APPLIED ELASTICITY

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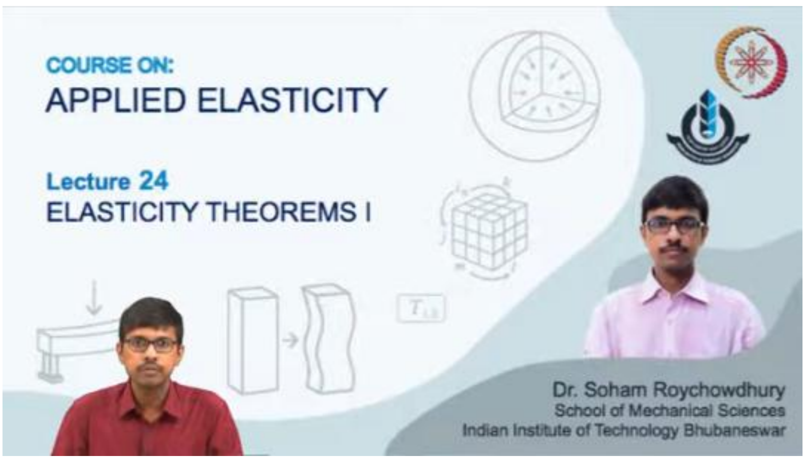
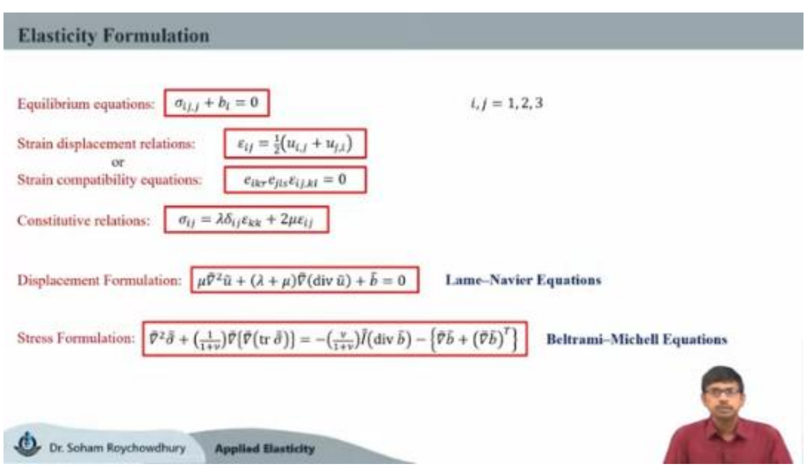
Week 5

Lecture 24: Elasticity Theorems I



Welcome back to the course on Applied Elasticity. In today's lecture, we are going to talk about the different elasticity theorems. In previous lectures, we discussed the field equations of elasticity and then, we covered the solution approach for both displacement boundary value problems and stress boundary value problems in elasticity.



Now, just to have a quick recap of the elasticity formulation for infinitesimally small linear elastic deformation problems. We have 15 field equations of elasticity. The first three are the equilibrium equations. Then we have six strain-displacement relations or six strain compatibility relations.

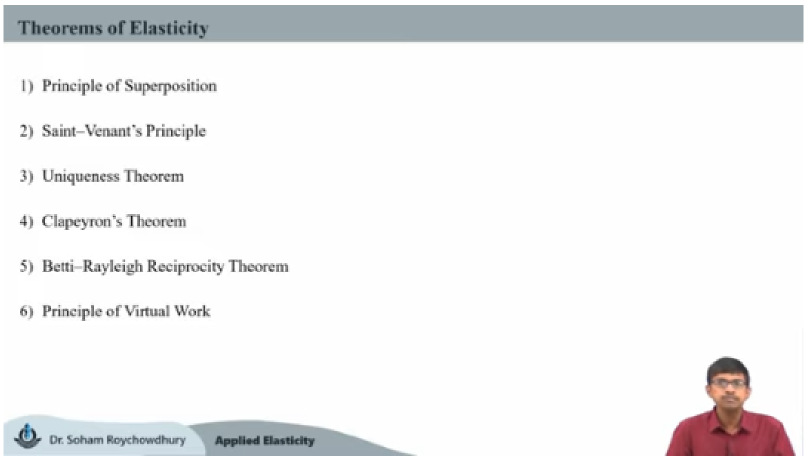
Then we have six constitutive relations. The constitutive relations relate the stress components to the strain components, and this form of constitutive relations is valid only for homogeneous linear elastic isotropic solids involving two material constants: λ and μ. They are known as Lame constants. Now, between strain-displacement and strain compatibility, we need to use either one set of six equations, depending on whether the strain components or displacement components are known and which ones we need to determine. For all these equations, i and j can take values of 1, 2, and 3.

Now, based on the boundary conditions required to solve the equilibrium equations. Because equilibrium equations are differential equations, we must have some boundary conditions for solving the equilibrium equation. Now, depending on the type of boundary conditions, which can be prescribed either on the displacement or on the tractions or stresses. Based on that, boundary conditions are classified into two categories. The first one is the displacement boundary condition, which is prescribed on the surface boundary displacements. And the second type is the traction boundary condition, for which the surface traction values are prescribed on the boundary surface.

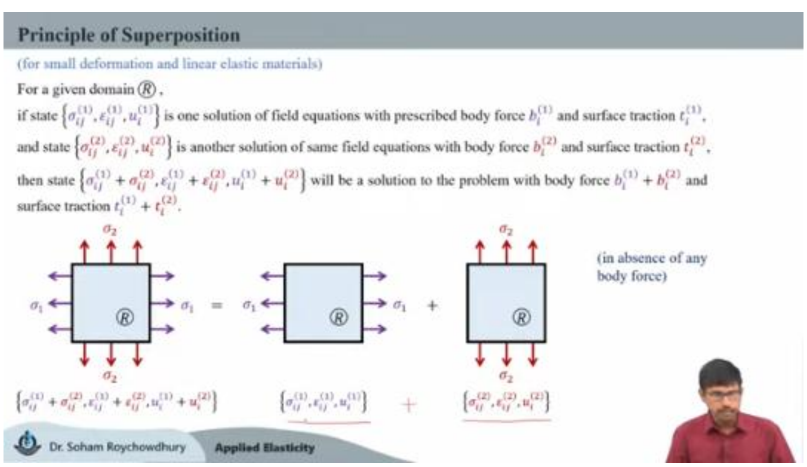
Depending on that, we can have two types of formulations. For the displacement boundary value problems, the expression or the equation is given like this: $\mu\tilde{\nabla}^2\tilde{u} + (\lambda+\mu)\tilde{\nabla}(\mathrm{div}\,\tilde{u}) + \tilde{b} = 0$. This is called the Lame-Navier equation for displacement boundary value problems.

Instead of displacement, if the boundary conditions are prescribed over the stresses or over the tractions, for that case, we have a stress-based formulation of elasticity, and the corresponding governing equation is given by $\tilde{\nabla}^2\tilde{\tilde{\sigma}} + \left(\frac{1}{1+\nu}\right)\tilde{\nabla}\left[\tilde{\nabla}\left(\mathrm{tr}\,\tilde{\tilde{\sigma}}\right)\right] = -\left(\frac{\nu}{1+\nu}\right)\tilde{\tilde{I}}\left(\mathrm{div}\,\tilde{b}\right) - \left[\tilde{\nabla}\tilde{b} + \left(\tilde{\nabla}\tilde{b}\right)^T\right]$, where $\tilde{\tilde{I}}$ is the identity tensor. This equation is called the Beltrami-Michell equation for solving stress boundary value problems of elasticity.

So, in the Lame-Navier equation, we have three equations involving three displacement components, which are used for displacement formulation with the known displacement boundary conditions. Whereas, for the Beltrami-Michell equations, we have six independent equations over six stress components, and those are used for solving the stress boundary value problems with the given traction boundary conditions on the surfaces. Now, while solving any one of these displacement or stress boundary value problems, we need to use various theorems of elasticity.



So, in today's lecture, we are going to start our discussion on different theorems related to elasticity. The first one is called the Principle of Superposition. This would be followed by Saint-Venant's principle. Then the Uniqueness Theorem of elasticity, Clapeyron's Theorem of elasticity, Betti-Rayleigh Reciprocity Theorem, and finally the principle of virtual work. We are going to discuss all these different theorems related to elasticity one by one.



First, let us start with the first one, which is the principle of superposition. This is valid only for small deformation problems and linear elastic solid materials, which are already our assumptions for the infinitesimally small theory of elasticity. So, these assumptions are also valid or are the same for the principle of superposition in elasticity.

Now, for a given domain R or within the total volume of the material, if we consider one solution of the field equations to be $\left\{\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)}\right\}$, then $\sigma_{ij}^{(1)}$is the first set of stress components, $\varepsilon_{ij}^{(1)}$ is the first set of strain components, and $u_i^{(1)}$ is the first set of displacement components, which altogether represent one possible solution of the field equations of elasticity for a problem subjected to prescribed body force $b_i^{(1)}$ and surface traction $t_i^{(1)}$.

Then, we consider another solution, $\left\{\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)}\right\}$, as the second solution of the same problem, same field equations, but with a different body force $b_i^{(2)}$ and a different surface traction $t_i^{(2)}$. So, we take the same body, the same continuum described by the same set of field equations, and then it is subjected to a first set of body force and surface traction as $b_i^{(1)}$and $t_i^{(1)}$, with the corresponding solution obtained as$\left\{\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)}\right\}$. Then, the same continuum is subjected to another set of body force and surface traction, $b_i^{(2)}$ and $t_i^{(2)}$, with the obtained solutions named as $\left\{\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)}\right\}$.

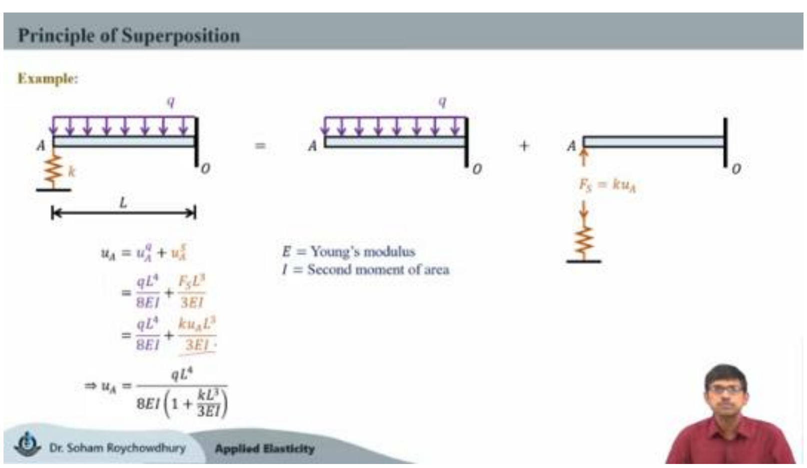
Now, the principle of superposition states that if you sum up these two solutions, then $\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$ is the resultant stress, and $\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$ is the resultant strain, and $u_i^{(1)} + u_i^{(2)}$ is the total displacement. This one, the summation of these two individual solutions would be another solution of the problem if the combined body force $b_i^{(1)} + b_i^{(2)}$ and combined surface traction $t_i^{(1)} + t_i^{(2)}$ are acting on the same continuum. Let us try to explain that with the help of one example.

First, we are considering this continuum which is a square subjected to σ_1, that is, stress along the 1 direction. 1 is horizontal direction, 2 is vertical direction let us say. So, this is subjected to one particular surface traction. $t_i^{(1)}$ is specified here as σ_1 on left and right faces. And when this body is subjected to this particular surface traction, the corresponding solution is obtained as let us say $\left\{\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)}\right\}$. We are not considering any body force; this is in absence of body force.

Now, let us consider the second type of loading on the same body. That is the vertical loading, σ_2. Surface tractions are applied on the top face and bottom face of σ_2 intensity acting on different surface boundaries. That is basically $t_i^{(2)}$ acting on the same continuum and let us say the resultant stress, strain, and displacements are obtained as $\left\{\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)}\right\}$.

Now, based on the principle of superposition, if you add these two loadings — so if you are having a body subjected to σ_1 along the x-direction and σ_2 along the y-direction at that same body with the same dimensions — the solution of the field equations of that problem should be the same as the summation of these two individual solutions. So, if you add these two, considering the body where both the surface tractions $t_1 + t_2$ are acting, then, the solution of stress would be $\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$, the solution of strain would be $\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$, and the displacement would be $u_i^{(1)} + u_i^{(2)}$.

This is the principle of superposition, and this is valid for small deformation, meaning the stress-strain-displacement relation being linear and for a linear elastic solid, meaning the constitutive equation being linear. Then only this principle of linear superposition holds true.



Now, taking one example problem, let us consider one cantilever beam fixed at one endpoint O and connected to a vertical spring of stiffness k at another endpoint A. The total length of the beam is taken to be L, and it is subjected to a uniformly distributed loading of intensity q. Using the principle of superposition, we can write this problem as the summation of two other problems where two loads are acting differently or we are having only one load acting on the body for each of those two individual problems.

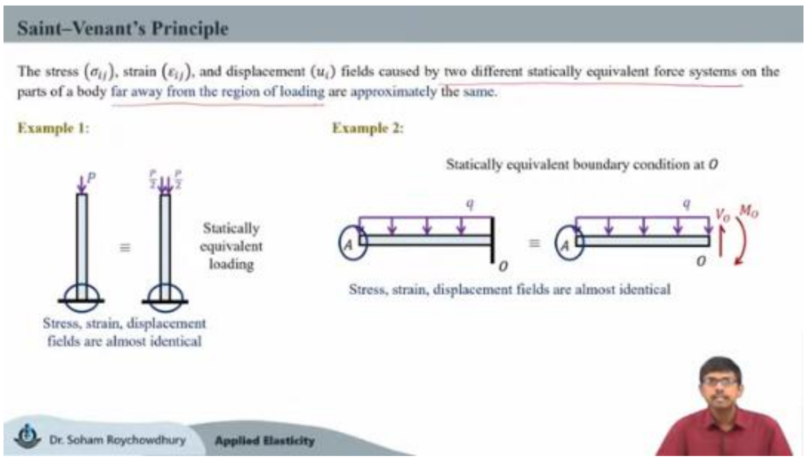
So, if you carefully observe, there are two types of external forces acting on this beam. One is the external distributed loading of intensity q, which is uniformly distributed throughout its length L. Another is the spring force coming at point A, which is equal to k times the displacement of point A, ku_A, let us say. So, we are considering the first problem where the same beam is subjected to only one loading, that is the distributed load of intensity q, plus the second problem where the same beam is subjected to only the spring force F_S, which is ku_A, where u_A is the deflection of the spring or vertical transverse deflection of that free endpoint A of the cantilever beam.

Now, using the principle of superposition, the total deflection of point A would be the summation of the individual deflections of these two different problems. In one case, it is the deflection due to the uniformly distributed loading of intensity q, and in another case, it is the displacement of the tip due to a concentrated load equal to the spring force of the system. So, u_A would be u_A due to q plus u_A due to the spring force F_S.

Now, E being the Young's modulus of the material of the beam and I being the second moment of area, we can write u_A due to the uniformly distributed loading q as $\frac{qL^4}{8EI}$. This is known to you from the elementary undergraduate beam deformation problems. So, any cantilever beam subjected to a uniformly distributed load of intensity q throughout its span will have a deflection of $\frac{qL^4}{8EI}$, whereas any cantilever subjected to a tip load P at the free end would have a tip displacement of $\frac{PL^3}{3EI}$.

Here, for the second case, that tip load P is nothing but the spring force F_S. So, u_A for the spring case is $\frac{F_S L^3}{3EI}$. F_S is once again related to the displacement u_A. So, F_S can be written as ku_A, the spring constant times the displacement of the spring, or the compression of the spring. Now, $u_A = \frac{qL^4}{8EI} + \frac{ku_A L^3}{3EI}$. Taking the second term to the left, and then solving for u_A, the displacement of point A can be obtained as $\frac{qL^4}{8EI\left(1+\frac{kL^3}{3EI}\right)}$.

So, using the principle of superposition, we can conveniently solve such problems where multiple loads are acting. We will be solving the same problem using individual forces and then combine the results of all those individual force cases to find out the total solution, where all those loads are acting simultaneously on the same continuum.



Now, moving to the next theorem, which is the Saint-Venant principle of elasticity. This states that the stress σ_{ij}, strain ε_{ij}, and displacement u_i, all these fields caused by two statically equivalent force systems, acting on parts of a body which are far away from the region of interest or region of loading, are approximately the same. We will explain this particular statement through examples.

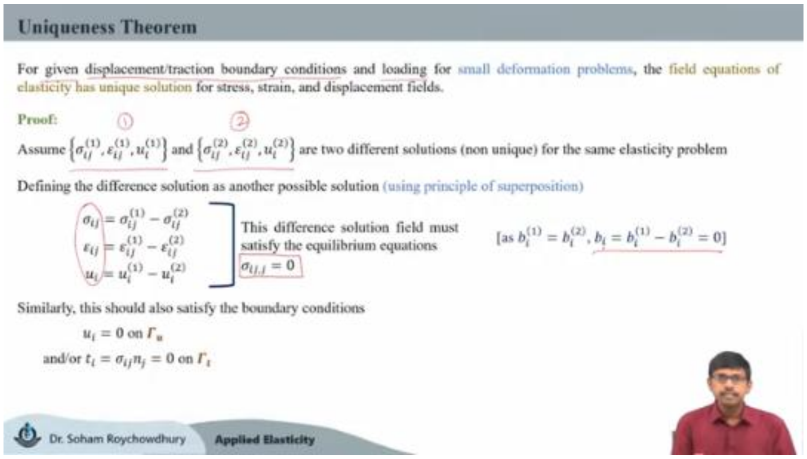
Let us consider a thin bar subjected to axial compressive loading P, at one of its ends, and the other end is a built-in end or cantilevered. Here, instead of P at this particular top point, let us apply $\frac{P}{2}$ and $\frac{P}{2}$, two different forces of half the magnitude of P at two adjacent points. So, instead of applying a single force P, we can apply $\frac{P}{2}$ and $\frac{P}{2}$ forces on two sides in a symmetric fashion so that no moment is created about the central vertical line about the central axis of the bar. So, moment is same and both $\frac{P}{2}$ and $\frac{P}{2}$ magnitude forces are just a very small distance away from each other. Then, these two force systems can be called equivalent force system; their effects must be same at the point of loading.

Under such cases, when two different statically equivalent force systems are acting on the body, then if you consider any region which is far away from the region of loading, there we cannot distinguish any effect or difference between these two cases even though the loads are different but statically equivalent. Then, if you consider any point which is far away from the point of loading - let us say at the fixed support end of the bar, at the bottom point - the equivalent forces are acting on the top and we are taking extremely different endpoints, the opposite endpoints, that is at the base, we would not be able to distinguish between the results at those two base points for two systems with statically equivalent loading. So, stress, strain, and displacement fields would be almost identical at the base of both the systems, when that base point is far away from the point of loading. This is called the Saint-Venant's principle.

If you take another example, let us take a cantilever beam subjected to uniformly distributed load of intensity q, fixed end is O, and free end is named as A. Now, we are removing this fixed end O and replacing that with equivalent force V_0 and equivalent moment M_0. This is a statically equivalent boundary condition at fixed end point O. V_0 and M_0 are so calculated that this first one, the given boundary condition is equivalent to the second free body diagram. These are called statically equivalent boundary condition at right end point O for the beam, which is loaded with uniform load distribution of intensity q.

Now, if I consider point A, which is far away from point O, exactly at the opposite end of the beam, the stress, strain, displacement fields would be identical for both of these two cases with statically equivalent boundary conditions at O. So, for the Saint-Venant's principle, it is basically stating that the effect of loading for two systems which are equivalent to each other. So, for two equivalent force systems acting on a body, their effect would only be confined to the region near the point of loading.

As you are going to the points which are away from that region, no difference can be felt in the stress, strain, or displacement fields for the two cases which are having statically equivalent loading.



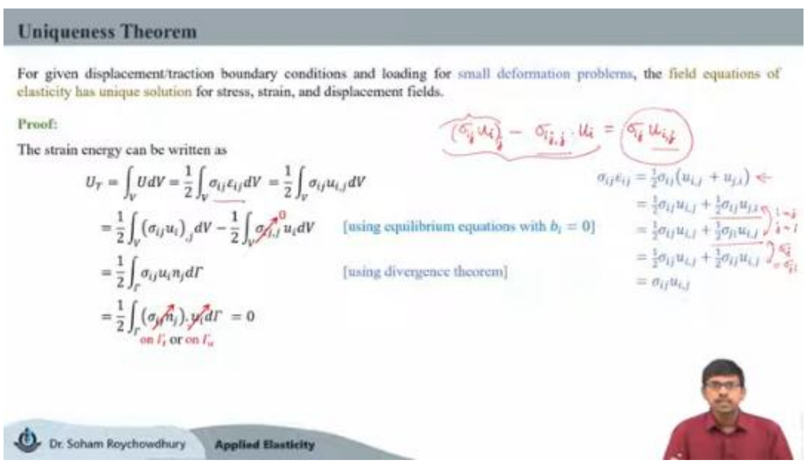
Now moving to the next theorem which is called uniqueness theorem. This is saying that for given traction or displacement boundary condition and loading for the small deformation problems, the field equations of elasticity can have only a single unique solution for stress, strain, and displacement fields. So, for any elasticity problem with a given displacement and traction boundary condition along with the given loading, with the assumption of small deformation, the field equations can have unique or single solution. We cannot have multiple solutions available for the field equations of elasticity. Now, we will try to prove that.

First, we will violate the statement of the uniqueness theorem. Let us assume non-unique solutions are existing. Two different set of solutions are chosen. First set is $\left\{\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)}\right\}$. Second set is $\left\{\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)}\right\}$. These are the two possible set of solutions for stress, strain, and displacements for the same elasticity problem.

Defining their difference as another possible solution using the principle of superposition, which states if this is solution 1, and this is solution 2, then their linear combination is another solution - this is the statement of principle of superposition. Let us say difference between these two solutions is another solution of the same problem. So, $\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $\varepsilon_{ij} = \varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)}$, $u_i = u_i^{(1)} - u_i^{(2)}$. This difference solution is defined to be another possible solution of the same problem and for this case, b_i would be equal to 0 because the body force acting in the first case and the body force acting in the second case must be the same, as that is given.

You look at the statement for the given displacement reaction boundary and loading. The loading means body forces, surface tractions, and boundary conditions over Γ_u and Γ_t should be the same for both problems 1 and 2. Thus, for the new state, the difference solution, b_i should be 0, as $b_i^{(1)} = b_i^{(2)}$. b_i being 0, this new difference solution must satisfy the boundary condition for the equilibrium equation of this form: $\sigma_{ij,j} = 0$.

In general, the equilibrium equation is $\sigma_{ij,j} + b_i = 0$, but as $b_i = 0$ here, the equilibrium equation reduces to the form of $\sigma_{ij,j} = 0$. Similarly, the difference solution should also satisfy the boundary conditions, which are $u_i = 0$ on Γ_u (the displacement boundary condition) and t_i equals some given $\sigma_{ij} n_j = 0$ on Γ_t (the traction boundary conditions).



Now, using the definition of the strain energy function, the total strain energy of the system can be written as the volume integral of the strain energy density, and the strain energy density is defined as half of the inner product of $\tilde{\tilde{\sigma}}$ and $\tilde{\tilde{\varepsilon}}$. Half of the integral of $\sigma_{ij}\varepsilon_{ij}$ over the entire volume is the total strain energy of the system. We can write $\sigma_{ij}\varepsilon_{ij}$ as $\frac{1}{2}\sigma_{ij}\left(u_{i,j} + u_{j,i}\right)$, using the strain-displacement relation.

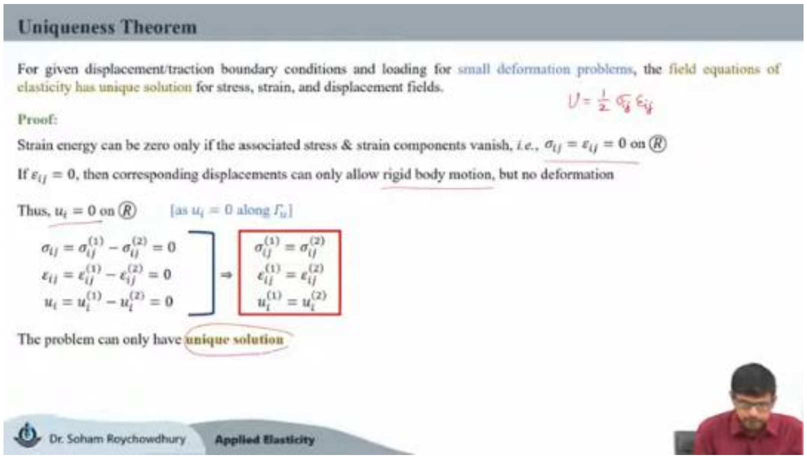
Writing ε_{ij} in terms of the displacement component, we can derive this particular equation. Expanding it into two terms and then, in the second term, interchanging the dummy indices i and j, we would get this. In the second term, we are interchanging i to j and j to i, only in the second term.

Finally, between this and this, we are using the symmetry of the Cauchy stress tensor, $\sigma_{ij} = \sigma_{ji}$. With that, we can show that $\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}u_{i,j}$. That is replaced here in the expression of strain energy. So, U_T would become $\frac{1}{2}\int_V \sigma_{ij}u_{i,j}dV$.

Now, writing this $\sigma_{ij}u_{i,j}$ as $\left(\sigma_{ij}u_i\right)_{,j} - \sigma_{ij,j}u_i$ because if you take the derivative of $\sigma_{ij}u_i$, this can have two terms. One term is the derivative of σ with respect to x_j keeping u_i constant. Another term is the derivative of u_i with respect to x_j keeping σ_{ij} constant. Thus, this one term $\sigma_{ij}u_{i,j}$ can be written as the difference between these two terms.

So, two volume integrals are coming, and using the equilibrium equation, with zero body force, for the given solution, body force b_i is zero, and the equilibrium equation is $\sigma_{ij,j} = 0$. So, the second term would go to zero, and we would have only one term. Using the divergence theorem, $\int_V \sigma_{ij}u_{i,j}dV$ can be written as $\int_\Gamma \sigma_{ij}u_i n_j d\Gamma$. Then, grouping the terms within the integral into two parts, we can write the total strain energy as $\frac{1}{2}\int_\Gamma \left(\sigma_{ij}n_j\right). u_i d\Gamma$.

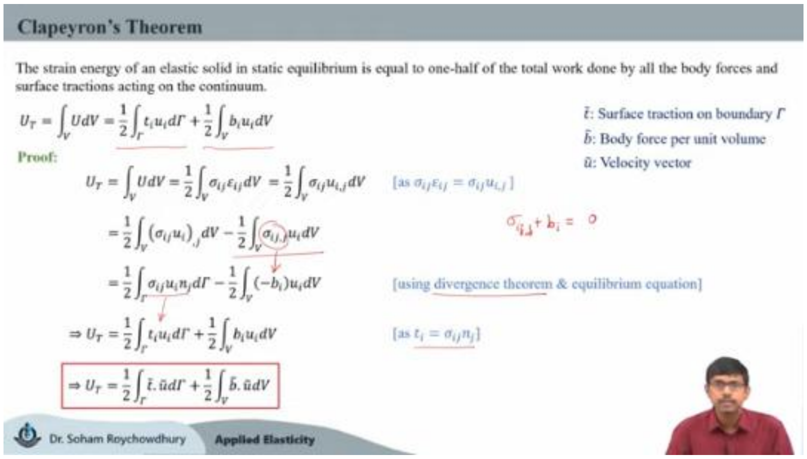
Now, if you recall the boundary conditions, on the boundaries, we have $\sigma_{ij}n_j$ to be 0, if the traction or stress boundaries are prescribed, that is on Γ_t. For the displacement boundaries, if Γ_u is prescribed, we have u_i to be 0. So, on Γ_t, the first term is 0, $\sigma_{ij}n_j$ is 0; on Γ_u, the second term u_i is 0. Whatever boundary conditions are specified, whether it is a stress boundary or displacement boundary, either one of the terms would go to 0. Thus, overall U_T would go to 0. The total strain energy for this difference solution state is equal to 0.



Now, if the total strain energy of this state is 0, the associated stress and strain components must vanish because $U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$. U being 0, both σ_{ij} and ε_{ij} components must go to 0. So, both σ_{ij} and ε_{ij} being 0 on the entire domain and using the strain-displacement relations, ε_{ij} being 0, no strains are allowed within the body. The body can only have some rigid motion without any elastic deformation.

Now, if you use the displacement boundary conditions at few points on the boundary, displacement boundary conditions are prescribed on Γ_u where u_i is 0. So, u_i being 0 at the boundaries and the body having only rigid body motion, we must have u_i to be 0. We cannot have different values of u_i within the body; along the boundary, u_i is 0, then for the entire body, displacement u_i must be 0. Hence, strain energy 0 refers to stress, strain, and displacement, all of them to be 0 within the entire boundary. So, σ_{ij}, ε_{ij}, u_i all are 0 means the difference solution is going to 0, thereby, resulting $\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)}$, $\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(2)}$, $u_i^{(1)} = u_i^{(2)}$.

That means the two states of solutions, whatever we had assumed to be non-unique, they are coming out to be the same. So, the elasticity problem or the solution of field equations of elasticity can have a single or unique solution, which is the statement of the uniqueness theorem, and thus it is proved.



Moving to the next theorem, Clapeyron's theorem, this states the strain energy stored within an elastic body which is in static equilibrium equals half of the summation of the total work done by all the body forces and surface tractions acting on the continuum.

Let us consider a body which is subjected to body force $\tilde{b}$ per unit volume, subjected to surface traction $\tilde{t}$ acting over boundary Γ, and velocity vector equal to $\tilde{u}$. The work done by body forces can be written as this last term $\frac{1}{2}\int_V b_i u_i dV$. Work done by the surface traction force is written as this term $\frac{1}{2}\int_\Gamma t_i u_i d\Gamma$ because traction surface tractions are acting over the boundary, and body forces are acting over the entire volume.

The right-hand side gives us half of the total work done by surface tractions and body forces. As per the statement of Clapeyron's theorem, this should be equal to the total strain energy stored within the body, that is the integral of U over the volume or U_T, the total strain energy stored. Now, if you try to prove this, U_T, total strain energy, is the integral of the strain energy density function, $\frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij} dV$. As we had already proved, $\sigma_{ij}\varepsilon_{ij}$ is the same as $\sigma_{ij}u_{i,j}$.

Replacing that here within the integrand and then writing $\sigma_{ij}u_{i,j}$ as $\left(\sigma_{ij}u_i\right)_{,j} - \sigma_{ij,j}u_i$, same as the previous proof of the uniqueness theorem, and then invoking the equations of equilibrium, the second integral term, would become like this because, using the equilibrium equation, we know that $\sigma_{ij,j} + b_i = 0$.

Thus, $\sigma_{ij,j}$ is written as minus b_i, and in the first term, which was a volume integral, $\frac{1}{2}\int_V (\sigma_{ij}u_i)_{,j} dV$, is converted into a surface integral with the help of the divergence theorem and written as $\frac{1}{2}\int_\Gamma \sigma_{ij}u_i n_j d\Gamma$, where n_j are the components of the unit surface normal vector. Now, $U_T = \frac{1}{2}\int_\Gamma t_i u_i d\Gamma + \frac{1}{2}\int_V b_i u_i dV$ by using the relation between the surface traction vector t_i and the stress component σ_{ij}, that is, $t_i = \sigma_{ij}n_j$.

That is replaced here, and thus, this becomes $\frac{1}{2}\int_\Gamma t_i u_i d\Gamma + \frac{1}{2}\int_V b_i u_i dV$, and this is equal to the total strain energy of the system. This is nothing but the statement of Clapeyron's theorem. In the vector form, we can write U_T as $\frac{1}{2}\int_\Gamma \tilde{t}.\tilde{u} d\Gamma + \frac{1}{2}\int_V \tilde{b}.\tilde{u} dV$.

In this lecture, we discussed four different theorems of elasticity known as the Principle of Superposition, Saint-Venant's principle, the Uniqueness Theorem, and the Clapeyron's Theorem.

Thank you.