APPLIED ELASTICITY

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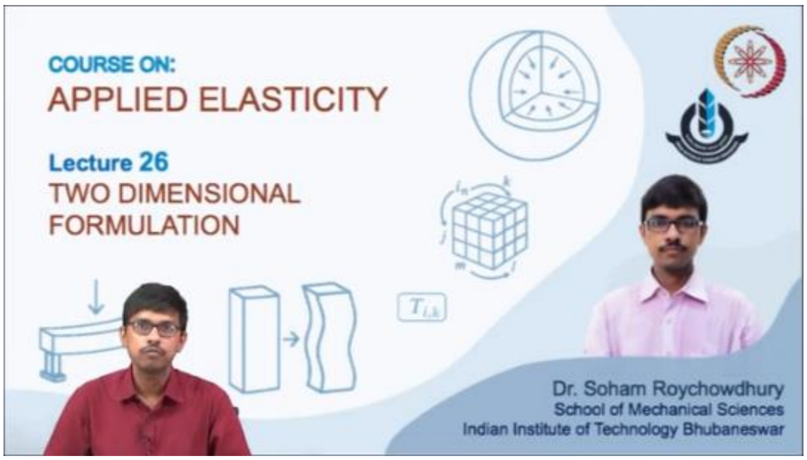
WEEK: 05
Lecture- 26



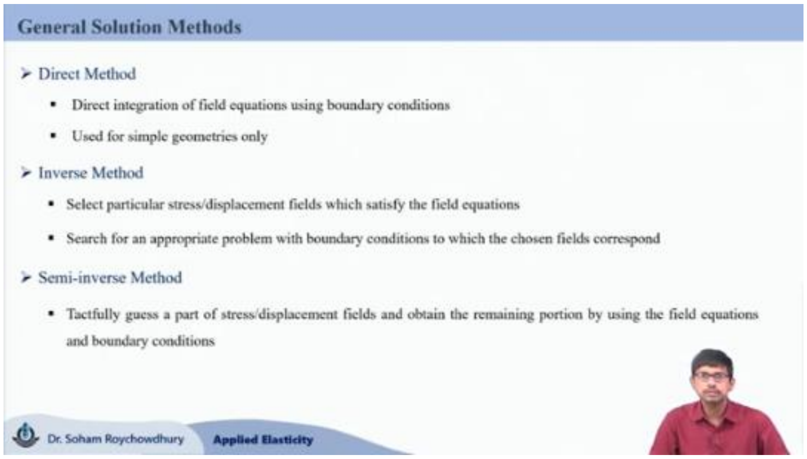
Welcome back to the course on applied elasticity. In today's lecture, we are going to talk about the two-dimensional formulation for elasticity problems.

So, in last week's lectures, we discussed the general field equations of elasticity, which can be used for solving any three-dimensional elasticity problem. We have seen that there exist 15 field equations in any elastic deformation problem in three dimensions. With the assumption of linear or small strain-displacement relationships, and the constitutive relation is taken to be a linear elastic solid.

So, for that, we got three equilibrium equations. $\sigma_{ij,j} + b_i = 0$, where σ_{ij} are the components of the stress tensor, and b_i are the components of the body force vector per unit volume. Now, i and j can take the values of 1, 2, and 3. Here, i is a free index, j is a dummy index for the first term, and thus i being the single free index, we will have three equilibrium equations after expanding this.

Considering the small strain assumption, we also got the strain-displacement relation to be $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$. So, with the assumption of small strain, we are neglecting the non-linear terms or higher-order terms of gradient of u in the strain-displacement relation, and thus we got this linear strain-displacement relation. i and j both are free indices here, and considering the symmetry of the ε_{ij}.

The infinitesimal small linear strain tensor has 6 expressions if you expand this particular strain-displacement relation. And finally, to relate the stress with the strain components, we have 6 constitutive equations which are given as $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ where C_{ijkl} are the components of a fourth-order elastic stiffness tensor. Now, if you assume a linear elastic isotropic solid, then the C_{ijkl} can be written in terms of only two independent nonzero material constants, commonly known as Lame constants. So, these 15 equations need to be solved for any three-dimensional elasticity problem with the assumption of small deformation, and with that, we could get the 15 unknowns, which include 6 stress components, 6 strain components, and 3 displacement components. For solving the equilibrium equation, as it is a differential equation, we need boundary conditions—either stress boundary conditions or displacement boundary conditions.



Now, for many cases, we approximate these 3D problems into 2D problems and then proceed toward solving the equations. There are different solution methods available for solving these elasticity problems or obtaining the solution of the field equations of elasticity. So, we will first discuss the general solution methods available for solving these field equations. The first method is called the direct method of solution, where we directly integrate the field equations, meaning the equilibrium equations.

And using the other two, that is, the constitutive equation and strain-displacement equation, to rewrite all the variables in terms of any one set of variables, either stress or displacement. Based on the stress boundary value problem or displacement boundary value problem. By using the proper boundary conditions which are given. So, by directly integrating the field equation, we should be able to solve the problem. This is called the direct method of solution. However, the problem associated with this method is that it is applicable only for simple geometries.

As we are going to get the closed-form solution by directly integrating the field equation, those are possible to achieve only for simple, standard geometries. If the body has complex geometry, complex boundary conditions, or complex loading, then this method may not work for solving such problems. Now, coming to the next solution method, that is called the inverse solution method. So, in this particular method, first, we select a particular stress field or a particular displacement field which satisfies the field equations of elasticity.

So, based on our intuition, we are choosing a stress or a displacement field, depending on the type of boundary value problem we have. If it is a stress boundary value problem, we may choose the stress field if it is a displacement boundary value problem, we may choose the displacement field. With the chosen fields, the field equations are automatically satisfied. Then, after having these chosen fields which satisfy the field equations,

We now search for an appropriate problem with appropriate boundary conditions to which our chosen fields, chosen stress or displacement fields, correspond. So, it is called the inverse method. Here, we are not starting with the problem. Here, we are starting with the solution.

We are starting with some assumed solution which satisfies the field equation and then try to see for what type of problem with what type of boundary condition the chosen solution fields can be used. So, since we are starting from the solution and then searching for the problem statement, this is called the inverse method of solution. Now, coming to the third one, this is the semi-inverse solution method for elasticity. Here

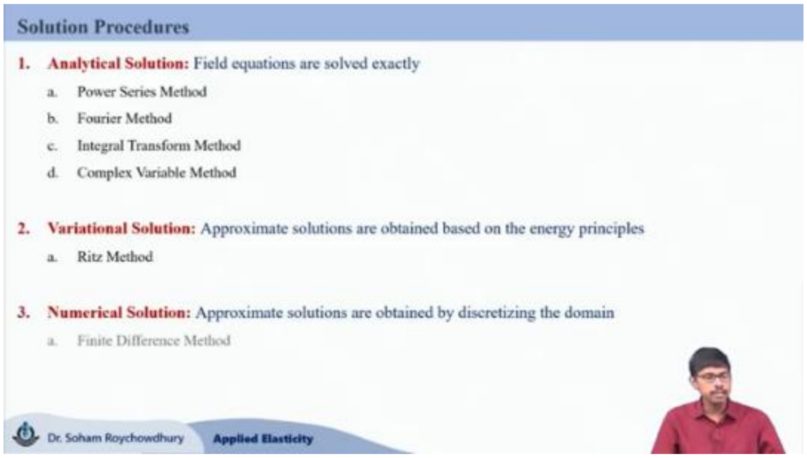
we are guessing some part of the stress or displacement field. So, this has to be done intelligently. So, we have to choose some part of the stress and some part of the displacement field and the remaining part of the stress or displacement field is obtained by using the field equations of elasticity and the boundary conditions of the given problem. So, this is called the semi-inverse method.

So, here what is happening is that instead of completely guessing the displacement field, which we are doing for the inverse method, here in the semi-inverse method, we are partially guessing the solution. But the rest of the part is obtained based on the given field equations and boundary conditions. So, it is not like a blind approach as the inverse method was. So, we are not completely choosing something blindly and then trying to fit it for a specific problem.

Here, we are guessing some part of the solution, and then the remaining part we are trying to search so that it fits with the given problem of deformation of a continuum along with the given boundary conditions. So, these are the three different solution methods available for solving any elasticity problem. Now, coming to the solution techniques or solution procedures. So, the first solution procedure is called the analytical solution procedure, where we are achieving the exact solution of the field equations.

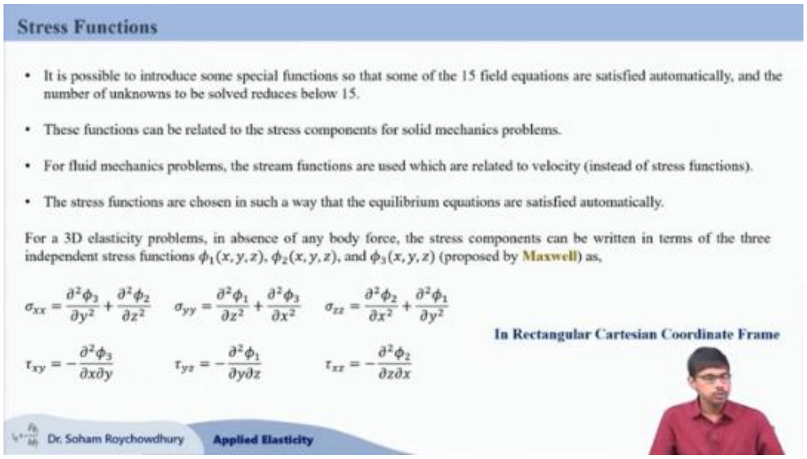
Coming to the second one, this is called the variational solution procedure, where the approximate solutions are obtained based on the variational principles or energy-based principles. So, the energy-based principle, which we had discussed in one of the previous lectures, was the principle of virtual work, which is also known as the principle of minimum potential energy. So, with the help of such principles, applying those for the case of elastic deformation problems.

Using that variational approach, we can get the solution, and those solutions are mostly approximate solutions. Coming to the third type of solution procedure or technique, that is called the numerical solution technique, where the entire domain, the entire continuum, is discretized into small regions, small pieces, and then approximate solutions are obtained for the discretized domain. As you are going for the discretization of the domain, this particular solution technique, the numerical one, is useful if you are going for a solution using computer coding.



Now, starting with the first one, there are different solution procedures or techniques available under the analytical solution scheme. So, those are a few of them: the power series method, Fourier method, integral transform method, and complex variable method. So, these are all using different methods, different tools. Some are using a particular approach, like the complex variable-based approach, whereas the second one is using the Fourier-based approach, the Fourier series-based approach,

and the first one is using the polynomial power series for getting the analytical solution of the problem. For the variational solution method, we are using the Ritz method, where the variational principle is used for finding the equation, and then with Ritz discretization, we can get the approximate solution. And different types of numerical solution techniques are also available, such as the finite difference method, finite element method, and boundary element method, all of which are quite popular. We will move forward to the concept of stress functions.



So, it is possible to introduce some of the special functions. If we are going to solve the problem either using the inverse method or the semi-inverse method. For those two approaches—those two methods—we are going to guess some part of the stress field or displacement field, and for that, the stress function approach is used. So, here, it is possible to introduce some special functions. So, that some of the field equations of elasticity—out of those available 15 equations—are satisfied automatically, and the number of unknowns to be solved reduces below 15. So, that simplifies the overall solution technique. If anyone is going for an analytical solution method with a direct approach,

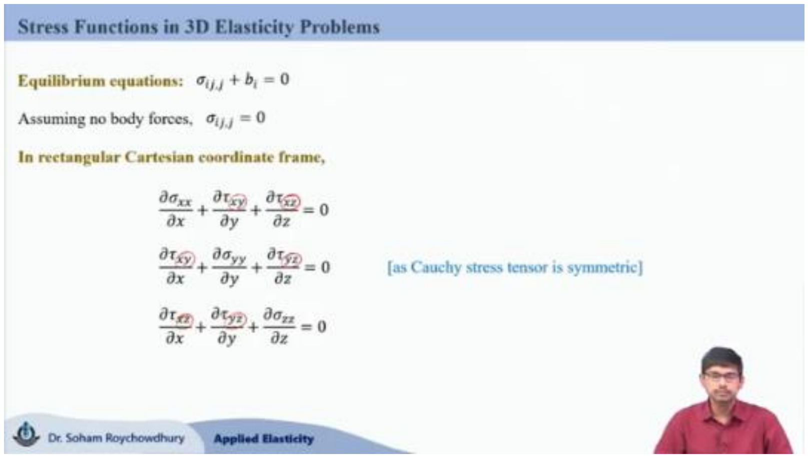
we have to integrate all three equilibrium equations involving the other 12 constitutive and strain-displacement equations and then need to solve them completely to get the exact solution, which is quite hectic and computationally heavy. Thus, it is possible to reduce the number of field equations—the number of unknowns—below 15 by satisfying some of the field equations directly, and this can be done by a proper choice of some special functions. Those functions are called stress functions, which can be related to the stress components for any elastic deformation problem or any solid mechanics problem.

If you are solving a fluid mechanics problem, there are also similar functions that can be assumed. But those are not known as stress functions. For fluid problems, instead of stress functions, we use the stream function. These are related to the velocities. Instead of stresses, we consider the velocity field for the fluid mechanics problem.

And we define stream functions, which are related to the velocity components. So, with the introduction of stress functions for elastostatic problems in solid mechanics or elastic deformation, the equilibrium equations are satisfied automatically. So, we will be left with 12 other equations apart from the 3 equilibrium equations after the introduction of the stress function. Thus, the solution process would be easier or simpler.

Now, for a three-dimensional elasticity problem, if you neglect the presence of body forces or consider a system without body forces, the stress components—six components of the Cauchy stress tensor—can be written in terms of three independent stress functions: ϕ_1, ϕ_2, and ϕ_3. These are three stress functions proposed by Maxwell, where they are all functions of $x, y,$ and z in a 3D elasticity problem described in a rectangular Cartesian coordinate system.

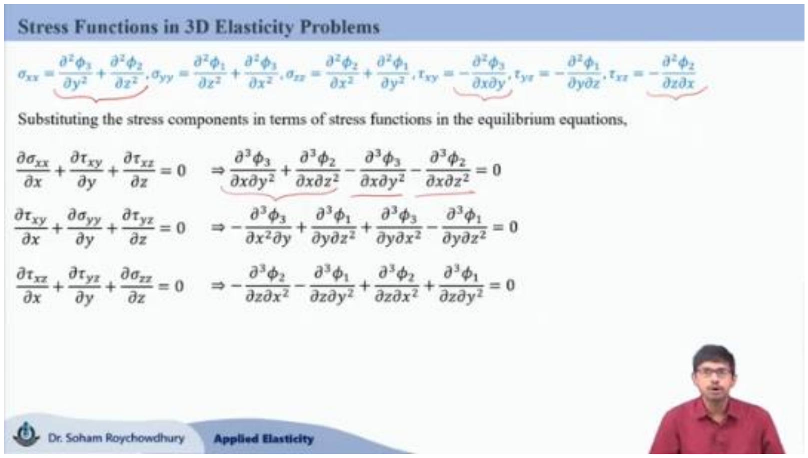
So, we can write the normal stress components as the derivatives of these three stress functions in the given fashion. Where $\sigma_{xx} = \frac{\partial^2 \phi_3}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2}$, $\sigma_{yy} = \frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_3}{\partial x^2}$, and $\sigma_{zz} = \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}$. Now, similarly, the shear stress components can be defined as $\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \partial y}$, $\tau_{yz} = -\frac{\partial^2 \phi_1}{\partial y \partial z}$, and $\tau_{xz} = -\frac{\partial^2 \phi_2}{\partial z \partial x}$. So, these are the definitions of Maxwell's stress functions with respect to the rectangular Cartesian coordinate frame.



And our objective of defining these stress functions is to satisfy the equilibrium equations directly and reduce the number of field equations or the number of unknowns for solving this problem below 15. Now, if you look at the equilibrium equations for the 3D problem, $\sigma_{ij,j} + b_i = 0$, where i and j vary from 1 to 3 for the rectangular Cartesian coordinate system, and i and j would be taking x, y, and z one after another. Now, as we are assuming the problem without any body force, with the assumption of zero body force ($b_i = 0$), we can write this equilibrium equation as $\sigma_{ij,j} = 0$. And in a rectangular Cartesian coordinate frame, i and j taking x, y, and z respectively, we would get the first equilibrium equation explicitly as $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$. Then, the second equation of equilibrium would be $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$. And the third one would be $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$. So, these are the equilibrium equations in the absence of any body force in the Cartesian coordinate system, and we had also assumed the symmetry of the Cauchy stress tensor.

Due to that, both this term (τ_{xy}) and this term (also τ_{xy}) are the same. Instead of yx, I am writing both of them as xy. The same applies for xz and zx, as well as τ_{yz} and τ_{zy}, which are the same. Also, the shear stress components are named with tau here, and the name sigma is preserved only for the normal stress components. We are writing σ_{ij} to represent all the components of the stress tensor here. For the sake of clarity and better understanding, I am using the commonly available nomenclature of shear stress as τ and normal stress as σ.

So, these are the three equations of equilibrium for the rectangular Cartesian coordinate system without any body force.

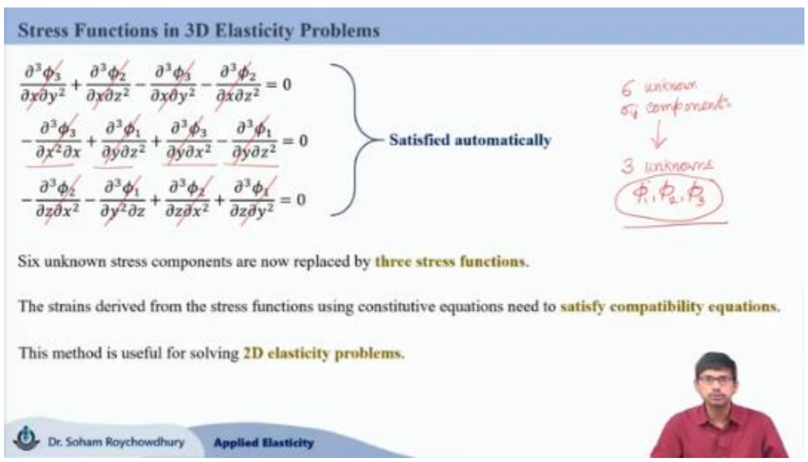


Now, we are going to substitute these stresses in terms of the derivatives of stress functions, as proposed by Maxwell, into these three equations of equilibrium in the Cartesian coordinate system. So, these are the three equations of equilibrium, and at the top, I have written all six stress components: σ_{xx}, σ_{yy}, σ_{zz} (three normal stress components) and τ_{xy}, τ_{yz}, τ_{xz} (three shear stress components) in terms of ϕ_1, ϕ_2, ϕ_3 (three Maxwell stress functions), which are functions of x, y, and z. Now, if you substitute in the first expression, the first term is $\frac{\partial \sigma_{xx}}{\partial x}$, which is $\frac{\partial}{\partial x}$ of this term, resulting in two terms: $\frac{\partial^3 \phi_3}{\partial x \partial y^2} + \frac{\partial^3 \phi_2}{\partial x \partial z^2}$. So, these are the 2 terms coming from $\frac{\partial \sigma_{xx}}{\partial x}$. Now, considering the second term of the first equilibrium equation $\frac{\partial \tau_{xy}}{\partial y}$. So, substituting τ_{xy} as this we would be getting the second term as $-\frac{\partial^3 \phi_3}{\partial x \partial y^2}$.

Similarly, simplifying the third term by substituting τ_{xz} as this, it would be $-\frac{\partial^3 \phi_2}{\partial x \partial z^2}$. So, on the left-hand side, all the stress components σ_{xx}, τ_{xy}, τ_{xz} are substituted in terms of stress functions, and then the respective partial derivatives are taken and then Similarly, we can write the other two equations of equilibrium like this.

So, the left-hand side of all three equations of equilibrium are written as partial derivatives of three stress functions ϕ_1, ϕ_2, and ϕ_3. So, I have rewritten those equations here. This is the first equation. Now, if you carefully check the first term of the left-hand side, it is the same as the third term of the left-hand side.

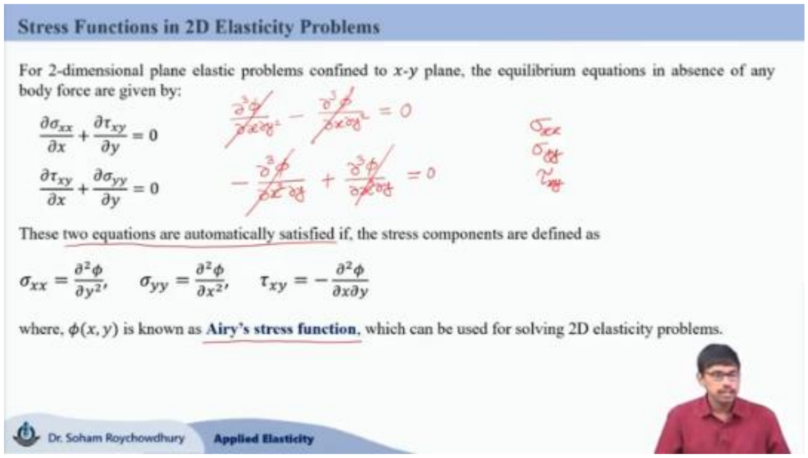
So, they would cancel each other. Also, the second term and fourth term are the same with a negative sign on one of the terms. So, they would also cancel each other. So, the first equilibrium equation is automatically satisfied if we choose σ_{xx}, τ_{xy}, , and τ_{xz} in this particular fashion as functions of ϕ_1, ϕ_2, and ϕ_3. The same exercise can be done for the other two equilibrium equations. In the second equilibrium equation, the first term is the same as the third term with a negative sign in one, and the second term would also cancel with the fourth term. Same for the third equation as well. Here also, the first term would cancel with the third. And the second would cancel with the fourth, so all three equilibrium equations are satisfied automatically. For this particular type of choice of stress components in terms of the Maxwell stress function, ϕ_1, ϕ_2, ϕ_3, thus the six stress components which were earlier there before the introduction of the stress function, are getting replaced with only three unknown stress functions. So, earlier the unknowns were six; six σ_{ij} components were there as unknowns. Now, instead of six, we are having three unknowns. What are they? The three unknowns are phi 1, phi 2, and phi 3. These three stress functions are three unknowns, and three of the equations are directly satisfied. So, the number of unknowns we are reducing by directly satisfying three equilibrium equations. So, with the introduction of the stress



function, the number of unknowns reduces, and some of the equations are automatically satisfied, which helps in the solution method. So, strains derived from these stress functions should also satisfy the compatibility equation, and that can be done with the help of the constitutive equation.

So, whatever stresses we had defined as partial derivatives of these three stress functions using the constitutive equations. Those stress components may be converted into strain components. And those strains must satisfy the strain compatibility equation to ensure the existence of a unique displacement field that we need to ensure. If you are using this stress function approach. Now, this method is extremely useful for solving two-dimensional elasticity problems. In three dimensions, we have three stress functions.

Whereas, for two dimensions, the number of stress functions would be further less, as we will see just after this.



So, this method is extremely useful for solving two-dimensional elasticity problems. And for two-dimensional elasticity problems in the absence of body forces, we have just two equilibrium equations. What are they? So, considering a two-dimensional elasticity problem confined to only the x-y plane.

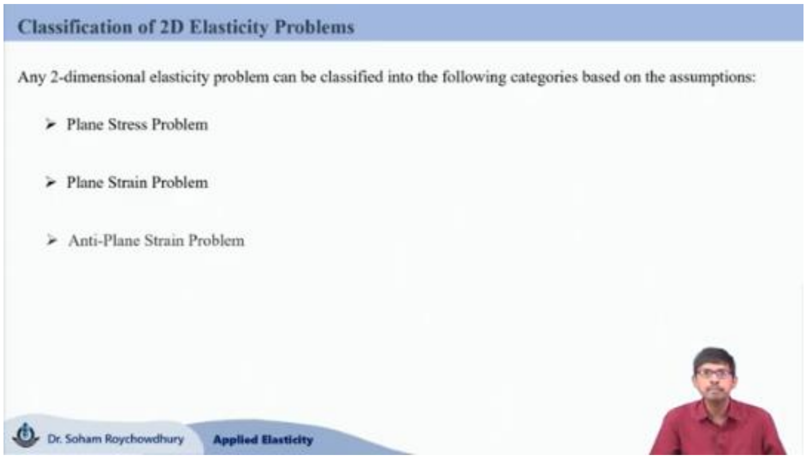
The first two equilibrium equations would be: $\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0$. These are the only two equilibrium equations available for a two-dimensional plane elastic problem confined to the x-y plane. Now, for solving this. Two equations automatically, let us define the stress components, the non-zero stress components. Now, here there are only three non-zero stress components existing: σ_{xx}, σ_{yy} (normal), and τ_{xy} (one shear).

These are the three non-zero stress components for a plane 2D elasticity problem. So, let us define these three stress components as: $\sigma_{xx}=\frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy}=\frac{\partial^2 \phi}{\partial x^2}$, $\tau_{xy}=-\frac{\partial^2 \phi}{\partial x \partial y}$, where ϕ is a function of x and y and that is a stress function. The name of this stress function is Airy's stress function, which is used only for the two-dimensional elasticity problem. For the three-dimensional elasticity problem, we need three stress functions: ϕ_1, ϕ_2, ϕ_3, as proposed by Maxwell.

Whereas, For the two-dimensional elasticity problem, only one stress function is sufficient to define all the non-zero stress components, and that stress function is a function of x and y only, independent of z. If that plane elastic problem is confined to the x-y plane, we call that function $\phi(x,y)$ to be the Airy stress function. Now, let us see with this choice of stress function for the 2D elasticity problem, are you able to directly satisfy the equilibrium equations or not?

So, if you substitute this in the equilibrium equation, you can verify that those would be directly satisfied. So, in the first equation, substituting $\sigma_{xx}=\frac{\partial^2 \phi}{\partial y^2}$, the first equation would be $\frac{\partial^3\phi}{\partial x\partial y^2}$. Then, writing $\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$, the second term would be $-\frac{\partial^3\phi}{\partial x\partial y^2}$, and as these two terms are going to cancel each other, this would directly come out to be 0. Similarly, the second equation can also be shown to be satisfied directly.

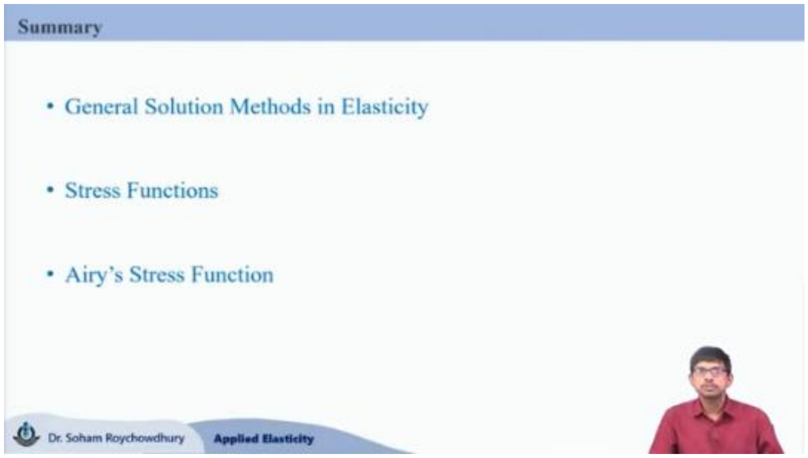
So, both the terms would be $\frac{\partial^3\phi}{\partial x^2\partial y}$, one with a plus sign and one with a minus sign; they would cancel, and it would be automatically satisfied. So, both the equilibrium equations are automatically satisfied for this form of stress components chosen with the help of the Airy stress function, which will directly satisfy the equilibrium equations. And these are widely used for solving different types of two-dimensional elasticity problems.



Now, there are different types of two-dimensional elasticity problems which are basically classified based on the assumptions. So, in any general problem, general elastic deformation problems should be of three dimensions. Now, for special geometries or special types of loadings or special boundary conditions, it is possible to approximate the three-dimensional elasticity problem to a two-dimensional elasticity problem. So, based on such assumptions,

three types of two-dimensional elasticity problems can be classified. So, the first two are most common. The first one is called the plane stress problem, which is valid for thin elastic continua where the thickness is small in one direction; thus, for such cases, we can consider the problem to be a plane stress problem. The next is the plane strain problem, which is valid; this approximation is valid for very large bodies.

So, these two are the most common 2D approximations through which we can reduce a three-dimensional elasticity problem into a 2D planar elasticity problem. And for solving both of them, the Airy stress functions are widely used. We will look into the detailed solution procedure for both plane stress formulation and plane strain formulation in the upcoming lectures. Apart from these two, there is one more formulation which is called the anti-plane strain formulation—anti-plane strain problem—that we will also discuss briefly in the upcoming lectures.



So, in total, in this lecture, we have discussed the general solution methods and different solution approaches for elasticity.

Then, we introduced the concept of stress functions, which are the Maxwell stress function for 3D elasticity problems and the concept of Airy stress functions for 2D elasticity problems. Thank you.