APPLIED ELASTICITY

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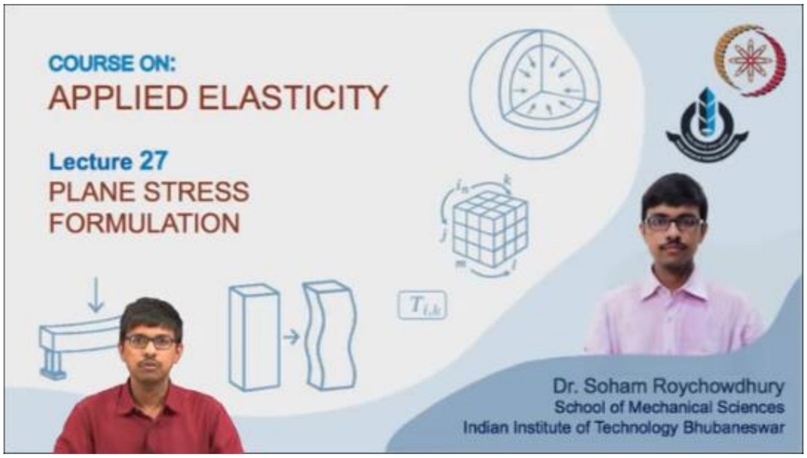
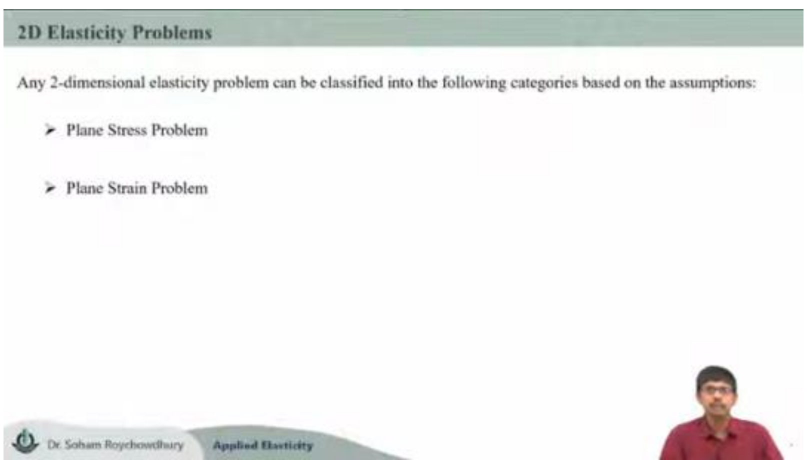
Lecture- 27

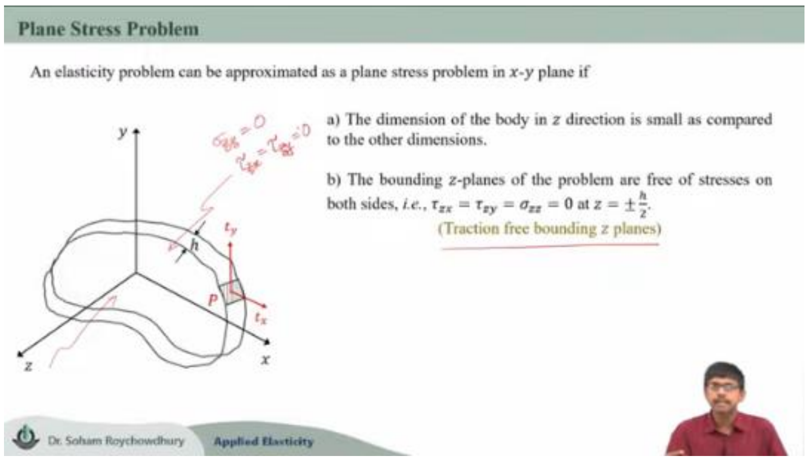


Welcome back to the course on applied elasticity. In today's lecture, we are going to discuss the formulation of the plane stress problems in applied elasticity.



So, in the previous lecture, we discussed the 2D elasticity problems and how these problems can be solved with the help of a stress function known as the Airy stress function. So, there are different categories of 2-dimensional elasticity problems.

Depending on the assumptions by which a 3D problem is reduced to a 2D problem. So, the first class of problem is the plane stress type 2D problem, and the next one is the plane strain type 2D problem. These are the two most common assumption sets with which a three-dimensional elasticity problem can be reduced to a two-dimensional problem. Apart from these two, we also have another one, which is not that much used in practice, but that is named the anti-plane strain problem. In today's lecture, we are going to talk about the formulation of the first case, that is, the plane stress problem.



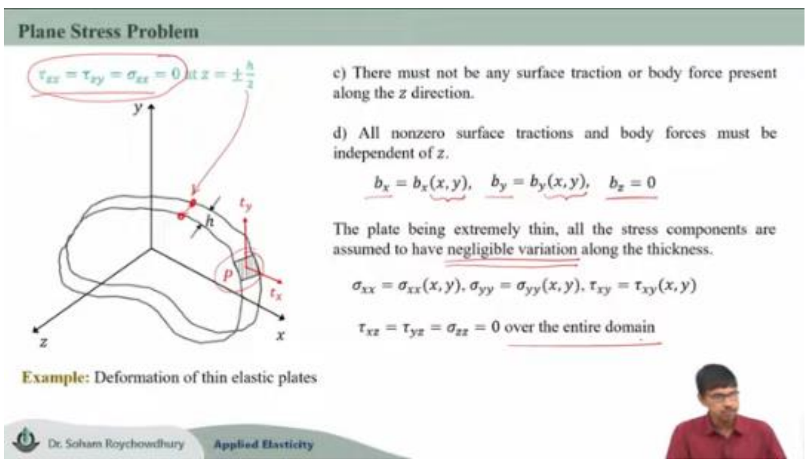
So, first, looking into the assumptions of the plane stress problem, under what assumptions we can call an elasticity problem, we can approximate a problem of deformation as a plane stress problem. So, let us consider a problem to be defined as a plane stress problem in the x-y plane. So, x-y is called in-plane; all the stress and strain components on the x-y plane are called in-plane stress and strain components, whereas, the z components on the z plane, the stress and strain components, are known as out-of-plane stress or out-of-plane strain components.

So, a problem would be called a plane stress problem in the x-y plane. if the dimension of the body is very small in the z direction. This means the thickness of the body along the z direction is much smaller compared to the dimensions of the body along the x and y directions. So, let us consider a thin plate which is aligned on the x-y plane. and its thickness h, which is along the z direction, is much smaller compared to the other dimensions along the x and y axes.

So, if you consider any point P on the side face of the plate of this thin elastic continuum, then t_x and t_y. These are the two stress resultants which are acting on the side faces. So, this is the first assumption of the plane stress problem. Now, the next one is the bounding

z planes. So, the front face of this continuum and the back face of the continuum are nothing but the positive z plane and the negative z plane.

Those two faces must be free of any kind of stresses; that is called traction-free bounding z planes. So, considering this origin to be at the mid-plane of this thin elastic plate, we can consider or denote the top or the front face, of the continuum as z equals to $+\frac{h}{2}$ plane, whereas the back face can be denoted as $z = -\frac{h}{2}$ plane. So, for both $z = \pm\frac{h}{2}$ planes, all three stress components which can be defined on the z plane— τ_{zx}, τ_{zy}, and σ_{zz}—are all 0. So, for the front and the back face, we have all three stress components—normal stress σ_{zz} to be 0, and both the shear stresses τ_{zx}, and τ_{zy} to be 0. So, all three out-of-plane stress components are forced to be 0 for z equals to plus minus h by 2. The traction-free boundary z planes are assumed.



Now, moving forward, the next assumption is there should not be any surface traction or body force present along the thickness direction.

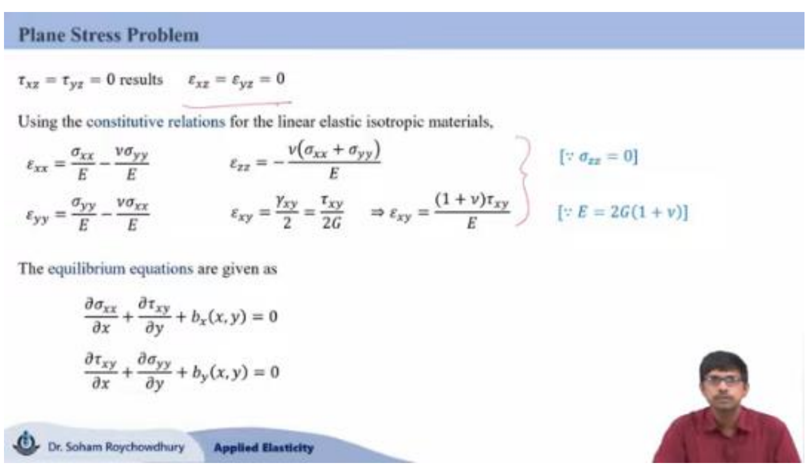
Along the z direction, the component of surface traction is neglected, and the component of body force is neglected. That is why in the diagram, if you look at the surface traction of point P, only t_x and t_y components are shown. No t_z would be there at any point P on the continuum. And if you are having any non-zero surface traction or non-zero body force component present within the domain along x and y direction, those components should also be independent of z, meaning you can have non-zero components of body force or surface traction along the x and y direction. So, b_x can be non-zero, b_y can be non-zero components of body force along the x and y directions respectively, but they can only be functions of x and y, independent of z. So, b_x and b_y are functions of x and y only, not dependent on z, whereas b_z, the out-of-plane component of body force, is zero. Similarly, similar properties are valid for t_x, t_y, and t_z, the surface traction components.

So, the plate being extremely thin and the edge being very small, The stress components—all the stress components—are assumed to have negligible variation along the thickness. So, along this thickness h, for a particular point, you keep the x and y coordinates constant, just varying the z coordinate from plus h by 2 to minus h by 2. You are traveling along the thickness of this thin continuum.

None of the stress values are going to vary, as the plate is assumed to be very thin. With this assumption, we can write the stress fields like this over the entire domain. So, σ_{xx}, σ_{yy}, and τ_{xy} are the three non-zero in-plane stress components, which are functions of x and y only, independent of z for this kind of problem. Whereas the three out-of-plane stress components—τ_{xz}, τ_{yz}, and σ_{zz}—are all zero over the entire domain. Why so?

From the second assumption, we had the stress components to be zero only for the front face, $z = +\frac{h}{2}$, and the bottom or back face, $z = -\frac{h}{2}$. With this assumption of negligible variation of any stress components along the thickness, all the stresses will go to zero. So, at this point and this point, they are directly zero based on this assumption. Due to no variation of the stresses within this small thickness h, we can consider all these three stresses to be zero for all values of z.

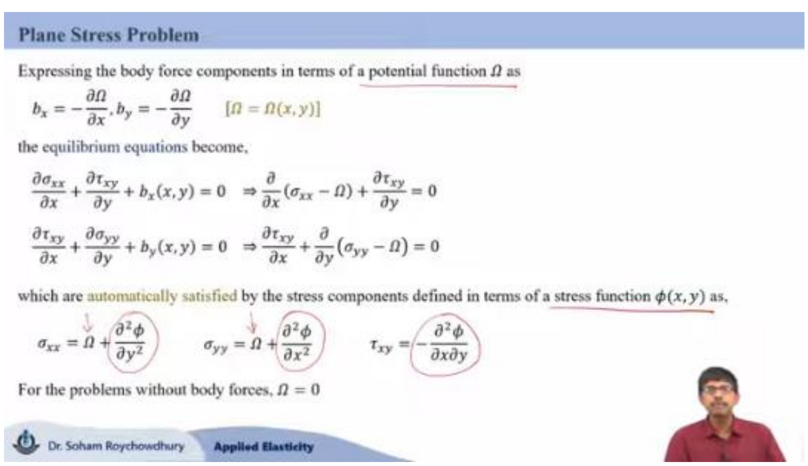
So, over the entire domain, out-of-plane stress components are 0, whereas in-plane stress components are functions of x and y only. So, an example of this kind of plane stress problem is the deformation of thin elastic plates or thin elastic shells.



So, for this plane stress problem, where the out-of-plane stress components τ_{xz} and τ_{yz} are taken to be 0 from the plane stress assumption, we must have the out-of-plane shear strain components to be 0; ε_{xz} and ε_{yz} would also be 0. Now, using the constitutive relation for linear elastic isotropic homogeneous solids, we can write the rest of the strain components—non-zero strain components—as ε_{xx}, the normal strain along the x direction, as $\frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E}$. Here, $\sigma_{zz} = 0$ for the plane stress assumption, and thus ε_{xx} is independent of σ_{zz}. Similarly, ε_{yy}, the normal strain in the y direction, is $\frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E}$. Then, ε_{zz}, the normal strain in the out-of-plane z direction, is $-\frac{\nu(\sigma_{xx}+\sigma_{yy})}{E}$. And the only non-zero shear strain, ε_{xy}, is equal to $\frac{\tau_{xy}}{2G}$.

It can also be written as $\frac{\gamma_{xy}}{2}$. γ_{xy} is the in-plane engineering shear strain; ε_{xy} is the in-plane tensorial shear strain. Now, using the relation between E and G, we can write E, Young's modulus, as $2G(1+\nu)$, and substituting that here, ε_{xy}. The in-plane tensorial shear strain can be written as $\frac{(1+\nu)\tau_{xy}}{E}$.

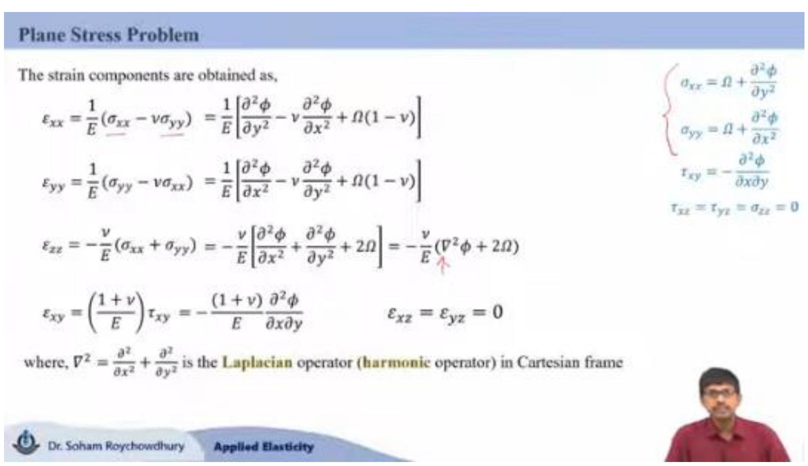
So, these are the four non-zero strain components, and these two are zero strain components for the plane stress problem. Now, for the plane stress problem, the equilibrium equations will look like this. Here, one of the equilibrium equations in the z-direction would be automatically satisfied, and the terms with τ_{xz} and τ_{yz} will go to zero in the first two equilibrium equations, leaving us with only two equilibrium equations for the plane stress problem. $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + b_x(x,y) = 0$, and $\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + b_y(x,y) = 0$, where b_x and b_y are the body force components per unit volume along the x and y directions, respectively. They can only be functions of in-plane variables x and y; b_x and b_y cannot be functions of z for the plane stress problem. So, we need to solve these equilibrium equations to solve our 2D plane stress problem, as the third equation is automatically satisfied.



Now, we are defining the body force components b_x and b_y as functions of a potential capital Ω. So, we define b_x and b_y as $-\frac{\partial\Omega}{\partial x}$ for b_x and $-\frac{\partial\Omega}{\partial y}$ for b_y, where Ω is a potential function that depends only on x and y and is independent of z. So, if you substitute this form of b_x and b_y, earlier we had two body forces, b_x and b_y. Now, they are related through this potential function, and the number of quantities involving the body forces in both equations is reduced from two to one. So, substituting $b_x = -\frac{\partial \Omega}{\partial x}$, the first equilibrium equation becomes $\frac{\partial}{\partial x}(\sigma_{xx} - \Omega) + \frac{\partial \tau_{xy}}{\partial y} = 0$ and substituting $b_y = -\frac{\partial \Omega}{\partial y}$ in the second equilibrium equation, that becomes $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - \Omega) = 0$. Now, you are going to introduce the concept of stress function. We are going to write these stress components as partial derivative functions of some stress function phi, so that both the equilibrium equations are automatically satisfied.

So, that was the concept for introducing the stress function, by which we are reducing the number of unknowns through the substitution of stress components in terms of the stress functions. So, here these two equations can be automatically satisfied if we define the stress components in terms of a stress function ϕ as $\sigma_{xx} = \Omega + \frac{\partial^2 \phi}{\partial y^2}$ and $\sigma_{yy} = \Omega + \frac{\partial^2 \phi}{\partial x^2}$.

$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$. So, if you recall the discussion on Airy stress function from the previous lecture, σ_{xx} was just defined as $\frac{\partial^2 \phi}{\partial y^2}$, σ_{yy} was $\frac{\partial^2 \phi}{\partial y^2}$, and τ_{xy} was this only. Here we are considering the non-zero body forces, and due to the presence of non-zero body forces these additional terms of capital omega are added to the normal stresses, but there is no effect of body force on the shear stress. If you are having a plane stress problem without any body force, then you can neglect this term, capital Ω, set it to 0, and stress functions are exactly similar to the Airy stress function.

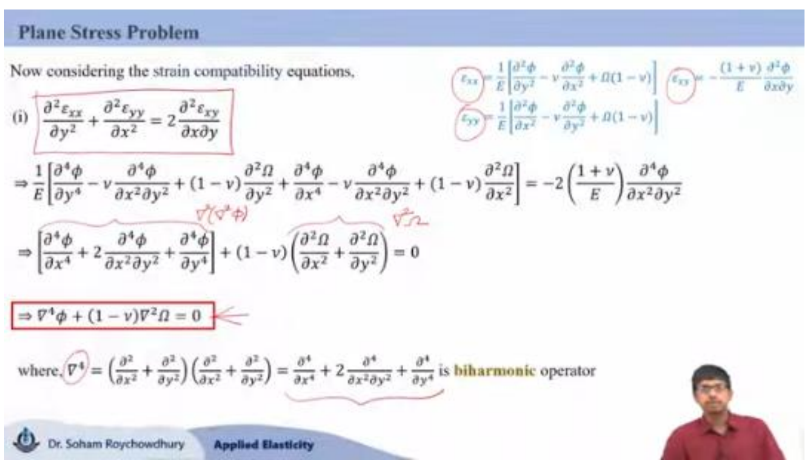


Now, moving forward, I am writing the strain components in terms of the stress function. So, we have already expressed the strain components as stress. The function of σ, the stress components, with the help of the constitutive equation. Now, replacing σ_{xx} and σ_{yy} in terms of this potential function, capital Ω, and stress function ϕ.

We can get ε_{xx} as $\frac{1}{E}\left[\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2} + \Omega(1-\nu)\right]$. Similarly, we can simplify ε_{yy}, ε_{zz}, and ε_{xy} as this, where. All these four non-zero strain components are now written in terms of two functions. One is the stress function phi, another is the potential function Ω corresponding to the body forces, and the other two strains are. ε_{xz} and ε_{yz}, both of them are 0.

So, this operator is the Laplacian operator, which is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for the Cartesian coordinate. This is also named as the harmonic operator. So, we have expressed all the strain components in terms of stress functions and the potential function Ω. Now, for solving this strain-stress problem. We are going to check whether these stress-strain components.

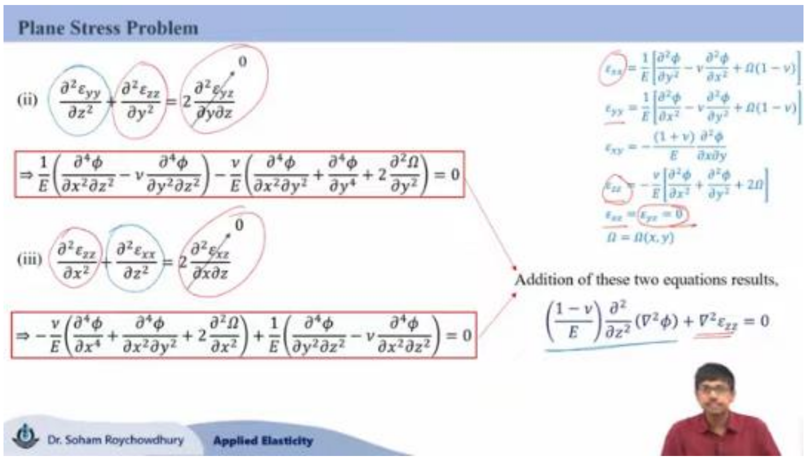
These obtained strain components in terms of ϕ are satisfying the strain compatibility equations or not. If these are satisfying the strain compatibility, then only we can find out the unique displacement field from these obtained strain fields. So, starting from the first strain compatibility equation, strain compatibility equations are given as curl of curl of epsilon to be 0. If you expand that, we would be getting 6 independent strain compatibility equations. So, on all 6 of them, we are going to substitute the expressions of ε_{xx}, ε_{yy}, ε_{xy}, and ε_{zz}, whereas ε_{xz} and yz are 0.



So, the first incompatibility equation is $\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}$. Now, substituting the expression of ε_{xx}, yy, and ε_{xy} on both sides of this equation and expanding it, it would look something like this. Now, we will try to further simplify it by combining all the terms with phi in one bracket and all the terms with capital omega in another bracket. So, the ϕ terms are $\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}$. Plus the next term involving Ω is $(1-\nu)\left(\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2}\right)$.

Now, this $\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}$ is nothing but the $\nabla^2 \Omega$, whereas the first one, $\frac{\partial^4 \phi}{\partial x^4}$, this particular term is $\nabla^4 \phi$. So, we can say the first term is the $\nabla^4 \phi + (1-\nu)\nabla^2 \Omega = 0$. So, this is the governing equation we are getting to satisfy the strain compatibility equation where $\nabla^4 \phi$ is the biharmonic operator, which is given in this particular form in the rectangular Cartesian coordinate system.

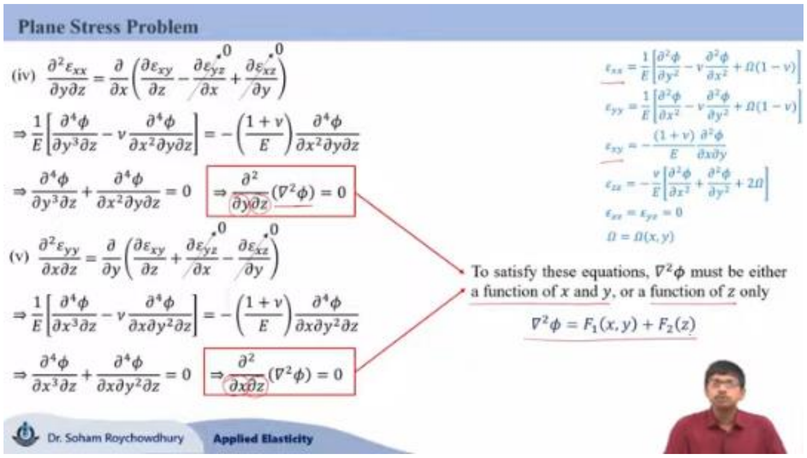
So, any choice of ϕ with which we are going to solve our plane stress problem must satisfy this governing equation. If ϕ and Ω satisfy this equation, then this first strain compatibility equation would be satisfied. So, the chosen form of ϕ must satisfy this.



Now, moving forward to the second strain compatibility equation, which is $\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$. Now, the $\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$ term is 0 because we already have this ε_{yz} component to be 0; thus, the right-hand side of this term is directly going to 0. Now, substituting ε_{yy} and ε_{zz} on the left-hand side and expanding it, the first equation would look like this. This is the second strain compatibility equation. Similarly, if you consider the third strain compatibility equation, which has ε_{xz} on the right-hand side, and once again, that is 0. So, this right-hand side term would also go to 0.

So, substituting ε_{zz} and ε_{xx} on the left-hand side of the third strain compatibility equation and simplifying it, it would look like this. So, these two equations are almost similar in form. Now, we can add these two equations, and with that, it can be written concisely like this. $\left(\frac{1-\nu}{E}\right)\frac{\partial^2}{\partial z^2}(\nabla^2 \phi) + \nabla^2 \varepsilon_{zz}$. So, if you look at the left-hand side of these two equations, the right-hand sides are both 0.

Now, as we are adding them, This $\frac{\partial^2 \varepsilon_{zz}}{\partial y^2}$ and $\frac{\partial^2 \varepsilon_{zz}}{\partial x^2}$. Addition of those two results in this term, $\nabla^2 \varepsilon_{zz}$. Whereas, addition of the other two terms, this one and this one, if you replace those and simplify by substituting ε in terms of stress function, that would result in this. So, we are getting this particular expression. This is another governing equation which is required to be satisfied from compatibility equations 2 and 3.

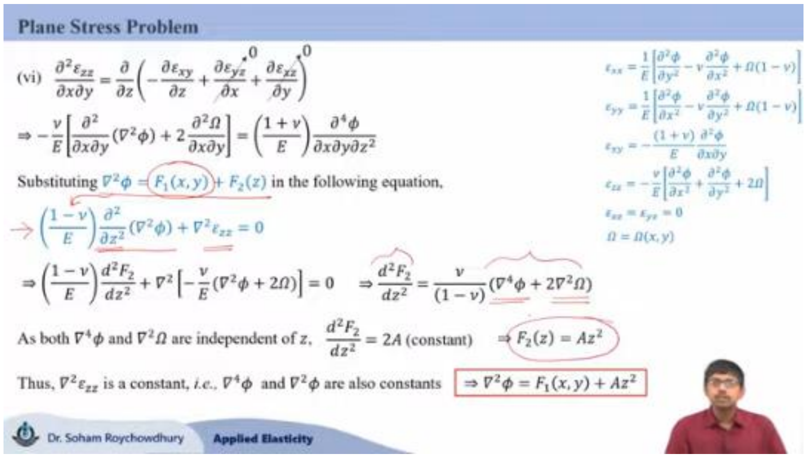


Now moving forward to the next compatibility equation, which looks slightly different. So, the first three compatibility equations are from the same branch, looking in a similar fashion, whereas the fourth, fifth, and sixth look different compared to those first sets. So, this is the third one where ε_{yz} and ε_{xz} terms are once again set to 0 on the right-hand side. We have only one non-zero term on the left and one non-zero term on the right. Now, substituting ε_{xx} and ε_{xy} on both terms, we would get this, which can be further simplified into this final form of $\frac{\partial^2}{\partial y \partial z}(\nabla^2 \phi)$.

So, this is the form of the fourth compatibility equation. In a similar fashion, taking the fifth compatibility equation, setting the right-hand side ε_{yz} and ε_{xz} terms to 0, and then substituting the expressions of ε_{yy} and ε_{xy}. The fifth compatibility equation can be simplified to a form like this: $\frac{\partial^2}{\partial x \partial z}(\nabla^2 \phi) = 0$. So, one equation gives $\frac{\partial^2}{\partial y \partial z}(\nabla^2 \phi) = 0$, another equation gives $\frac{\partial^2}{\partial x \partial z}(\nabla^2 \phi) = 0$. Combining these two equations, we are going to choose a form for $\nabla^2 \phi$ with which this can be satisfied. Both of these can be satisfied. To satisfy both of them, this $\nabla^2 \phi$ can either be a $F_1(x, y)$. If $\nabla^2 \phi$ is only a $F_1(x, y)$, not z, then due to the presence of this $\frac{\partial}{\partial z}$ component, both terms will go to 0.

Alternatively, if the $\nabla^2 \phi$ is a function of z only, independent of x and y, then due to the presence of the ∂y term, the first equation would be satisfied. Due to the presence of the ∂x term, the second equation would be satisfied. Thus, ϕ can either be a $F_1(x, y)$ only, independent of z, or it can be a $F_2(z)$ only, independent of x and y.

So, these are the two possible choices of ϕ. So, in two possible choices of $\nabla^2\phi$, not ϕ. So, this $\nabla^2\phi$ in general can be taken as a superposition or summation of these two possible solution sets. So, f1(x,y), just a function of x and y, plus f2(z), where f2 is just a function of z independent of x and y. So, we are going to choose our $\nabla^2\phi$ as f1(x,y) plus f2(z), with which both of these two equations would be satisfied automatically.

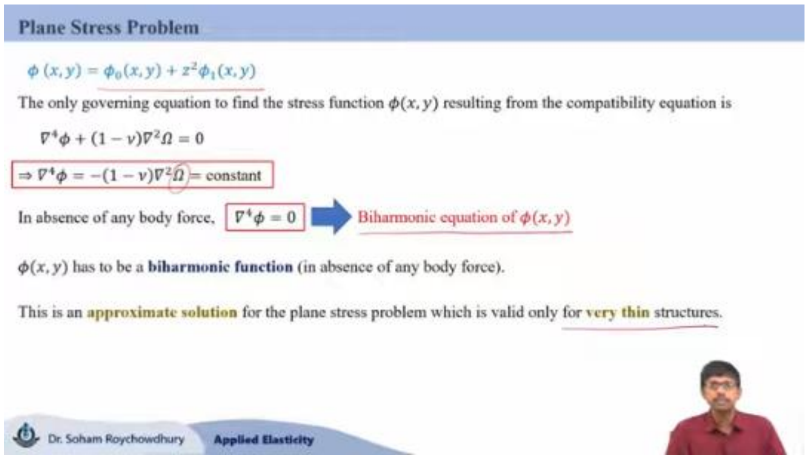


Now, moving to the last strain compatibility equation, setting ε_{xz} and ε_{yz} to 0 and then substituting the expressions of ε_{zz} and ε_{xy}, we would get this form of the equation. So, out of 6 strain-displacement compatibility equations, the first one gave us one basic governing equation, then combining the second and third, we get another equation; combining the fourth and fifth, we get del^2 phi, Laplacian of phi, to be f1(x,y) plus f2(z). Now, replacing that f1(x,y) plus f2(z) form of Laplacian of phi,

in the equation obtained by adding compatibility equation 2 and compatibility equation 3. So, these we got from 4 and 5; this expression is substituted in the expression obtained from compatibility 2 and 3. So, if you substitute and simplify, it would look like this. So, I am taking the f2 term on one side, It would be $\frac{d^2F_2}{dz^2} = \frac{\nu}{(1-\nu)}(\nabla^4\phi + 2\nabla^2\Omega)$.

Now, note that. As we are having $\frac{d^2}{dz^2}$ of the Laplacian of phi, there would be no contribution from the f1(x,y) term because f1 is independent of z. As you are taking the double derivative with respect to z, the contribution from the f1 part would go to 0. This equation would be used for finding f2. Now, if you look at this equation, the left-hand side is a function of z, whereas the right-hand side contains phi and omega.

If you recall, both phi and omega are functions of in-plane coordinates x and y, independent of z. So, in this equation, the left-hand side is a function of z, and the right-hand side is a function of x and y, and they are equal. This is possible only if this particular equation is constant, and thus, both sides are equal to a constant. Let us assume that constant to be 2a, which would make f2(z) equal to a times z squared and since both sides are constant, we must have these two terms, $\nabla^4\phi$ and $\nabla^2\Omega$, to be constant, and also this term, $\nabla^2\varepsilon_{zz}$, to be constant.



Hence, the Laplacian of phi, $\nabla^2\phi$, can be chosen as $F_1(x,y) + Az^2$, where f2(z) is already obtained to be a a quadratic function of z, so Az^2. Now, if the Laplacian of phi is $\nabla^2\phi = F_1(x,y) + Az^2$, we can choose our ϕ to be $\phi_0(x,y) + z^2\phi_1(x,y)$. So, there are two parts of ϕ. So, there are two parts of ϕ.

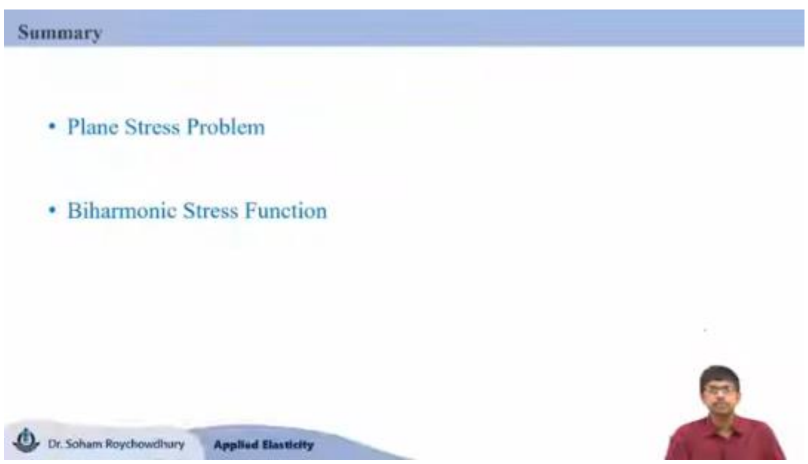
One is independent of z, which is ϕ_0. The other is dependent on z with a quadratic function of z. That part is named ϕ_1. So, if you take the $\nabla^2\phi_0$, the Laplacian of this term would result in F_1, and the Laplacian of this term should result in this constant A. Thus, ϕ_0 and ϕ_1 must be chosen in this form.

Now, this form of ϕ satisfies all the compatibility equations. If you substitute this into the last compatibility equation, the sixth one, it would also automatically be satisfied, as you can check. This is also symmetric about the $z = 0$ plane due to the presence of the z squared term, so the solution would be symmetric about the mid-plane. From this, we can obtain the stress components σ_{xx}, σ_{yy}, σ_{zz}, and τ_{xy}, which have a parabolic variation in z. But we want all these stresses to be constant along the z-axis for the plane stress problem.

So, that can be ensured only if we take the value of this constant A to be 0, and that is possible if the thickness of the body is very small. Thus, in the stresses, the z-dependent terms can be made negligible compared to the z-independent terms by taking a very small value of this constant a, and that is valid only for a very thin continuum where the thickness along the z-direction is small. So, that is why the assumption of thin plates or thin elastic continuum is extremely important for using this plane stress formulation. So, this ϕ , which is $\phi_0 + z^2\phi_1$, for solving this ϕ or choosing ϕ_0 and ϕ_1,

The only governing equation which we have is derived from the first strain compatibility equation as $\nabla^4\phi + (1-\nu)\nabla^2\Omega = 0$. $\nabla^4\phi = -(1-\nu)\nabla^2\Omega$ equals to constant. So, this is the only governing equation from which we can choose the form of ϕ, which would look something like this. By taking the coefficient of the z-square term to be 0, we can approximate this as a plane stress problem. If you do not have any body force, this capital omega term would go to 0, and the equation becomes del^4 phi equals 0 (biharmonic).

The operator acting over the stress function is 0. Thus, the stress function must satisfy the biharmonic condition. This is the condition for any stress function which is required to solve a plane stress problem in the absence of any body force. So, $\phi(x, y)$ must be a biharmonic function. Note that this solution is an approximate solution because we are solving some of the strain compatibility equations approximately with the assumption of a very thin structure. So, we are not obtaining the exact solution for the plane stress problem. We are getting an approximate solution for this particular problem.



So, in this lecture, we had discussed about the assumptions and formulation of plane stress problem and derived that we should be having a biharmonic stress function for solving the plane stress problem in absence of any body force. Thank you.