

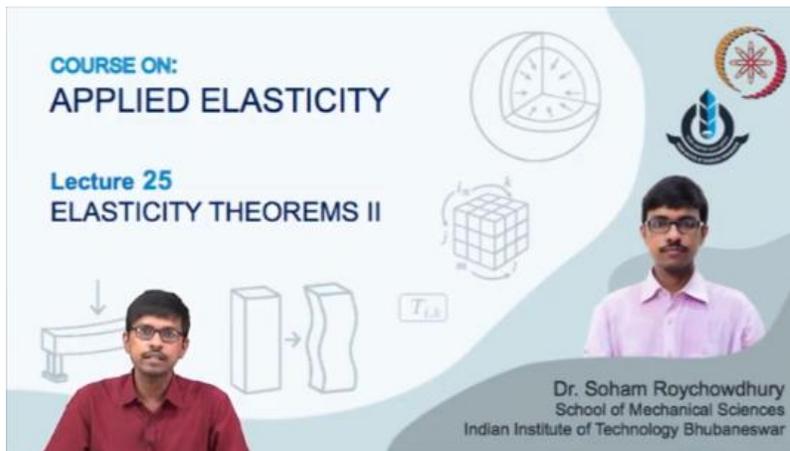
# APPLIED ELASTICITY

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## Lecture 25: Elasticity Theorems II



Welcome back to the course on Applied Elasticity. In this lecture, we are going to continue our discussion on different theorems in the theory of elasticity. In the last lecture, we had started our discussion on various elasticity theorems, and now in this lecture, we will continue with some other theorems of elasticity.



If you recall, in the previous lecture, we had discussed four different theorems of elasticity, starting from the principle of superposition, which is valid for linear infinitesimally small theory of elasticity problems, where if you have two different solutions available from the same continuum under two different forces acting on it, the superimposed solution — the linear superposition of those two solutions — can be one of the solutions of the same problem when both sets of forces are acting on the body simultaneously.

Then after that, we had discussed the Saint-Venant's principle, which states the effect of loading for two different force systems that are statically equivalent to each other, the effects are only confined near the zone of loading, and for points far away from the point of loading, there would be no difference between these two cases. This is known as the Saint-Venant's principle.

Then, coming to the next one, the uniqueness theorem states that for a continuum subjected to given boundary conditions, external forces, body forces, and surface tractions, the stress, strain, and displacement fields can only have one set of unique solutions. This is known as the uniqueness theorem of elasticity.

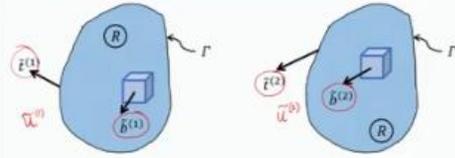
Finally, Clapeyron's theorem states that the total work done by all the forces acting on a continuum divided by 2 is equal to the total strain energy stored within the continuum during this deformation.

Now, we are going to discuss two other theorems of elasticity in today's lecture. The first one is Betti-Rayleigh's reciprocity theorem, and the second one is the principle of virtual work. Sometimes, that is also called the principle of minimum potential.

### Betti-Rayleigh Reciprocity Theorem

For any continuum, the work done by the first system of forces  $\{\tilde{t}^{(1)}, \tilde{b}^{(1)}\}$  acting through displacement  $\tilde{u}^{(2)}$  corresponding to second force system is equal to the work done by the second system of forces  $\{\tilde{t}^{(2)}, \tilde{b}^{(2)}\}$  acting through displacement  $\tilde{u}^{(1)}$  corresponding to first force system.

$$\int_{\Gamma} \tilde{t}^{(1)} \cdot \tilde{u}^{(2)} d\Gamma + \int_V \tilde{b}^{(1)} \cdot \tilde{u}^{(2)} dV = \int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma + \int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV$$



Two equilibrium states of same body under two different force systems  $\{\tilde{t}^{(1)}, \tilde{b}^{(1)}\}$  and  $\{\tilde{t}^{(2)}, \tilde{b}^{(2)}\}$

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First, starting with the reciprocity theorem — Betti-Rayleigh's reciprocity theorem. Here we are considering a continuum which is subjected to two different force systems. One particular force system consists of a set of surface traction and body force. As we are considering two sets of forces or force systems which are acting on the same continuum, and under both of these two force systems, the body is in equilibrium.

So, for this continuum, whenever it is being acted upon by the first force system, there would be some kind of displacement field resulting for this continuum, which we are denoting as  $\tilde{u}^{(1)}$ . So,  $\tilde{t}^{(1)}$  and  $\tilde{b}^{(1)}$  are the first system of forces: surface traction and body force for the first force system, which results in displacement  $\tilde{u}^{(1)}$  in the continuum. Then, the same continuum, the same elastic body, is subjected to a second force system, where  $\tilde{t}^{(2)}$  is the surface traction and  $\tilde{b}^{(2)}$  is the body force, corresponding to the resulting displacement  $\tilde{u}^{(2)}$  for the second case.

Now, this reciprocity theorem states that for the same continuum, the work done by the system of first forces acting through the displacement of the second system (that is,  $\tilde{u}^{(2)}$ ) is equal to the work done by the second force system acting through the displacement of  $\tilde{u}^{(1)}$  corresponding to the first system of forces. So, here the same continuum is considered, and two equilibrium states of the same continuum or same body are shown in this figure, but the forces acting on these two are different.

For the first figure, the surface traction acting on the surface is equal to  $\tilde{t}^{(1)}$ , and the body force acting per unit volume is equal to  $\tilde{b}^{(1)}$ . Whereas, for the second case, the surface

traction acting is  $\tilde{t}^{(2)}$  on the boundary, and the body force per unit volume is  $\tilde{b}^{(2)}$  within the body. Let us say  $\tilde{u}^{(1)}$  is the displacement for the continuum in the first case, and  $\tilde{u}^{(2)}$  is the displacement of the continuum in the second case. Now, as per the statement of this reciprocity theorem, the work done by the first system of forces  $\tilde{t}^{(1)}$  and  $\tilde{b}^{(1)}$  when it is acting upon the second system displacement  $\tilde{u}^{(2)}$ , that would be equal to the work done by the second system of force  $\tilde{t}^{(2)}$  and  $\tilde{b}^{(2)}$  when it is acting with  $\tilde{u}^{(1)}$ , the first displacement of the first force system.

So, basically, we are taking the opposite displacement and taking the force system from another one. We are not taking the corresponding displacement of the individual force system; we are taking the opposite displacement and then finding the work done. If you do so for both the systems, then those two work done quantities should be the same as per this theorem.

So, if you are mathematically expressing this particular statement, it would be like this. The left-hand side terms are the work done by the first force system when they are acting upon by second displacement  $\tilde{u}^{(2)}$ . The first term is a surface integral  $\tilde{t}^{(1)} \cdot \tilde{u}^{(2)}$ . So, work done by all the surface tractions for the first case multiplied with the displacement of the second case integrated over total surface boundary plus volume integral of  $\tilde{b}^{(1)} \cdot \tilde{u}^{(2)}$ . This is the work done by the body forces of first force system multiplied with displacement corresponding to second force system.

These total work done of left hand side should be same as the work done by the second force system acted upon first system displacement  $\tilde{u}^{(1)}$ . First term on the right hand side is  $\int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma$ . This is surface traction work done for second surface traction force multiplied with first displacement, and the last term on right hand side is  $\int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV$ . This is the body force work done that is work done by the second body force multiplied with the first corresponding displacement integrated over total volume. This is the statement or expression for the reciprocity theorem.

### Betti-Rayleigh Reciprocity Theorem

$$\int_{\Gamma} \tilde{t}^{(1)} \cdot \tilde{u}^{(2)} d\Gamma + \int_V \tilde{b}^{(1)} \cdot \tilde{u}^{(2)} dV = \int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma + \int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV$$

Proof:

Work done by  $\{\tilde{t}^{(1)}, \tilde{b}^{(1)}\}$  acting through  $\tilde{u}^{(2)}$  is

$$\begin{aligned} W_{12} &= \int_{\Gamma} \tilde{t}^{(1)} \cdot \tilde{u}^{(2)} d\Gamma + \int_V \tilde{b}^{(1)} \cdot \tilde{u}^{(2)} dV = \int_{\Gamma} \sigma_{ij}^{(1)} n_j^{(1)} u_i^{(2)} d\Gamma + \int_V b_i^{(1)} u_i^{(2)} dV \\ &= \int_V \left( \sigma_{ij}^{(1)} u_{i,j}^{(2)} \right) dV + \int_V b_i^{(1)} u_i^{(2)} dV \quad [\text{using divergence theorem}] \\ &= \int_V \sigma_{ij,j}^{(1)} u_i^{(2)} dV + \int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)} dV + \int_V b_i^{(1)} u_i^{(2)} dV \\ &= \int_V \sigma_{ij,j}^{(1)} u_i^{(2)} dV \quad [\sigma_{ij,j}^{(1)} + b_i^{(1)} = 0 \text{ from equilibrium equations of first state}] \\ &= \int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV \quad [\text{as } \varepsilon_{ij} = u_{i,j}] \end{aligned}$$



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Now, we will try to prove it. So, this was the statement written. For the proof, we are first considering the left hand side - the work done by first force system  $\{\tilde{t}^{(1)}, \tilde{b}^{(1)}\}$  when it is acting through the displacement  $\tilde{u}^{(2)}$ . This work done I am naming as  $W_{12}$ ; so first subscript of  $W_{12}$  is work done for the first force system when it is acting through displacement  $\tilde{u}^{(2)}$ . So, first subscripts refers to the force, and second subscript refers to the displacement in this  $W$  term.

Now,  $W_{12}$  is  $\int_{\Gamma} \tilde{t}^{(1)} \cdot \tilde{u}^{(2)} d\Gamma + \int_V \tilde{b}^{(1)} \cdot \tilde{u}^{(2)} dV$ . First term is the work done by surface traction  $\tilde{t}^{(1)}$ , and second term is work done by body force  $\tilde{b}^{(1)}$ , both due to displacement  $\tilde{u}^{(2)}$  corresponding to second force system. Writing  $\tilde{t}$  as  $\tilde{\sigma}\tilde{n}$  using the relation between the surface traction and the Cauchy stress components, we can write  $\int_{\Gamma} \sigma_{ij}^{(1)} n_j^{(1)} u_i^{(2)} d\Gamma$  as the total first term and second term becomes  $\int_V b_i^{(1)} u_i^{(2)} dV$ . Using the divergence theorem, this surface integral is changed to a volume integral and it becomes  $\int_V \left( \sigma_{ij}^{(1)} u_{i,j}^{(2)} \right) dV + \int_V b_i^{(1)} u_i^{(2)} dV$ .

This comma  $j$  term can be expanded; we can take the derivative once for  $\sigma_{ij}^{(1)}$  term and then another once for  $u_i^{(2)}$  term. First term is obtained by taking the  $x_j$  derivative for  $\sigma_{ij}^{(1)}$ . So, it becomes  $\int_V \sigma_{ij,j}^{(1)} u_i^{(2)} dV$  plus second term is taking the derivative over  $u_i^{(2)}$ . Second term would be  $\int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)} dV$  plus the last term  $\int_V b_i^{(1)} u_i^{(2)} dV$  remains as it is. Using the equilibrium equation for the first state, which is  $\sigma_{ij,j}^{(1)} + b_i^{(1)} = 0$ , using this the

combination of these two terms would result 0. That is from the equilibrium of the continuum when it is subjected to first system of force.

Thus,  $W_{12}$  will be left out with only one non-zero term that is  $\int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)} dV$  integrated over the total volume of the continuum. Now using the relation  $\varepsilon_{ij} = u_{i,j}$ . This we had derived and discussed multiple times in past few lectures, where  $\varepsilon_{ij}$  can be written as  $u_{i,j}$  for the small displacement linear elasticity problem, and also assuming strain tensor to be symmetric, this condition can be proved as we had discussed. Substituting that here,  $W_{12}$  becomes  $\int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV$ .

**Betti-Rayleigh Reciprocity Theorem**

$$\int_{\Gamma} \tilde{t}^{(1)} \cdot \tilde{u}^{(2)} d\Gamma + \int_V \tilde{b}^{(1)} \cdot \tilde{u}^{(2)} dV = \int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma + \int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV$$

Proof:  $W_{12} = \int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV$

Similarly,  $W_{21} = \int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma + \int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV = \int_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} dV$

Now,  $\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \{C_{ijkl} \varepsilon_{kl}^{(1)}\} \varepsilon_{ij}^{(2)} = C_{klij} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)}$  [as  $C_{ijkl} = C_{klij}$  from Major Symmetry]

$$= \{C_{klij} \varepsilon_{ij}^{(2)}\} \varepsilon_{kl}^{(1)} = \sigma_{kl}^{(2)} \varepsilon_{kl}^{(1)}$$

Thus,  $\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}$

$$\Rightarrow \int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV = \int_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} dV \Rightarrow W_{12} = W_{21}$$


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Now, coming to the right hand side of the reciprocity theorem. So, left hand side of the reciprocity theorem is  $W_{12}$  which we had expressed in this form:  $\int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV$ . Coming to the right hand side, in a similar fashion, we define this as  $W_{21}$ , that is work done by the second set of forces acting through the displacement  $\tilde{u}^{(1)}$ , corresponding to the first force system, that is equal to  $\int_{\Gamma} \tilde{t}^{(2)} \cdot \tilde{u}^{(1)} d\Gamma + \int_V \tilde{b}^{(2)} \cdot \tilde{u}^{(1)} dV$ .

Following a similar approach as discussed for the left hand side, this can be shown to be  $\int_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} dV$ . Our objective is to show that  $W_{12}$  and  $W_{21}$ , that is, this expression and this expression are same. If you are able to show that, then this particular Betti-Rayleigh reciprocity theorem would be proved. Let us start with the constitutive equation for proving this.

So, if you are considering this first integrand term within the  $W_{12}$  expression, which is  $\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}$ , we know that for linear elastic solid materials the constitutive relation is  $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ , where  $C_{ijkl}$  are the components of the fourth order elastic stiffness tensor and note that this  $\overset{\infty}{C}$  is material property. It is independent to first force system or second force system; material properties are not dependent on the forcing or boundary conditions of the system. This is always same.

So,  $\sigma_{ij}^{(1)}$  is written here as  $C_{ijkl} \varepsilon_{kl}^{(1)}$  and  $\varepsilon_{ij}^{(2)}$  is also existing. We also know that this  $\overset{\infty}{C}$  matrix or elastic stiffness matrix is a symmetric one, which is showing two types of symmetry: one is called major symmetry of  $C_{ijkl}$  another one is minor symmetry of  $C_{ijkl}$ . Here, we are going to use the major symmetry property of  $C_{ijkl}$  which states  $C_{ijkl} = C_{klij}$ .

The first two and last two set of indices can be flipped together and using that, this  $C_{ijkl}$  is written as  $C_{klij}$ . Combining this  $C_{klij}$  term with  $\varepsilon_{ij}^{(2)}$  term, we can write these two together as  $C_{klij} \varepsilon_{ij}^{(2)}$ , and this is nothing but  $\sigma_{ij}^{(2)}$ . So, if you recall,  $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ . The first two subscripts of  $C$  are the same as the two subscripts of  $\sigma$ , whereas the last two subscripts of  $C$  are the same as the two subscripts of  $\varepsilon$ . From here, the last two subscripts are  $ij$  for  $C$ , they are the same as the subscripts of  $\varepsilon$ . The first two subscripts of  $C_{kl}$  should, thus be present in the subscripts of  $\sigma$ . So,  $C_{klij} \varepsilon_{ij}^{(2)}$  would be  $\sigma_{kl}^{(2)}$ . Thus, we have shown that this  $\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{kl}^{(2)} \varepsilon_{kl}^{(1)}$ .

Now, on the right-hand side, both  $k$  and  $l$  are dummy indices; their names can always be changed to  $ij$ . We are substituting  $k$  with  $i$  and  $l$  with  $j$  on the right-hand side, and with that,  $\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}$  would be equal to  $\sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}$ . With this, we have shown that the integrand present within the expression of  $W_{12}$  and  $W_{21}$  are the same. So, if these two quantities are the same, both of them integrated over the total volume would also be the same.  $W_{12} = W_{21}$ , and with that, we are able to prove the Betti-Rayleigh reciprocity theorem.

### Principle of Virtual Work

The **virtual displacement** is a small fictitious displacement for any material point without changing the forces acting at that point and without disturbing the boundary conditions of the body.

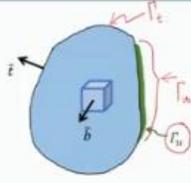
The **virtual work** is the work done by forces due to corresponding virtual displacements.

This principle states that the **total virtual work** done due to all the admissible virtual displacements of a **body in equilibrium is zero**.

$$\delta u_i = \{\delta u_1, \delta u_2, \delta u_3\} \rightarrow \text{Components of virtual displacement vector } \delta \mathbf{u}$$

$$\delta \epsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \rightarrow \text{Virtual strain}$$

Also  $\delta u_i = 0$  along  $\Gamma_u$  as the virtual displacements cannot violate the kinematic displacement boundary conditions.



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Moving to the next theorem, the principle of virtual work. Here, we are considering a continuum which is subjected to body force  $\tilde{b}$  per unit volume within it, and surface traction  $\tilde{t}$  is acting over some portion of the boundary, which is  $\Gamma_t$ . For the remaining portion of the boundary, we have some displacement boundary condition prescribed over  $\Gamma_u$ . So, you can see for this region of the boundary, the displacement is fixed to be 0, and thus, we call this part to have the displacement boundary condition. Whereas, for the remaining part in this region, the surface traction  $\tilde{t}$  is prescribed, where we can say the traction or stress boundary condition exists, that is  $\Gamma_t$ .

Now, the principle of virtual work, first, it involves the definition of virtual displacement. The virtual displacement is defined to be a very small fictitious displacement at any material point, which occurs without any change of the forces acting at that point and without violating the boundary conditions of the continuum. So, at any point within the body, along with the actual displacement, we can add some small fictitious displacement.

That displacement would be known as virtual displacement, but while applying that displacement, we need to take care of two points. The first is that the forces acting at that point are not going to change, and the second point is that while applying this virtual displacement, we are not disturbing or violating any of the boundary conditions of the body. So, if such virtual displacements are possible to exist, which do not violate the boundary conditions of the body, those are called admissible virtual displacements. Those do not violate any of the boundary conditions and, thus, are termed as admissible virtual displacements.

Now, virtual work is defined to be the work done by the forces multiplied by the corresponding virtual displacement. So, at a point, if you assume some virtual displacement, that multiplied by the force acting at that point would be equal to the virtual work done due to that particular virtual displacement. If you are considering multiple virtual displacements at multiple points on the body, then the total virtual work done would be the summation of virtual work done for each of the virtual displacements multiplied by the respective forces acting at those respective points.

Coming to the principle of virtual work, which states that the total virtual work done due to all admissible virtual displacements of a continuum in equilibrium is 0. You can apply  $n$  number of virtual displacements at different points within a body, which are all admissible, meaning they do not violate any of the boundary conditions. Then, if you calculate the total virtual work done on the body and it comes out to be 0, only then is the body in equilibrium. So, if the body is in equilibrium, we must have the total virtual work to be 0, which results from all possible admissible virtual displacements that do not violate the boundary conditions.

Let us say  $\delta u_i$ , where this  $\delta$  operator is used for defining or expressing virtual displacement and virtual work done. So,  $\delta u_i$  is the virtual displacement vector at any point, which has 3 components:  $\delta u_1$ ,  $\delta u_2$ , and  $\delta u_3$ . Now, due to this virtual displacement, some amount of virtual strain will be generated. The actual displacement was resulting in actual strain. If you add this admissible virtual displacement, it would result in some amount of additional virtual strain. The virtual strain due to virtual displacement  $\delta u_i$  can be written as  $\delta \varepsilon_{ij}$ , which is equal to  $\frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$ .

Note that  $\delta u_i$  is equal to 0 along the displacement boundary or  $\Gamma_u$ , as the virtual displacement must be admissible; it cannot violate the kinematic displacement boundary condition. For this reason,  $\Gamma_u$ , where displacement is already specified to be 0, at those set of points, we cannot add any virtual displacement that would cause a violation of the boundary condition. So,  $\delta u_i$  must be 0 over the surface or the boundary where displacement boundary conditions are prescribed.

## Principle of Virtual Work

Proof:

Virtual work done by the body forces and surface tractions can be written as,

$$\begin{aligned} \delta W &= \int_{\Gamma} \tilde{t}_i \delta u_i d\Gamma + \int_V \tilde{b}_i \delta u_i dV = \int_{\Gamma} t_i \delta u_i d\Gamma + \int_V b_i \delta u_i dV \\ \Rightarrow \delta W &= \int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma + \int_V b_i \delta u_i dV = \int_V (\sigma_{ij} \delta u_i)_{,j} dV + \int_V b_i \delta u_i dV \quad [\text{using divergence theorem}] \\ \Rightarrow \delta W &= \int_V \sigma_{ij,j} \delta u_i dV + \int_V \sigma_{ij} \delta u_{i,j} dV + \int_V b_i \delta u_i dV \\ \Rightarrow \delta W &= \int_V \sigma_{ij} \delta u_{i,j} dV \quad (\sigma_{ij,j} + b_i) = 0 \text{ from equilibrium equation} \\ \Rightarrow \delta W &= \int_V \sigma_{ij} \delta \epsilon_{ij} dV \quad [\text{as } \epsilon_{ij} = u_{i,j} \Rightarrow \delta \epsilon_{ij} = \delta u_{i,j}] \end{aligned}$$



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Moving forward to the proof of this principle, the virtual work done by all the forces, that is, body force and surface tractions, can be written as  $\delta W = \int_{\Gamma} \tilde{t}_i \delta u_i d\Gamma + \int_V \tilde{b}_i \delta u_i dV$ . The first term is the work done by the surface traction due to corresponding virtual displacements, and the second term is the work done by the body forces due to corresponding virtual displacements. Writing this in the indicial form, it would be  $\int_{\Gamma} t_i \delta u_i d\Gamma + \int_V b_i \delta u_i dV$ .

Using the relation between surface traction with the Cauchy stress components,  $t_i$  is rewritten as  $\sigma_{ij} n_j$ , and thus the virtual work done,  $\delta W$ , becomes the  $\int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma + \int_V b_i \delta u_i dV$ . Using the divergence theorem by converting this first surface integral into a volume integral, we get the volume integral of  $(\sigma_{ij} \delta u_i)_{,j}$  plus  $b_i \delta u_i$  volume integral.

The first term can be divided by using the chain rule of differentiation. First, we are taking the differentiation of  $\sigma_{ij}$ , and in the next term, we are taking the differentiation of  $u_i$ . Thus, on the right-hand side, we get three terms. Combining the first term and the third term – the  $\sigma_{ij,j}$  term and the  $b_i$  term - these two, if we sum them up, that should be equal to 0 from the equilibrium equation. As the principle of virtual work states, when the body is in equilibrium, the total virtual work done is 0.

We are using that part of the statement: the body is assumed to be in equilibrium, thus this quantity  $\sigma_{ij,j} + b_i$  must be 0. So, the first and third term in combination would go to 0.  $\delta W$  is  $\int_V \sigma_{ij} \delta u_{i,j} dV$ . We already know and have proved this multiple times that

$\varepsilon_{ij} = u_{i,j}$ , and a similar relation must be valid between the virtual displacement and virtual strains, thus  $\delta\varepsilon_{ij} = \delta u_{i,j}$ .

**Principle of Virtual Work**

**Proof:**

$$\delta W = \int_V \sigma_{ij} \delta\varepsilon_{ij} dV$$

Under the assumption of the existence of a strain energy function  $U(\tilde{\varepsilon})$   $\left[ \sigma_{ij} = \frac{\partial U(\tilde{\varepsilon})}{\partial \varepsilon_{ij}} \right]$

$$\delta W = \int_V \sigma_{ij} \delta\varepsilon_{ij} dV = \int_V \frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} dV = \delta \int_V U dV$$

Using the definition of virtual work  $\delta W$ ,  $\left[ \delta W = \int_{\Gamma} t_i u_i d\Gamma + \int_V b_i u_i dV \right]$

$$\int_V \delta U dV - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_V b_i \delta u_i dV = 0$$

$$\Rightarrow \delta(U_T - W) = 0 \quad \Rightarrow \delta\Pi = 0 \quad \text{at equilibrium} \quad \left[ \because U_T = \int_V U dV \right]$$

where total potential energy,  $\Pi = U_T - W = \text{Strain energy} - \text{External work done}$

This is also known as the **Principle of Minimum Potential**.



So, under the assumption of the existence of a strain energy density function, i.e.,  $U$ , strain energy density per unit volume, we can write  $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$ , and with the help of this, the virtual work expression  $\delta W$  can be rewritten as  $\int_V \sigma_{ij} \delta\varepsilon_{ij} dV$ , and replacing  $\sigma_{ij}$  as  $\frac{\partial U}{\partial \varepsilon_{ij}}$ , the total  $\delta W$  would be  $\int_V \frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} dV$ .

Now, we can combine these two terms into a single term of  $\delta U$ . So,  $\frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij}$  can be written as  $\delta U$ , where  $\delta U$  is the variation of the strain energy density function of the body. Using the definition of the virtual work,  $\delta W$ , which is the total virtual work done by the surface traction  $\tilde{t}$  and body force  $\tilde{b}$ , replacing that on the left-hand side, we can rewrite this expression as  $\int_V \delta U dV - \int_{\Gamma} t_i u_i d\Gamma - \int_V b_i u_i dV = 0$ . The first term refers to the change or variation in the total potential strain energy of the system, whereas the next two terms in total give us the overall virtual work done.

This equation can be written as  $\delta U_T - \delta W = 0$ , where  $U_T$  is the total strain energy stored, defined as the integral of the strain energy density over the total volume of the material. Defining  $U_T - W$  as the total potential of the system, denoted as  $\Pi$ , we must have  $\delta\Pi$  to be 0 at equilibrium. This is the statement coming from the principle of virtual work.

So,  $\delta\Pi = 0$  when the body is at equilibrium, where  $\Pi$  is defined as the total potential energy of the system, equal to strain energy minus external work done. So, when the body is in equilibrium, the variation of the total potential is 0, meaning the total potential is reaching a minima. The first variation of the total potential must be 0 when the continuum is in equilibrium, and thus, this principle is also known as the principle of minimum potential energy. The potential energy should be minimum for a system when it is at equilibrium.

**Derivations from the Principle of Virtual Work**

Derivation of equilibrium equations & boundary conditions from the principle of virtual work:

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_V b_i \delta u_i dV = 0$$

$$\Rightarrow \int_V \sigma_{ij} \delta u_{i,j} dV - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_V b_i \delta u_i dV = 0 \quad [\text{as } \epsilon_{ij} = u_{i,j} \Rightarrow \delta \epsilon_{ij} = \delta u_{i,j}]$$

$$\Rightarrow \int_V (\sigma_{ij} \delta u_{i,j}) dV - \int_V \sigma_{i,j,j} \delta u_i dV - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_V b_i \delta u_i dV = 0$$

$$\Rightarrow \int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma - \int_V (\sigma_{i,j,j} + b_i) \delta u_i dV - \int_{\Gamma} t_i \delta u_i d\Gamma = 0 \quad [\text{using divergence theorem}]$$

$$\Rightarrow \int_V (\sigma_{i,j,j} + b_i) \delta u_i dV + \int_{\Gamma} (t_i - \sigma_{ij} n_j) \delta u_i d\Gamma = 0$$


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Using this principle of virtual work, we can derive the equation of equilibrium along with the boundary condition. This is called the variational approach to solving any solid mechanics problem. Starting with the statement of virtual work, which is  $\int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_V b_i \delta u_i dV = 0$ .

Using  $\delta \epsilon_{ij}$  as  $\delta u_{i,j}$ , we are rewriting the first term, then we are writing this first term  $\sigma_{ij} \delta u_{i,j}$  as the summation of two terms. This is  $(\sigma_{ij} \delta u_i)_{,j} - \sigma_{i,j,j} \delta u_i$ . If you combine these two terms, you would get back this. So, the first integral is distributed into two terms, and the third and fourth terms remain as they are.

Now, we are using the divergence theorem for the first term, and thus, it is rewritten as  $\int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma$ . The second term and fourth term are combined together and written as  $\int_V (\sigma_{i,j,j} + b_i) \delta u_i dV - \int_{\Gamma} t_i \delta u_i d\Gamma$ . Once again, the first and last terms are surface integral terms, and those two can once again be combined.

So, there would be two terms: one is the volume integral term, and another is the surface integral term. The volume integral term would be  $\int_V (\sigma_{ij,j} + b_i) \delta u_i dV$  plus the surface integral  $\int_\Gamma (t_i - \sigma_{ij} n_j) \delta u_i d\Gamma$ . This is the expression we get by simplifying the principle of virtual work.

**Derivations from the Principle of Virtual Work**

Derivation of equilibrium equations & boundary conditions from the principle of virtual work:

$$\int_V (\sigma_{ij,j} + b_i) \delta u_i dV + \int_\Gamma (t_i - \sigma_{ij} n_j) \delta u_i d\Gamma = 0$$

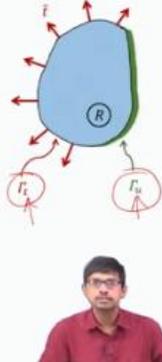
To satisfy the above equation for arbitrary virtual displacement  $\delta u_i$ ,

$$\sigma_{ij,j} + b_i = 0 \text{ over entire } \mathcal{R}$$

and

either  $t_i = \sigma_{ij} n_j$  along  $\Gamma_t$  or  $\delta u_i = 0 \Rightarrow u_i = 0$  (constant) along  $\Gamma_u$

Natural boundary conditions      Essential boundary conditions



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Now, this quantity must be 0 when the body is in equilibrium under the action of different forces and subjected to two types of boundary conditions: displacement boundary condition,  $\Gamma_u$ , and traction boundary condition,  $\Gamma_t$ . To satisfy this above equation for any arbitrary virtual displacement, because virtual displacements are small fictitious displacements, which are arbitrary; it can be any value, but small and admissible to the boundary condition. For any arbitrary  $\delta u_i$ , the first volume integral term can be 0 only if the coefficient of  $\delta u_i$  goes to 0, meaning we must have  $\sigma_{ij,j} + b_i = 0$  over the entire volume, i.e., over the entire domain.

With that, the first integral can be ensured to be 0, and this is nothing but the equilibrium equation of the system. Along with that, we must also satisfy the second integral to be 0. The second integral is defined over the surface boundary, not over the entire domain. So, on the surface boundary, either you should have this term equal to 0, or you should have this term equal to 0 on the surface boundary.  $t_i = \sigma_{ij} n_j$ , which is, the given value of  $\sigma_{ij} n_j$  should be equal to  $t_i$  or surface traction, that is the condition for the surface traction boundary, and along with that, the second part  $\delta u_i$ , that is defined for the displacement boundary.

So, either  $t_i = \sigma_{ij}n_j$  on  $\Gamma_t$ , or we must have  $\delta u_i$  to be 0 on  $\Gamma_u$ .  $\delta u_i = 0$  refers to the case where  $u_i$  is constant, and that constant can be chosen as 0. So,  $u_i = 0$  occurs on the displacement boundary  $\Gamma_u$ . Thus, either we should have this or this, and with this only, we can satisfy the second surface integral going to 0.

The first one is called the natural boundary condition or traction boundary condition, whereas the second one, which is prescribed on displacement, is called the essential or geometric boundary condition. For any continuum deformation problem, it is essential to have at least one essential boundary condition with which the supports are defined. A body without any essential boundary condition means a body without support. It is not possible to solve such problems.

So, essential boundary conditions must be present within the continuum and with this approach, we are able to show both the equilibrium equation as well as the two types of boundary conditions can be obtained using the principle of virtual work. This is called the variational approach for deriving the equation of motion and boundary conditions of any body or continuum.

Summary

- Betti-Rayleigh Reciprocity Theorem
- Principle of Virtual Work

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In this lecture, we discussed two theorems: one is the Betti-Rayleigh reciprocity theorem, and another is the principle of virtual work, which is also known as the principle of minimum potential.

Thank you.