APPLIED ELASTICITY

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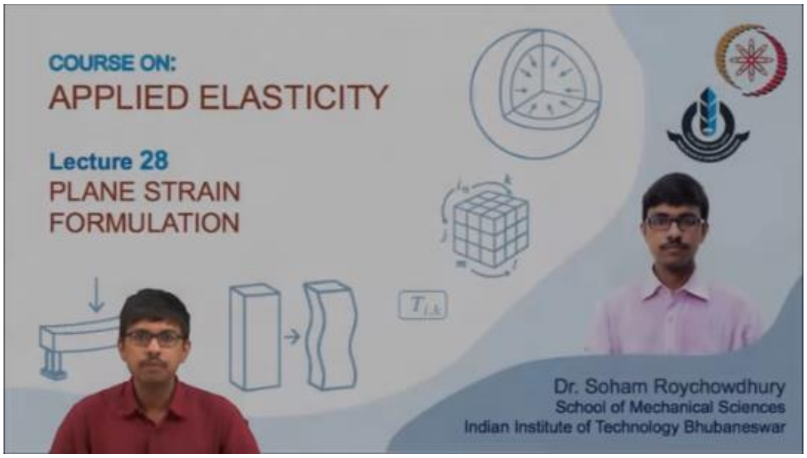
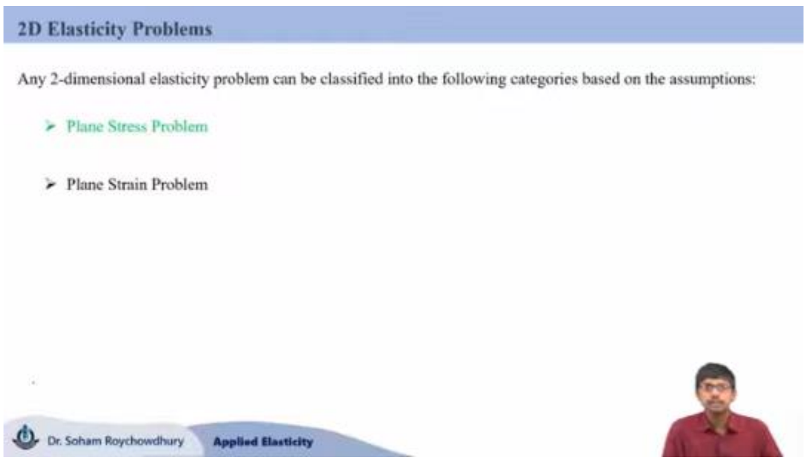
Lecture- 28

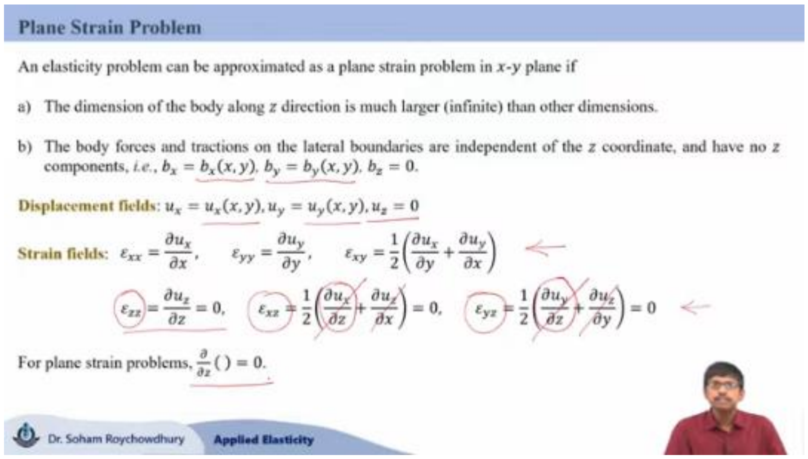


Welcome back to the course on applied elasticity. In today's lecture, we are going to discuss the plane strain formulation of elasticity problems. In the previous lecture, we discussed the plane stress problem.



Now, we know that there are two basic assumptions using which we can reduce a three-dimensional elasticity problem into a two-dimensional elasticity problem.

One is called plane stress, and the other is called plane strain. So, the plane stress problem was discussed in the previous lecture. We are going to talk about the plane strain problem in the present lecture. Apart from these two, there exists another case of anti-plane strain problem, which we will discuss in the next lecture.



Now, coming to the assumptions of the plane strain problem, an elasticity problem can be approximated as a plane strain problem in the x-y plane. The length of the body, the dimension of the body along the z direction, is much larger compared to the other dimensions along the x and y directions. So, theoretically, the body is infinitely large along one of the directions, and we call that direction the z direction; then only the problem can be approximated as a plane strain problem in the x-y plane. For plane stress, the thickness was assumed to be very small, whereas for plane strain, we are assuming the length along the z-direction or out-of-plane direction to be extremely large, theoretically infinite. Now, similar to the plane stress problem, we are also making an assumption that the surface traction and the body forces are independent of the z-coordinate. And the z-component of the surface traction and the body forces must be zero. So, b_x and b_y, the x and y components of the body force, can be non-zero but they must not be dependent on z. b_x can only be a function of x and y, b_y can also only be a function of x and y, whereas the z-component of the body force, b_z, must be zero.

The same assumption is required to be imposed for the surface tractions on the side faces or the lateral boundaries. Now, coming to the displacement field for this plane strain problem. For the plane strain problem, the displacement fields are as follows. Two in-plane displacements, u_x and u_y, exist as non-zero components, and both of them are functions of only in-plane variables x and y, independent of z. And the out-of-plane displacement, u_z, is taken to be zero because the body is infinitely large along the z-direction.

Thus, the strain in that direction, the change in length in the z-direction, is negligible or cannot be felt by the infinite body. So, u_z is equal to zero. And as it is a very large, long body, none of the quantities are varying along z. We are not going to consider the variation of any quantity along z for the plane strain problem.

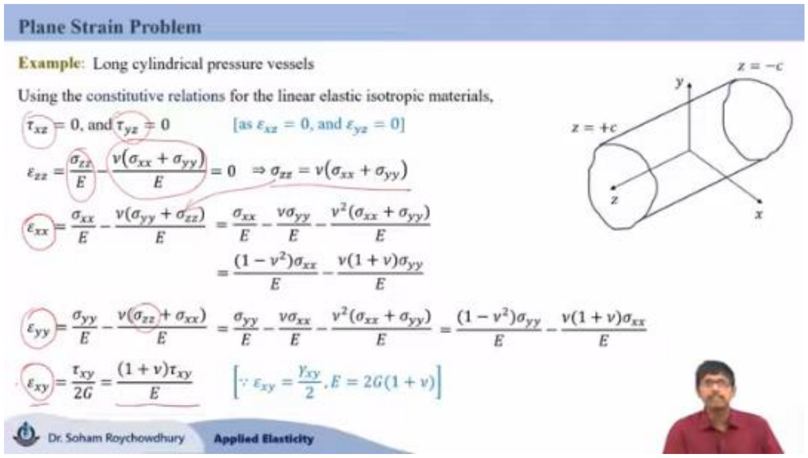
So, the partial derivative of any quantity with respect to z, $\frac{\partial}{\partial z}$ of any quantity is going to be 0 for the plane strain problem, which is valid for the long elastic continuum along the z direction. Now, using these displacement fields, if you substitute them in the strain-displacement relation, the strain components can be obtained like this. $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$. The in-plane shear strain $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$. So, these three are the in-plane strain components. These are all non-zero strain components.

Now, coming to the out-of-plane strain components, which are in the second row, ε_{zz} the out-of-plane normal strain, this is defined as $\frac{\partial u_z}{\partial z}$. Now, u_z being 0, this ε_{zz} is going to be 0. Now, considering two out-of-plane shear strains, ε_{xz} and ε_{yz}, they are defined as $\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$ and $\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$.

Considering u_z to be 0, both of these u_z terms would go to 0. Now, as u_x and u_y are functions of x and y only, independent of z, and here we are taking the derivative of u_x and u_y with respect to z. So, thus these two terms would also go to 0 because u_x and u_y are independent of z. So, their z-derivative, partial derivative of these quantities with respect to z, would also vanish.

So, all three out-of-plane strain components, normal strain σ_{zz} and two out-of-plane shear strains ε_{xz} and ε_{yz}, both of them are going to 0, and as I had already stated, length being very large along the z-direction, variation of the quantities along the z-direction, $\frac{\partial}{\partial z}$ of any quantity, is negligible or set to 0 for the plane strain problem. So, the assumption—these are the assumptions of the plane strain problem with which all three out-of-plane strain components, normal as well as shear components, are set to be 0, and $\frac{\partial}{\partial z}$ of any quantity is also 0.

We assume u_x and u_y to be the only two non-zero displacement components, which are functions of in-plane variables x and y.

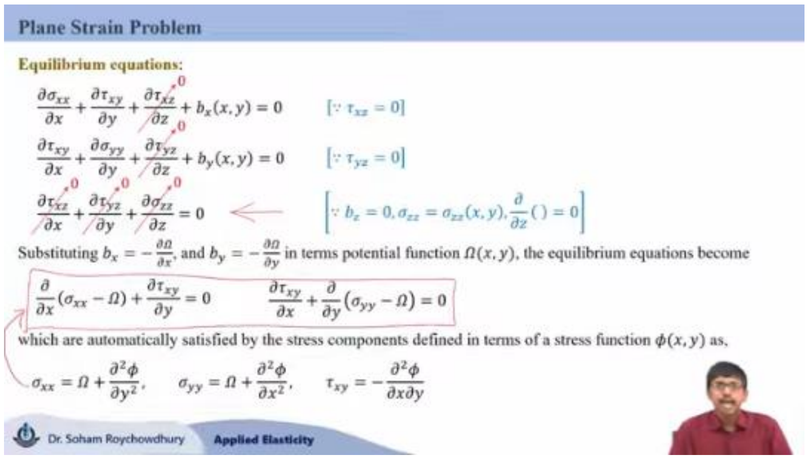


Now, moving forward, the example of this kind of problem is the long cylindrical pressure vessels or a very large cylinder subjected to uniform loading along the z-direction. So, you can consider this long cylinder where z is the axial direction, and z is varying from minus c to plus c, where c is a very large number.

So, for such cases, For different uniform loading acting on the plane in-plane loading in the xy-plane, forces or loading being confined to the xy-plane, we can consider these problems to be described as plane strain problems. Now, using the constitutive relation for linear elastic isotropic homogeneous solids, tau $\varepsilon_{xz} = 0$ and $\varepsilon_{yz} = 0$. Both the out-of-plane shear strain components being 0, we must have both the out-of-plane shear stress components τ_{xz} and τ_{yz} to be 0. Now, ε_{zz}, the out-of-plane normal strain, is also 0, which, using the constitutive equation, can be written as $\frac{\sigma_{zz}}{E} - \frac{\nu(\sigma_{xx}+\sigma_{yy})}{E}$. Now, As ε_{zz} is 0, we can write that this term equals this term, and canceling E from both sides, $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. So, out-of-plane stress is written as a function of two in-plane normal stresses, σ_{xx} and σ_{yy}, and out-of-plane shear stresses, both τ_{xz} and τ_{yz}, are obtained to be 0 for the plane strain problem.

Now, moving to the in-plane strain components, ε_{xx}, which is $\frac{\sigma_{xx}}{E} - \frac{\nu(\sigma_{yy}+\sigma_{zz})}{E}$. Here, I am replacing σ_{zz} in terms of σ_{xx} and σ_{yy}, and if you simplify this, ε_{xx} would come out to be a function of σ_{xx} and σ_{yy} only as $\frac{(1-\nu^2)\sigma_{xx}}{E} - \frac{\nu(1+\nu)\sigma_{yy}}{E}$. In the same fashion, the other in-plane normal strain, ε_{yy}, can be written as $\frac{\sigma_{yy}}{E} - \frac{\nu(\sigma_{zz}+\sigma_{xx})}{E}$.

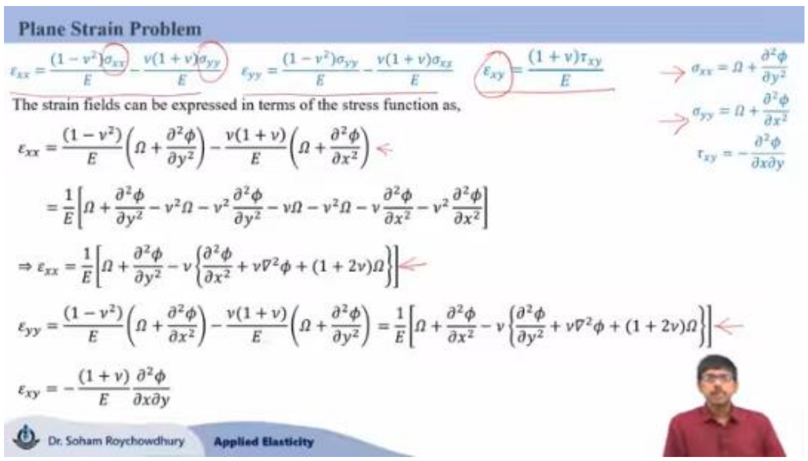
Substituting σ_{zz} in this form and simplifying it, we would get ε_{yy} as $\frac{(1-\nu^2)\sigma_{yy}}{E} - \frac{\nu(1+\nu)\sigma_{xx}}{E}$. And finally, the in-plane shear strain components ε_{xy} can be written as $\frac{\tau_{xy}}{2G}$, where ε_{xy} is the tensorial in-plane shear strain and gamma xy is the engineering shear strain. So, $\frac{\gamma_{xy}}{2}$ equals ε_{xy}, and using the relation between Young's modulus and G, Young's modulus E and modulus of rigidity G as $E = 2G(1+\nu)$. We can rewrite $\varepsilon_{xy} = \frac{(1+\nu)\tau_{xy}}{E}$. So now, we have written all the nonzero strain components. Three non-zero strain components are there, all are in-plane components: ε_{xx}, ε_{yy}, ε_{xy}, confined to the xy plane only. These are written in terms of three in-plane stress components: σ_{xx}, σ_{yy}, and τ_{xy}.



Now we are going to substitute these into the equilibrium equation. So, the first equilibrium equation is $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x$, where body force b_x may be non-zero but independent of z, only a function of x and y. Similarly, we can write the second and third equilibrium equations. Note that for the third equilibrium equation along the z-direction, there would be no body force, by one of the assumptions of the plane strain problem, which states that b_z must be 0. Now, as we have τ_{xz} to be 0, two terms are going to 0. $\tau_{yz} = 0$, two more terms are going to 0, and we also have $\frac{\partial}{\partial z}$ of any quantity to be 0. Considering that this last term, which is $\frac{\partial \sigma_{zz}}{\partial z}$, would also go to 0. σ_{zz} is a function of x and y only. So, thus, this term will also go to 0. So, if you carefully look, the last equilibrium equation is already satisfied automatically.

We would be left with only the first two equilibrium equations along the x and y directions, and now writing the writing the b_x and b_y, the body forces, in terms of a potential function capital Ω as $-\frac{\partial \Omega}{\partial x}$ equals to b_x and $-\frac{\partial \Omega}{\partial x}$ equals to b_y. This same fashion was used for writing the body forces for the plane stress problem in terms of a potential function capital Ω. So, rewriting the equilibrium equation after substituting b_x and b_y in terms of capital Ω, we would get these two new forms of equilibrium equations as $\frac{\partial}{\partial x}(\sigma_{xx} - \Omega) + \frac{\partial \tau_{xy}}{\partial y} = 0$, and $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - \Omega) = 0$. If you recall or compare this with the plane stress-type governing equations for a 2D plane stress problem, these were exactly identical. Now, these two equations can be automatically satisfied if you introduce a stress function phi x, y.

In this particular form, where the stress components are defined as $\sigma_{xx} = \Omega + \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy} = \Omega + \frac{\partial^2 \phi}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$. So, with this definition of stress function, stress components in terms of a stress function ϕ and the potential function Ω representing the body forces, we can automatically satisfy both of these two equilibrium equations. So, substituting this here, both these equations will get automatically satisfied. This form of stress function stress components as a function of stress function ϕ are also exactly the same as those for the plane stress problem.



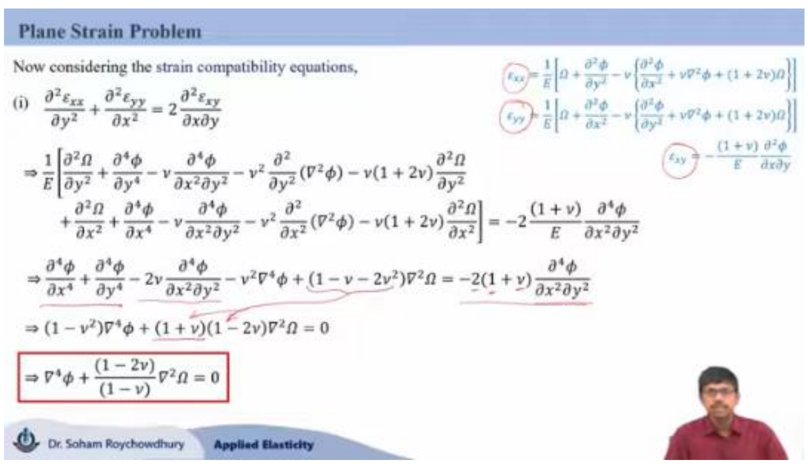
Now, moving forward, we are going to express the strain fields, non-zero strain components, in terms of stress functions. So, earlier we had obtained the strain fields as a function of stress fields here. So, these are the three equations which we had obtained by using the constitutive equation.

The strain components were related with three in-plane stress components: σ_{xx}, σ_{yy}, and τ_{xy}. In place of σ_{xx} and σ_{yy}, I am going to replace these expressions, which are σ_{xx} and σ_{yy} in terms of stress function ϕ and the potential function Ω. So, replacing $\sigma_{xx} = \Omega + \frac{\partial^2 \phi}{\partial y^2}$ and $\sigma_{yy} = \Omega + \frac{\partial^2 \phi}{\partial x^2}$ in the first expression of ε_{xx}, it would come out to be of this particular form.

Now, this can be further expanded and written in this particular form, having a total of 8 terms. In the same fashion, we can rewrite ε_{yy} as well. So, if you do so, ε_{yy} will look like this. Now, ε_{xx} is $\frac{1}{E}\left[\Omega + \frac{\partial^2 \phi}{\partial y^2} - \nu\left\{\frac{\partial^2 \phi}{\partial x^2} + \nu\nabla^2\phi + (1+2\nu)\Omega\right\}\right]$. Whereas, ε_{yy} in terms of the stress function would become $\frac{1}{E}\left[\Omega + \frac{\partial^2 \phi}{\partial x^2} - \nu\left\{\frac{\partial^2 \phi}{\partial y^2} + \nu\nabla^2\phi + (1+2\nu)\Omega\right\}\right]$.

So, these are the two final expressions of the in-plane normal strains ε_{xx} and ε_{yy} as functions of ϕ and Ω. Now, coming to the in-plane shear strain component ε_{xy}, in the expression of ε_{xy}, we are replacing $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$, and thus it would become $-\frac{(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}$. This is the in-plane strain component. So, now we have replaced all three in-plane strain components: ε_{xx}, ε_{yy}, and ε_{xy}.

In terms of the stress function ϕ and the potential function capital Ω. Apart from these three in-plane strains, we have three out-of-plane strain components: ε_{xz}, ε_{yz}, and ε_{zz}. However, for the plane strain problem, all three of them are taken to be 0. Now, our next objective is to satisfy the strain compatibility equation with this form of strain components so that we can ensure the existence of a unique displacement field. To ensure a unique displacement field, this form of strain components must satisfy all six strain compatibility equations.



Now, we will take all the compatibility equations one after another. So, first starting with the first incompatibility, which is $\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$.

Now, replacing ε_{xx}, ε_{yy}, and ε_{zz}. In this form of as a function of ϕ and Ω in this first compatibility equation on both sides, it would look like this. This big expression would appear where we have 10 terms on the left-hand side and one term on the right-hand side coming from ε_{xy}. Now, if you start combining some of the terms on the left-hand side, they can be combined into harmonic or biharmonic operators and thus, After simplification, this expression would look like this: $\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} - 2\nu\frac{\partial^4 \phi}{\partial x^2 \partial y^2} - \nu^2\nabla^4\phi + (1-\nu-2\nu^2)\nabla^2\Omega = -2(1+\nu)\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$. Now, further, this particular term $2\nu\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$ will get cancelled with this $2\nu\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$ from both sides, and the remaining term minus 2 times 1 into $\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$ can be taken on the left-hand side and combined with these two terms, which would result in another $\nabla^4\phi$, the harmonic or biharmonic of ϕ. Thus, simplifying this further by taking this term on the left-hand side, the equation would be like this: $(1 - \nu^2)\nabla^4\phi + (1+\nu)(1-2\nu)\nabla^2\Omega$. So, this particular term is now factorized into two terms: $(1+\nu)$ and $(1-2\nu)$.

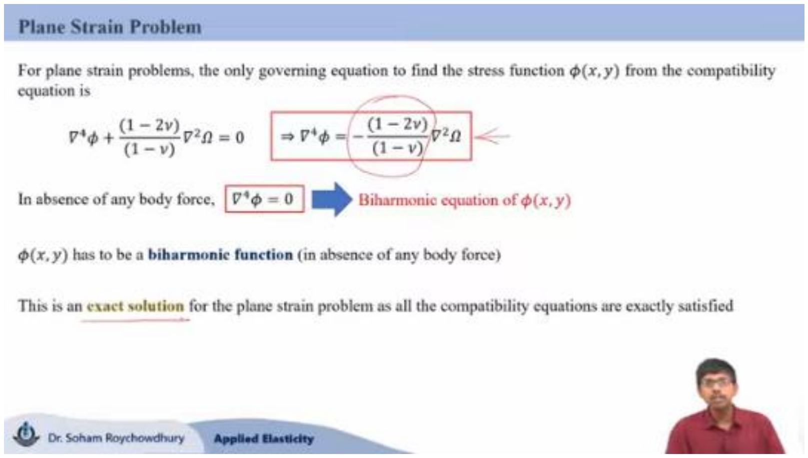
Now, the first term of $1-\nu^2$ can also be factorized as $1+\nu$ and $1-\nu$, and the $1+\nu$ factor will get canceled from both terms as that is a common factor. Hence, finally, this equation—the first incompatible equation—would reduce to this form: $\nabla^4\phi + \frac{(1-2\nu)}{(1-\nu)}\nabla^2\Omega = 0$. So, for a given body force field, we know omega, then the stress function must satisfy— ϕ must satisfy this particular expression. If you have zero body force, then Ω would go to 0, and this equation will be $\nabla^4\phi$ equals 0.

Now, coming to the other five strain compatibility equations apart from the first one. From the first one, we got a condition which is required to be satisfied by the stress function phi. Now, the other five strain compatibility equations are shown here.

Now, here we we will show that all these 5, all these 5 strain compatibility equations are directly satisfied without any further constraint for a plane strain problem confined to xy plane. So, considering epsilon xz to be 0, we have ε_{xz}, ε_{yz} and ε_{zz}, all 3 out of plane strain components to be 0. So, first imposing this ε_{xz} to be 0, the four terms, these four terms would be going to 0.

Imposing ε_{yz} to 0, further four more terms, this one, this one, this one and this one, this four would also be going to 0. Then imposing ε_{zz} to be 0, these three terms, this one, this one and this one. These three terms would further go to 0. And finally, for the plane strain problem, we have $\frac{\partial}{\partial z}(\,) = 0$.

Partial derivative with respect to z for any quantity is 0. So, with that, all rest of the terms, this is having $\frac{\partial^2}{\partial z^2}$, this is having $\frac{\partial^2}{\partial z^2}$, $\frac{\partial}{\partial z}$ here, $\frac{\partial}{\partial z}$ here, $\frac{\partial}{\partial z}$ here, Here also, here also, all these terms would go to 0 and you can see all these 5 strain compatibility equations are automatically satisfied. So, we are having only one strain compatibility equation to be satisfied by 5 stress functions; all the rest are directly satisfied for the plane strain problem.



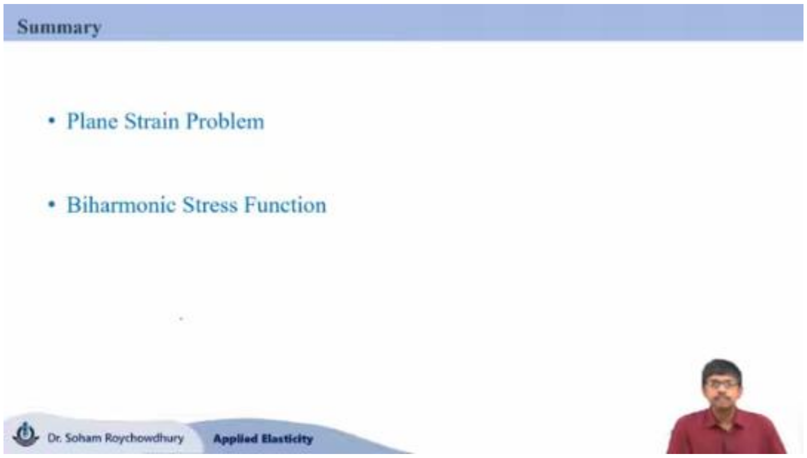
So, for the plane strain problem, the only governing equation to determine the stress function from the compatibility equation, this is. The $\nabla^4\phi + \frac{(1-2\nu)}{(1-\nu)}\nabla^2\Omega = 0$. And thus, we can write the $\nabla^4\phi = -\frac{(1-2\nu)}{(1-\nu)}\nabla^2\Omega$. So, this is the governing equation for any plane strain problem which has a non-zero body force.

And in the absence of body force, the right-hand side term would go to 0 as capital Ω is 0. Thus, $\nabla^4\phi = 0$. This is called the Biharmonic equation of $\phi(x,y)$. So, in the absence of body force, the stress function for the plane strain problem must satisfy the biharmonic condition.

This condition was the same for the plane stress problem as well in the absence of body force. However, if you have body force, then the solution or choice of ϕ would be different. If you look at this factor, present for the plane stress case and plane strain case both are dependent only on Poisson's ratio ν but these factors have two different forms; these are two different functions of Poisson's ratio, ν. So, in the presence of body force, the choice of stress function would be different for plane stress and plane strain problems. However, If body forces are absent, we can choose the same stress function for solving plane stress or plane strain problems, and that choice must satisfy the biharmonic condition: $\nabla^4\phi = 0$. So, in the absence of any body force, $\phi(x,y)$ has to be a biharmonic function for plane strain problem solution.

Another difference between plane stress and plane strain formulation is that the plane stress problem solution was an approximate solution. We were able to solve, we were able to satisfy the strain compatibility equations approximately with the assumption of a very thin continuum, by neglecting the z-dependent terms in the stress components.

Here, it is an exact solution. This is the only governing equation coming from the first strain-displacement compatibility equation. If you are able to satisfy this exactly, then the obtained solution of this plane strain problem would be an exact solution because all the remaining five strain compatibility equations are exactly satisfied.



So, in this lecture, we talked about the plane strain formulation, starting with this assumption and then proved that, in the absence of any body forces, the stress function for a plane strain problem should be a bi-harmonic stress function. Thank you.