix, 5 plus 5 plus 18 resulting 28. Going to the next one $S_{ij}S_{ji}$ here you can directly write as S times S. The product of this $S_{ij}S_{ji}$ can be written as S times S or square of the given matrix, and then ii component of that, that is summation of the diagonal terms of S times S.

So finding the square of this given S matrix that would be like this, and summation of all its diagonal term will be 7 plus 1 plus 15, that would be 23. So, here if you compare first two a and b, just by change in location of the one index, j getting replaced with i, i getting replaced with j in one of the term, not in both. We are having different answers if you expand those expressions.

Now, coming to the third one, $S_{ij}a_i a_j$ here simplifies this with the help of the definition of a second-order tensor. We need to rewrite it in this fashion, S_{ij} is written as S_{ji}^T, which is acting over a_i, and then we have the product of a_j. So, S_{ji}^T, if you find the transpose of the given X matrix and multiply it with the given a, it would result in a vector: 10, 2, and 15. Then, once again, we take another dot product with the a vector, which would result in a scalar of 59.

Now, in the last part, the positions of the i and j subscripts are flipped in S. So, instead of S_{ij}, this is $(S_{ji}a_i)a_j$. So, $S_{ji}a_i$ can be simplified first. This will result in a vector: 7, 8, 12, and then we take the dot product of that with a_j, given a. If you expand and take this dot product, it would come out to be 59. So, here you can see $S_{ij}a_i a_j$ is the same as $S_{ji}a_i a_j$.

Why so? Because here we are flipping the indices i and j in all the terms. If you replace i with j and j with i, in the first one, it will become $S_{ji}a_j a_i$, which is the same as the second one. So, if you flip all the indices in all the terms and then compare them, they are equal.

whereas in the first case a and B second case only one term indices only in only one term was flipped and that is why those answers were different. Now, coming to thenext example, T_{ij} in terms of the tensor $\tilde{\tilde{T}}$, its component T_{ij} can be defined

as $\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_j$. Using these definition of tensor component, you are asked to prove that

T_{ij} component of product of two tensor $\tilde{\tilde{T}}\tilde{\tilde{S}}$ is equals to $T_{ik}S_{kj}$. So, we are starting with the definition of tensor component for this product tensor $(\tilde{\tilde{T}}\tilde{\tilde{S}})_{ij}$ component of $\tilde{\tilde{T}}\tilde{\tilde{S}}$ is $\tilde{e}_i$ dot $\tilde{\tilde{T}}\tilde{\tilde{S}}$ acting over $\tilde{e}_j$. Now, first $\tilde{\tilde{S}}$ will be acting over $\tilde{e}_j$ and then $\tilde{\tilde{S}}$ is written in terms of dyadic product as S_{mn} $\tilde{e}_m \otimes \tilde{e}_n$ this is equals to S tensor by definition of the dyadic product. Now, $\tilde{e}_m \otimes \tilde{e}_n$ times $\tilde{e}_j$ this particular part by definition of dyadic product

this can be written as $\tilde{e}_m$ times $\tilde{e}_n$ dot $\tilde{e}_j$. This is by definition of dyadic product which we had discussed in the previous lecture. Now, here we are having dot product of 2 unit vector $\tilde{e}_n$ and $\tilde{e}_j$ which is nothing but Kronecker δ δ_{nj} and here this Smn the scalar we have taken out. So, if you simplify this it would be $S_{mn}\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_m\delta_{nj}$.

δ_{nj} can once again be taken out and then writing $\tilde{\tilde{T}}$ as T_{kl} $\tilde{e}_k$ dot $\tilde{e}_l$ once again with the help of dyadic product and simplifying in the similar fashion as the previous step, this can be obtained as TklSmjei dotted with $\tilde{e}_k$ acting over $\tilde{e}_l$ dot $\tilde{e}_m$ then $\tilde{e}_l$ dot $\tilde{e}_m$ this is once again a Kronecker δ_{lm}. So, then finally, dot product of $\tilde{e}_i$ and $\tilde{e}_k$ these two unit vectors are remaining which will result δ_{ik}.

So, $T_{kl}S_{lj}\delta_{ik}$ and as multiplication of Kronecker δ_{ik} will result k replaced with i the final form will be $T_{il}S_{lj}$. So, thus $\tilde{\tilde{T}}\tilde{\tilde{S}}$ product ij component product of two tensor component is equals to $T_{il}S_{lj}$ which can also be written as $T_{ik}S_{kj}$ where l the dummy index is replaced with a new dummy index k. Coming to thenext example, here a tensor is defined with the help of dyadic product as

$\tilde{\tilde{T}}$ equals to 3 e_1 dyadic e_1 minus 4 e_1 dyadic e_2 and so on, and a vector $\tilde{a}$ is given as 4 e_1 plus 2 e_2 plus 5 e_3. We need to find out $\tilde{\tilde{T}}$ acting over $\tilde{a}$. Now, for such cases, first we will find out $\tilde{\tilde{T}}$ acting over each of the unit vectors and

we will use the definition of dyadic product. So, $\tilde{\tilde{T}}$ acting over e_1 is expanded with the help of definition of $\tilde{\tilde{T}}$. 3 times e_1 dyadic e_1 acting over e_1 minus 4 times e_1 dyadic e_2 acting over e_1 and so on. All the terms are written. Now, by the definition of dyadic product e_1 dyadic e_1 acting over e_1 can be written as e_1 times e_1 dot e_1.

Second case e_1 dyadic e_1 acting over e_1 can be written as e_1 times e_1 dot e_1 and so on. Now here you can see there are many dot products coming where one vector e_1 is dotted with another vector e_1 here. They are not same vector. Here also e_1 dot e_1, they arenot same vector. e_1 dot e_1, they are not same vector.

All this will go to 0 by definition of dot product of unit vectors which are orthogonal. And thus only non-zero terms are the first term and third term. These are two non-zero terms. and if you simplify that it would result 3 e_1 plus 2 e_2. So, $\tilde{\tilde{T}}$ acting over first unit vector e_1 is 3 e_1 plus 2 e_2.

In the same fashion, you can find $\tilde{\tilde{T}}$ acting over e_2 to be minus 4 e_1 plus e_2 and $\tilde{\tilde{T}}$ acting over e_3 to be only e_3. Now, using the definition of $\tilde{a}$, and $\tilde{\tilde{T}}$ acting over a, which will be T acting over 4 e_1 plus 2 e_2 plus 5 e_3. We know what $\tilde{\tilde{T}}$ acting over e_1 is, what $\tilde{\tilde{T}}$ acting over e_2 is, and what $\tilde{\tilde{T}}$ acting over e_3 is. If you replace all those three, the final form of $\tilde{\tilde{T}}$ $\tilde{a}$ can be obtained as 4 e_1 plus 10 e_2 plus 5 e_3.

And coming to the next problem, here also we need to prove this identity. Capital $\tilde{\tilde{A}}$ acting over small $\tilde{a}$ dot capital $\tilde{\tilde{B}}$ acting over small $\tilde{b}$ equals small $\tilde{a}$ dot $\tilde{\tilde{A}}^T\tilde{\tilde{B}}$ small $\tilde{b}$. Now, we are taking the left-hand side first, which is the dot product of two vectors: capital $\tilde{\tilde{A}}$ times small $\tilde{a}$ and capital $\tilde{\tilde{B}}$ times small $\tilde{b}$. And both of them can be expanded as capital A_{ij} small a_j capital B_{ik} small b_k. Whereas, on the right-hand side,

if you expand this dot product of these two vectors, it would be small a_i and then the i component of $\tilde{\tilde{A}}^T\tilde{\tilde{B}}$ small $\tilde{b}$. Now, the second term can further be simplified as $\tilde{\tilde{A}}^T$ B_{ik} small b_k. $\tilde{\tilde{A}}^T$ B_{ik} is basically the product of two tensors $\tilde{\tilde{A}}^T$ and $\tilde{\tilde{B}}$, which can be written as capital A_{ji} capital B_{jk}.

Thus, the total right-hand side expression becomes small a_i capital A_{ji} capital B_{jk} small b_k. And we can rewrite this in this particular fashion by flipping, by interchanging i and j, these two dummy indices as small a_j, A_{ij}, B_{ik}, small b_k. And thus, this becomes the same as the left-hand side expression.

Thus, this identity is proved. So, in total, in this particular lecture, we discussed the transformation of vectors and tensors with the help of a transformation matrix and then solved a few example problems on tensor algebra. Thank you.