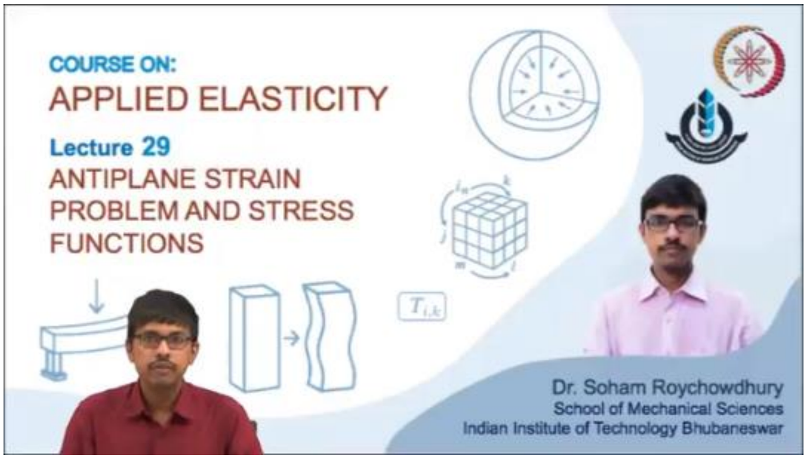
APPLIED ELASTICITY

Dr. SOHAM ROYCHOWDHURY

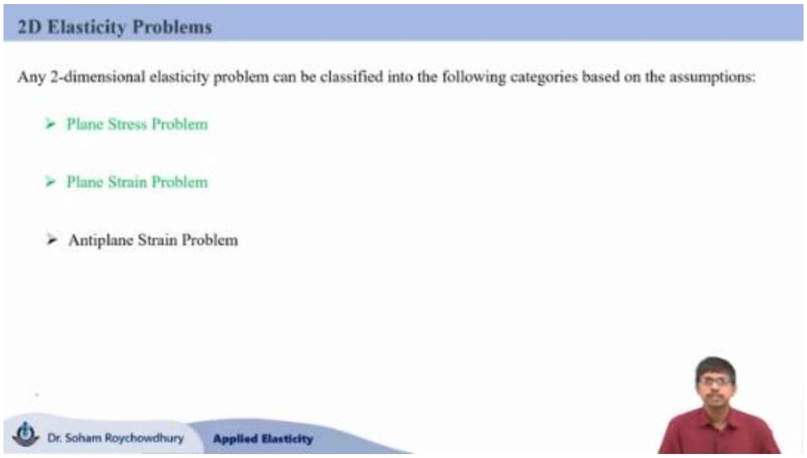
SCHOOL OF MECHANICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR

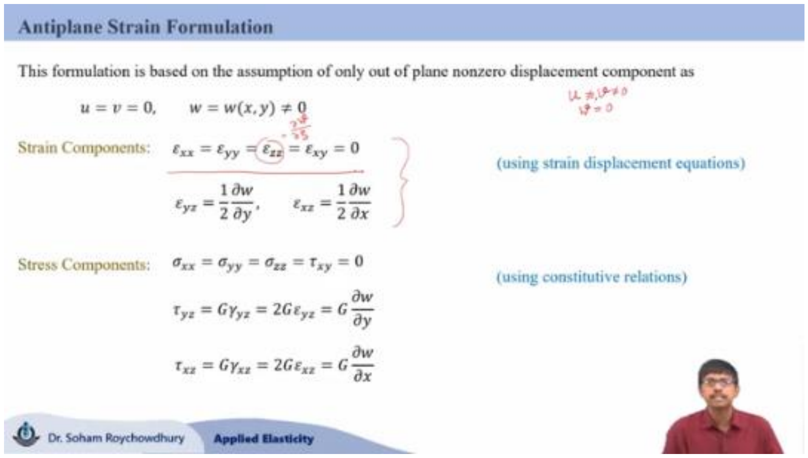
WEEK: 06
Lecture- 29



Welcome back to the course on applied elasticity. In today's lecture, we are going to talk about the anti-plane strain problem formulation and the concept of stress functions.



So, we are discussing the different types of problems of the three-dimensional elastic continuum problem and how it can be reduced to a two-dimensional elastic continuum problem under different assumptions. The two types of problems discussed in the previous lectures were the plane stress problem and the plane strain problem.



Now, after that, in today's lecture, we are going to talk about the formulation of the anti-plane strain problem. So, coming to the definition of the anti-plane strain problem formulation, this is based on the assumption that the displacement component is only non-zero along the out-of-plane direction while both the in-plane displacement components are taken to be zero.

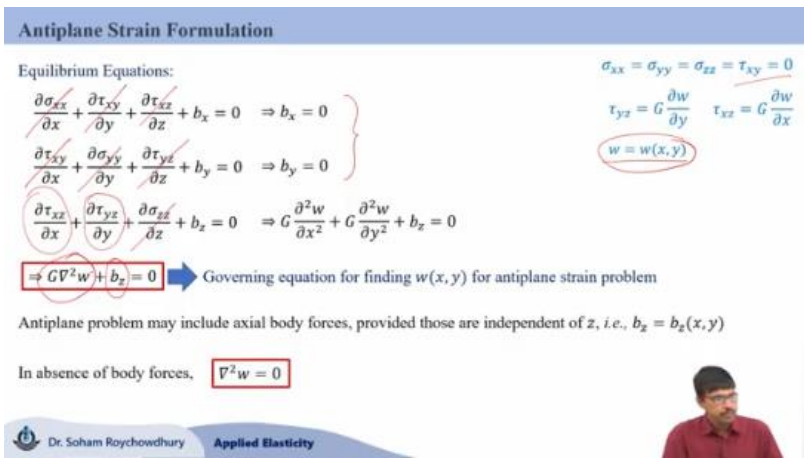
So, if we have a problem where u and v, that is, the two in-plane displacements u_x and u_y, are zero and w, the out-of-plane displacement component is the only non-zero one, then that is called the anti-plane strain formulation. If you recall the plane strain case, for that, u and v, or u_x and u_y, these were non-zero, whereas w or u_z that was taken to be 0 for the plane strain problem whereas in this one, you are just having the opposite choice of the displacement component u and v in-plane displacement components are 0 whereas out-of-plane displacement component w is non-zero, that is why it is named as the anti-plane strain formulation. The choice of displacement components are exactly opposite to the plane strain formulation problem.

Now, with this choice of the displacement fields, using the strain-displacement equations, we can get the strain fields to be ε_{xx}, ε_{yy}, ε_{zz}, and ε_{xy} to be 0. These four strain components would go to 0 because both u and v are 0 and w is a function of x and y only.

We are considering the out-of-plane displacement component to be non-zero, but that is dependent on only two in-plane variables x and y, not a function of z. And because of that, this ε_{zz}, which is equal to $\frac{\partial w}{\partial z}$, this would go to 0 as w is a function of x and y only.

Now, coming to the two non-zero strain components, those are ε_{xz} and ε_{yz}. So, ε_{xz} would be $\frac{1}{2}\frac{\partial w}{\partial x}$, and ε_{yz} would be $\frac{1}{2}\frac{\partial w}{\partial y}$.

So, these are the strain components for an anti-strain problem. Now, coming to the stress components with the help of the constitutive equation for a linear elastic isotropic solid, we would get four stress components to be zero, as we had these four strain components as zero. All three normal stresses—σ_{xx}, σ_{yy}, σ_{zz}—are zero, and the in-plane shear stress $\tau_{xy} = 0$, whereas τ_{xz} and τ_{yz} Two out-of-plane shear stresses would be non-zero, and τ_{xz} can be written as $G\frac{\partial w}{\partial x}$, and τ_{yz} can be written as $G\frac{\partial w}{\partial y}$, where G is the modulus of rigidity, one of the material properties. Now, with all these assumptions of the anti-plane strain formulation, if you write the equilibrium equation, the equilibrium equations along the x, y, and z directions in the rectangular Cartesian coordinate system can be written like this.



Now, we will impose the constraints or the assumptions for the plane strain formulation on these equilibrium equations. So, for the four stress components we have as zero, σ_{xx} is zero in the first equation, so this term would vanish.

σ_{yy} is zero in the second equation, so this term would vanish. σ_{zz} is zero in the third equation, so this would vanish. Also, τ_{xy} being zero, these two τ_{xy} terms would also go to zero. Now, we know that $\tau_{xz} = G\frac{\partial w}{\partial x}$, where w is a function of x and y only. Thus, this particular term would also go to zero because τ_{xz} is a function of w, which is a function of x and y.

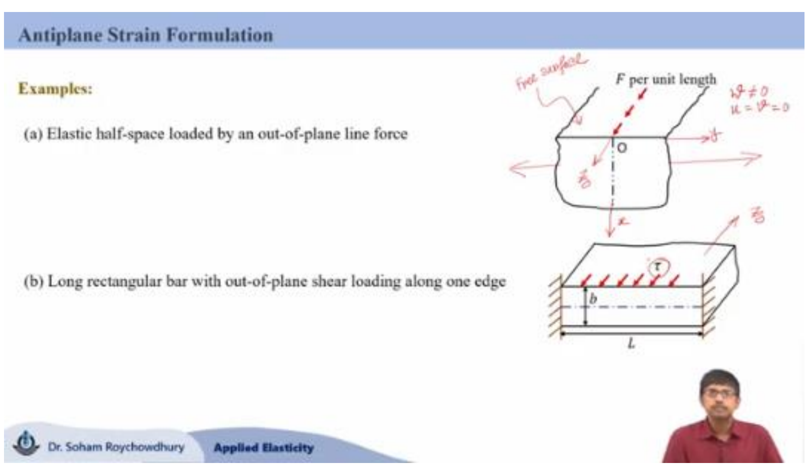
So, $\partial/\partial z$ of some function which is independent of z would also go to 0. So, the first equation results in $b_x = 0$. Similarly, as τ_{yz} is only a function of w or only a function of x and y independent of z, in the second equation, $\frac{\partial \tau_{yz}}{\partial z}$ would also be 0. So, thus the second equilibrium equation will give us $b_y = 0$. So, from the first two equilibrium equations, we are getting the corresponding body force components b_x and b_y to be 0.

So, this has to be imposed for the anti-plane strain problems. Now, coming to the third one, Both the terms $\frac{\partial \tau_{xz}}{\partial x}$ and $\frac{\partial \tau_{yz}}{\partial y}$, these two terms are non-zero terms, and along with that, we have a b_z term in the third equilibrium equation. So, simplifying that, it would be $G\frac{\partial^2 w}{\partial x^2} + G\frac{\partial^2 w}{\partial y^2} + b_z = 0$. Or introducing the Laplacian operator here, $G\nabla^2 w + b_z = 0$.

So, this is the governing equation for the anti-plane strain problem. Here, we are not introducing any concept of stress function, but the displacement field w, the out-of-plane non-zero displacement field w as a function of x and y. must be chosen based on this particular equation. That must satisfy this equation; then only the problem can be assumed to be of plane strain type problem. In the absence of the body force b_z, the $\nabla^2 w = 0$.

So, w must be a harmonic function for the anti-plane strain problem in the absence of any body force. And also note that this b_z can only be a function of x and y because w is a function of x and y, the first term of this equation is a function of x and y, the second term b_z must also be a function of x and y only. So, it is allowed to have the axial body force component b_z to be non-zero for the anti-plane strain problem.

However, that can only be a function of x and y and must be independent of z. And in the absence of body force, we will have the $\nabla^2 w = 0$. So, these are the governing equations for the anti-plane strain problem.

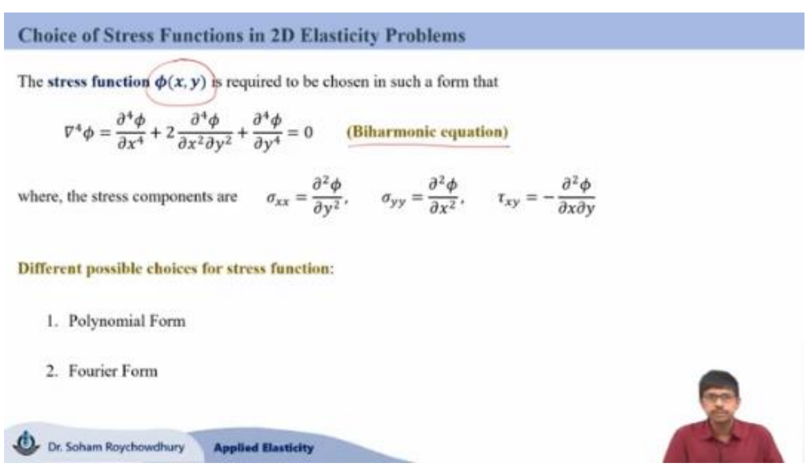


Now moving to the examples: for what type of problem can we impose this assumption of anti-plane formulation? Anti-plane strain formulation—so one example may be a large elastic half-space loaded by out-of-plane line loading. So, we are considering one elastic half-space. Half-space means this is extended in this direction and in this direction till infinity. It is a very large one. So, this is the free surface.

This is a free top surface. And below that free surface, the entire domain is filled with the elastic continuum. So, that is why we are considering and naming this as an elastic half-space. When such an elastic half-space is subjected to out-of-plane line loading, then we can approximate that as an anti-plane strain problem. In the direction of this F out-of-plane loading, we choose the axis z, and x and y are in-plane axes. So, x and y can be chosen along these particular directions. So, this is x, this is y, and z is coming out. Due to this type of loading, the only non-zero displacement component would be that along the z direction. No displacement would be caused along the x or y direction. So, u and v would be 0.

So, this particular problem can be approximated as an anti-plane strain problem. Another possible example is a long rectangular bar. This bar is long in the z direction, the axial direction. Now, at one of the edges of this bar with cross-section l and b, you are subjecting it to out-of-plane shear loading of intensity tau. Here also, this tau can only cause deformation or displacement along the w direction, along the z direction.

So, only the z component of displacement would be non-zero for this particular problem, and the u and v components of the displacements are zero. So, this can be treated as another example of an anti-plane strain problem.

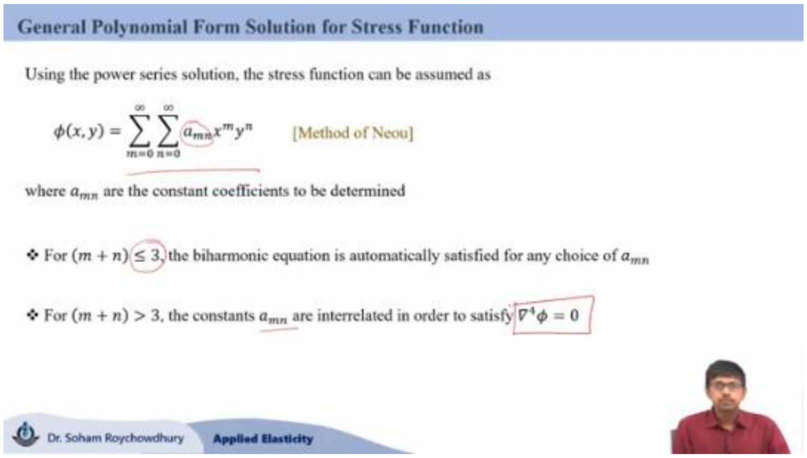


Now, moving forward after the discussion of these three types of assumptions—plane stress, plane strain, and anti-plane strain— with which we can reduce a 3D problem to a 2D elasticity problem, and out of these three, the plane stress and plane strain are the most common cases: one is valid for very thin elastic continuum, another is valid for very large continuum.

Plane stress is for thin bodies; plane strain is for large bodies with a very long length along the z-direction—theoretically infinite for both of them. We need to choose a stress function phi, which is a function of x and y only—plane variables—in such a form that this choice of ϕ must satisfy the bi-harmonic condition: $\nabla^4\phi$ should be equal to 0. This is in the absence of any body force within the problem. So, $\nabla^4\phi$ should be 0, or $\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}$ should be 0.

This biharmonic condition or biharmonic equation must be satisfied for any stress function phi if we are solving the problem in two dimensions. If you are approximating the 3D problem into a 2D problem, either using the plane stress assumption or the plane strain assumption, the biharmonic of ϕ equals to 0 condition must be satisfied by the choice of stress function. If the choice of ϕ satisfies this, then only we can proceed further with that ϕ.

And obtain the stress components as $\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$. These are the three non-zero in-plane stress components. Now, for the choice of phi, we have two different options. The first one is the polynomial form of solution of the stress function, whereas the second one is the Fourier series form of solution of the stress functions.



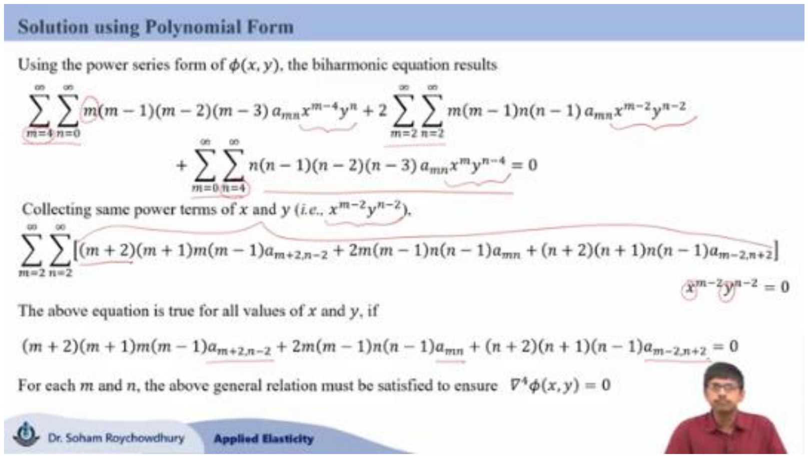
So, we are first going to talk about the polynomial form of solution of the stress function, and we will continue our discussion on the Fourier form of the stress function in the next lecture. So, starting with the general polynomial solution form of the stress function $\phi(x, y)$. For the polynomial form, we are using a power series solution as the stress function, as our choice of stress function.

So, let us assume $\phi(x, y)$ is as a series solution of both the variables x and y. So, there are two summation signs: the first one is for x with the index m, and the second one is for y with the index n. So, $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m y^n$. This is the power series solution which can be assumed for any stress function ϕ, and m and n are the, these are the two numbers of our choice. We can choose any value of m and n. If both m and n are 0, then that stress function phi would come out to be a constant, which is not a valid case. This method was proposed by Neou, and with this power solution, it is possible to solve many problems. Note that this power solution is valid for the rectangular Cartesian coordinate system $x\ y$.

We will later discuss how to choose the stress function in polar coordinates in detail in the later part of this course. Now, we are considering the problems—problems in the rectangular Cartesian coordinate frame for which in general, the polynomial form of the solution of ϕ can be like this. Now, a_{mn} —this a_{mn} is a constant, a set of constants which are different for different index values m and n. We need to determine this based on the biharmonic condition.

Now, if $m + n$ So, the total number of or total power of the series solution is 3 or less, then the biharmonic equation is automatically satisfied for any choice of a_{mn}. So, the biharmonic operator contains fourth-order partial derivatives with respect to x or y or both, up to a. x 2, del 2, del y 2.

So, maximum fourth-order partial derivatives exist in the biharmonic operator. Thus, if the series solution has a power less than 4 or less than or equal to 3, then the biharmonic equation would be automatically satisfied because we will be taking the partial derivative 4 times, and all the terms would be set to 0; the biharmonic of 5 will be 0 automatically. However, if $(m + n) > 3$, 4 or more, then all these a_{mn} cannot be independent arbitrary choices. The choice of a_{mn} would be related, which can be obtained by setting $\nabla^4 \phi = 0$.



So, now, we are going to substitute this power series solution of ϕ in the expression of the biharmonic condition $\nabla^4\phi = 0$. So, if you do that, substituting the power series solution of ϕ in the biharmonic equation, you would get these three terms. The first term results from $\frac{\partial^4\phi}{\partial x^4}$; the second term comes from $2\frac{\partial^4\phi}{\partial x^2\partial y^2}$, and the third term comes from $\frac{\partial^4\phi}{\partial y^4}$. Now, if you carefully look at this and try to combine the common order terms of x and y, the first point to note is that these indices are different in all three. So, the range for m and n is different; first, we need to make them equal. So, let us say we are trying to make m and n both vary from 2 to infinity. So, here, this m 4 to infinity is required to be changed; here, in the last term, n 4 to infinity is also required to be changed, and with that, we can make these terms the same in all three.

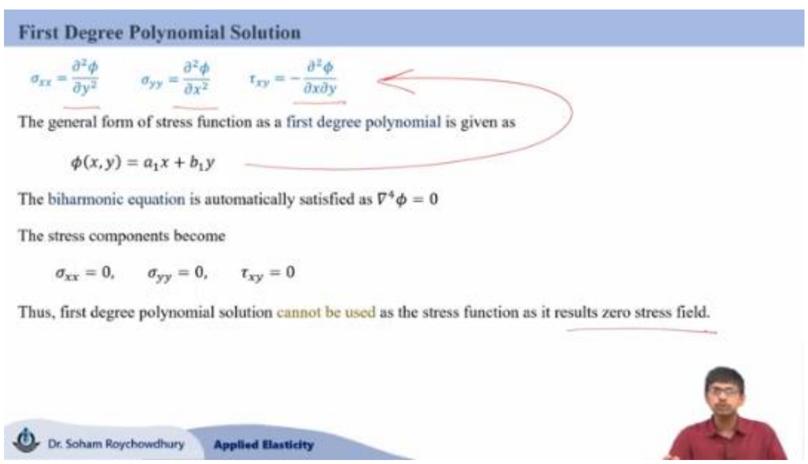
So, let us try to make x^{m-2} and y^{n-2} to be common terms, common functions of x and y in all. So, in the first one, m minus 4 should be replaced with m minus 2, and for that, if you are collecting those powers by setting all of them to be equal, m varying from 2 to infinity, the first term m would be changed to m plus 2. So, replacing m with m plus 2 in the first set of terms and replacing n with n plus 2 in the last, this third set of terms.

We can get x^{m-2} and y^{n-2} as common terms in all three of them, and thus this particular biharmonic equation can be transformed to this particular form with summation m n to infinity, m equals to 2 to infinity, summation n equals to 2 to infinity then this entire term within square brackets multiplied with x to the power m minus 2 and y to the power n minus 2 equals to 0. Now, this expression must be valid for all values of x and y. For any x and y, this equation should be valid because the biharmonic condition of ϕ should be satisfied over the entire domain for all x and y coordinate values, and thus that is true only if this entire quantity within the bracket goes to 0. So, that entire quantity is set to 0, and with that, you can see we are getting a relation, an interconnection between different

a_{mn}. We are having three constants involved in this equation: a_{mn}, $a_{m+2,n-2}$, $a_{m-2,n+2}$. These three constants are related through this biharmonic equation, and if m plus n is 4 or more, then this particular equation is required to be imposed.

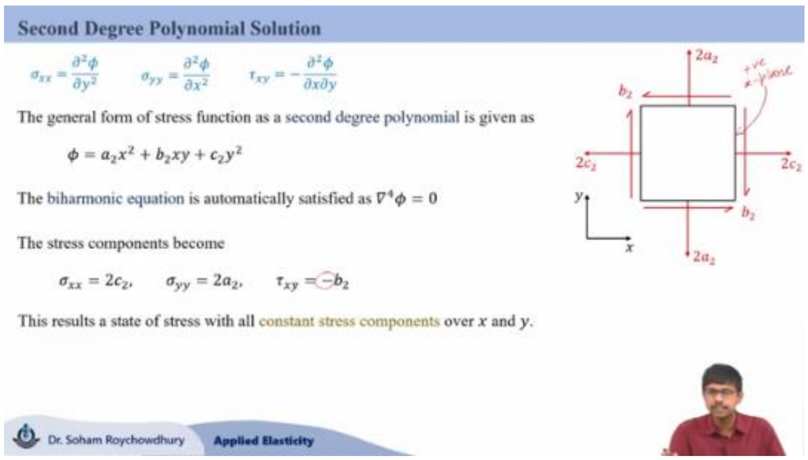
With this, some of the constants are interconnected. So, in general, if the above equation is satisfied, then the chosen form of power series solution or polynomial form of phi is going to satisfy the biharmonic equation $\nabla^4\phi(x, y) = 0$. So, polynomial functions can be used as stress functions, which must satisfy the biharmonic condition, and for that, we can have a degree less than 4, which will automatically satisfy the biharmonic constraint.

However, for degree 4 or more, a polynomial of degree 4 or more, we need to put some additional constraints on the constants a_{mn} to satisfy the biharmonic condition. Now, we will be taking different polynomials of first, second, third, and fourth order or fourth degree and check what kind of problems can be solved or represented if you are choosing this elementary polynomial solution for ϕ or stress function.



So, first, let us start with the first-degree polynomial solution of the stress function, where the stress components σ_{xx}, σ_{yy}, and τ_{xy} can be expressed in terms of ϕ as $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial x^2}$, and $-\frac{\partial^2\phi}{\partial x\partial y}$, respectively. So, for the first-degree polynomial, ϕ is a function of x and y, and they are directly proportional to ϕ, so $a_1x + b_1y$, a linear combination of two first-order terms, one for x and one for y. This is the most common or general form of the first-degree polynomial of the stress function. Now, choosing phi to be $a_1x + b_1y$ and then checking the biharmonic equation, this is automatically satisfied. Up to a third-degree polynomial, it would be directly satisfied. For a fourth-degree polynomial, we would need some additional constraints to satisfy the biharmonic condition. So, here, there is no need to substitute this ϕ and check whether it satisfies the biharmonic equation or not. As the order of the polynomial is below 4, it is directly satisfied.

Now, stress components for this case can be obtained by substituting this ϕ here in this set of equations. All of them would go to 0 because ϕ contains only first-order terms of x and y, and all these derivatives are second-order. So, all the stresses— σ_{xx}, σ_{yy}, and τ_{xy} —would go to 0, and hence, A first-degree polynomial cannot be used as a stress function, which results in a zero stress field. No stresses are produced with this first-degree polynomial as a stress function, and hence, it cannot be used to represent any elastic deformation problem.

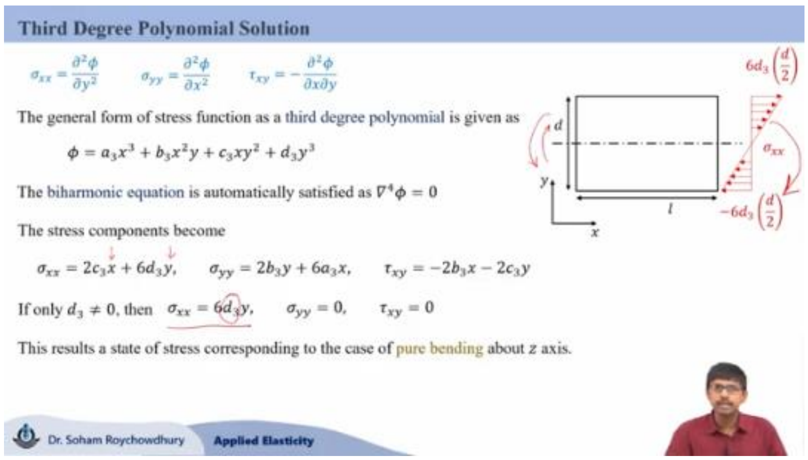


Now, moving to a second-degree polynomial, the most general form of a second-degree polynomial is $a_2x^2 + b_2xy + c_2y^2$ equals 0. So, we have three second-order terms: x^2, xy, and y^2. No other second-order terms can exist for a planar problem with x and y as the two variables. Here, a_2, b_2, and c_2 are three arbitrary unknown constants. Here also, as it is a second-degree polynomial with a degree less than 4, the biharmonic equation is directly satisfied.

And substituting this second-order polynomial in the stress component equation, you can obtain the stress component $\sigma_{xx} = 2c_2$, $\sigma_{yy} = 2a_2$, and $\tau_{xy} = -b_2$. a_2, b_2, c_2—all these three being some constants—all three stress components σ_{xx}, σ_{yy}, and τ_{xy} are constant for the second-degree polynomial stress function. Now, these results represent a state of stress with all constant stress components along the x and y directions.

So, if you think of an example, let us consider this small box on the xy plane, with sides aligned along the x and y axes. So, as $\sigma_{xx} = 2c_2$, the stress on both edges of this box would be equal to $2c_2$. And $\sigma_{yy} = 2a_2$, the vertical direction normal stress component would be $2a_2$. $\tau_{xy} = -b_2$, the shear stresses can be shown like this. This right-hand side one—this is a positive x plane.

And as the shear stress $\tau_{xy} = -b_2$ on the positive x plane, that is why I have drawn the arrow of b_2 to be downward, because then only it is along the negative y direction, and that will take care of this minus sign. So, this is a simple constant 2D element subjected to a constant state of stress. A planar 2D element subjected to constant shear stress τ_{xy} and constant normal stresses σ_{xx} and σ_{yy}. For such a problem, if you want to formulate it, then this second-degree polynomial is suitable as the choice of stress function.



Now, moving forward to the third-degree polynomial, The general form of a third-degree polynomial is given as $a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3$. A third-degree polynomial also satisfies the biharmonic condition directly. So, $\nabla^4\phi = 0$. Substituting this form of ϕ in the stress component equation, $\sigma_{xx} = 2c_3x + 6d_3y$, $\sigma_{yy} = 2b_3y + 6a_3x$, and $\tau_{xy} = -2b_3x - 2c_3y$.

If you carefully look at all three stress components here, all of them are linear functions of both x and y. So, for ϕ being a first-degree polynomial, the stresses were 0, ϕ being a second-degree polynomial, the stress components were constant, ϕ being a third-degree polynomial, all stress components are linear functions of the in-plane coordinates x and y. So, these particular third-degree polynomials can be used to represent pure bending problems.

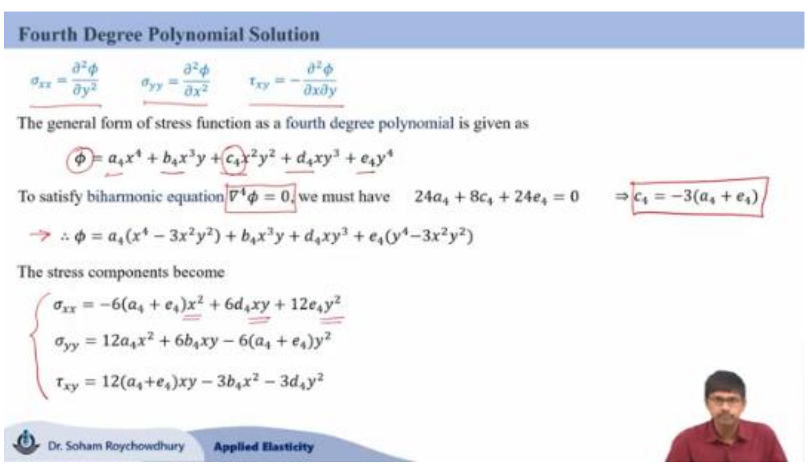
Depending on the axis of bending or the axis of moment, we can choose some of these constants a_3, b_3, c_3 to be 0 and some of them to be non-zero. So, let us choose one of the constants to be non-zero only, that is d_3. And a_3, b_3, c_3 are set to 0. With that, the only non-zero stress component is σ_{xx}, which is given by $6d_3y$.

σ_{yy} and τ_{xy} would both go to 0. So, for this particular case, we can represent the state of stress as a case of pure bending about the z-axis. So, if you think about this problem, let us consider this beam of length l, which is aligned along the x-axis, and width d, thickness t, which is aligned along the y-axis. z is the out-of-plane axis about which a bending moment is applied. So, width d is much smaller compared to length.

For clarity, I have just shown d on a bigger side, scaled it up. Now, it is subjected to some bending moment about the z-axis in this direction or in the opposite direction, either one. Now, for such cases, stresses would be generated only along the x-axis, which should vary linearly over the depth along d, along the y-axis. And here, this σ_{xx} seeks $6d_3y$ represents the same. For a pure bending problem, we will not have any normal strain along y, stress along the y direction, or any shear stress τ_{xy}.

So, thus this problem can be replicated with the help of this chosen third-degree polynomial stress function with only one constant d_3 being non-zero and the rest three constants set to zero. So, this linear variation of σ_{xx} can be obtained with the maximum and minimum values occurring at the top and bottom fibre with magnitude $6d_3\left(\frac{d}{2}\right)$, where d is the total height or thickness of the beam. So, the top fibre would be having tensile, the bottom fibre would be having compressive. If you are having a moment applied in this fashion.

If you keep the direction of the moment just the sign of d_3 would be flipped. And top fibers will be having compression, bottom fibers will be having tension. So any pure bending problem can be solved with the help of third-degree polynomial stress functions.

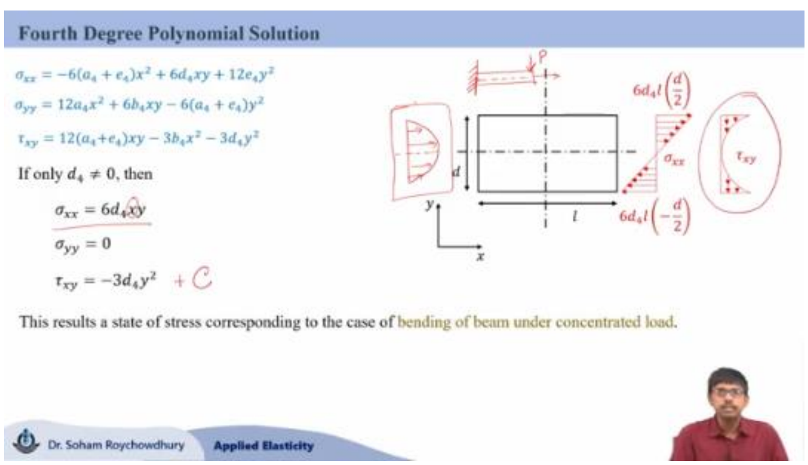


Now, moving to the fourth-degree polynomial stress function in its most general form, it can be written as $a_4x^4 + b_4x^3y + c_4x^2y^2 + d_4xy^3 + e_4y^4$.

Now, note that till third-order or third-degree polynomial stress functions automatically satisfy the biharmonic condition $\nabla^4\phi = 0$ directly. However, for fourth-degree onwards, Additionally, we need to satisfy the biharmonic equation, and that will give us some constraint between these five constants. Here, we have five constants: a_4, b_4, c_4, d_4, and e_4. These five constants will be linked by one equation, which we would be getting when we are substituting this chosen form of ϕ in the biharmonic equation.

So, if you do so, for the present case, this would be the condition coming: $24a_4 + 8c_4 + 24e_4 = 0$, and from this, one of the constants $c_4 = -3(a_4 + e_4)$. So, all five constants for a general fourth-degree polynomial stress function are not independent. One of the constants is related to the other two constants; c_4 is related to a_4 and e_4. In this particular fashion, then only the stress function of fourth-degree polynomial is going to satisfy the biharmonic condition.

Thus, the modified form of 5' for a fourth-degree polynomial can be written as this, where this c_4 is replaced with minus $3a_4$ minus $3e_4$, and thus the ϕ is modified as $a_4(x^4 - 3x^2y^2) + b_4x^3y + d_4xy^3 + e_4(y^4 - 3x^2y^2)$. This form of ϕ directly satisfies the biharmonic condition because we had already imposed the constraint obtained from $\nabla^4\phi$ as c_4 equals minus $3a_4$ minus $3e_4$. Now, finding the stress components by using this equation, σ_{xx}, σ_{yy}, and τ_{xy} can be obtained like this.



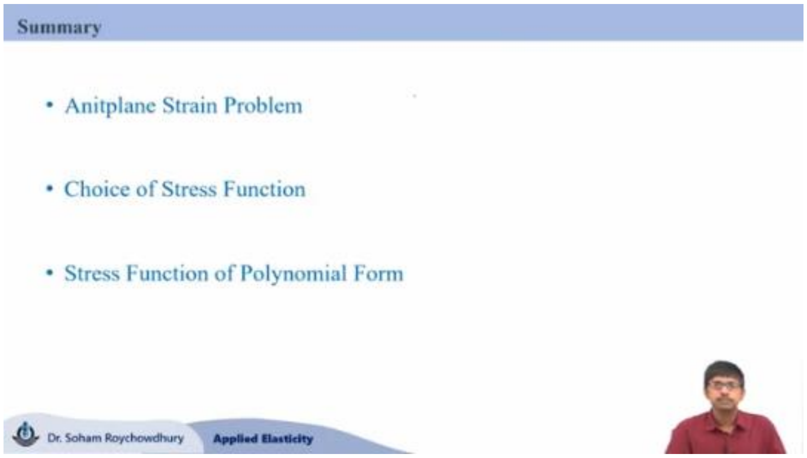
By using this particular form of phi, which is a fourth-degree polynomial. Now, we will try to see what type of problem can be solved with the help of this particular stress function. Check that or note that all these stress components are quadratic functions. Second-order functions of the in-plane variables x and y, so x^2, xy, and y^2 terms are present in the expressions of all three stresses: σ_{xx}, σ_{yy}, and τ_{xy}. They are related through different constants, so for the fourth-degree polynomial stress function choice, the stress variations would be quadratic in nature over x and y.

Now, if you choose only one of the constants d_4 to be non-zero, setting the rest of the constants to 0, $\sigma_{xx} = 6d_4xy$, $\sigma_{yy} = 0$, and $\tau_{xy} = -3d_4y^2$. Now, this particular form of stress components can be used for solving a bending beam problem under concentrated loading where the stress is not constant along the length for the pure bending problem. The bending moment acting, M, was constant throughout the length of the beam, and thus σ_{xx} was constant along the length of the beam. That was not varying with x. That was only varying linearly with y. However, for the beam subjected to concentrated transverse loading, for such cases, the bending moment varies linearly along x.

Along the length of the beam, and thus the in-plane normal stress (bending stress, σ_{xx}) should also be a linear function of x, along with a linear function of y. And these result in a quadratic function of y (a second-order function of y^2) in τ_{xy}, the transverse shear stress. So, if you are considering this beam, which has length l and depth d, the height d, the σ_{xx} at any value of x would be a linear function.

So, this varies linearly over y along the depth, and it would also vary linearly along x. So, if you have a cantilever beam subjected to an end load P, that would be a problem which can be solved with the help of this. So, as you move from the fixed edge to the free edge, the bending moment would change. At the free edge, the bending moment is 0. As you move towards the fixed edge, the bending moment will increase linearly.

So, through this x term, that effect can be taken care of, and the shear stress variation would look like this, which is parabolic in nature. Now, typically, the shear stress variation is like this. It is maximum at the top and minimum at the mid-plane, and maximum at the mid-plane and minimum at the top and bottom. So, that can be obtained just by superimposing a constant with this. If $\tau_{xy} = -3d_4y^2$ plus some other constant, let us say C, then we can get this kind of variation instead of this one. So, these can be obtained by using a combined stress function of second-order and fourth-order polynomials. We have seen that a second-degree polynomial results in constant tau, and a fourth-degree polynomial results in a quadratic variation of τ over y. So, if you combine both of them, then we can get this kind of stress distribution for tau xy over y. So, for many of the physical problems, we need to combine different orders of polynomials, different degrees of polynomials, so that the obtained stress fields can exactly mimic the actual stresses generated in that particular problem.



So, in this lecture, we discussed the anti-plane strain formulation, then the different possible choices of stress functions, and then looked into different polynomial stress functions of first, second, third, and fourth degree. Thank you.