APPLIED ELASTICITY

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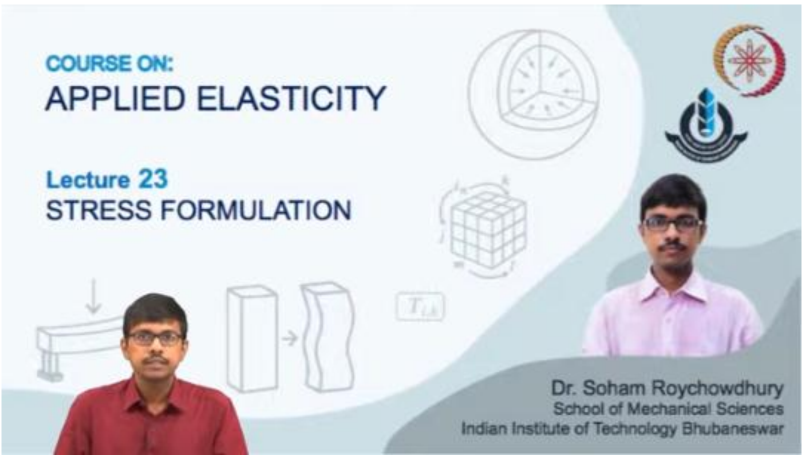
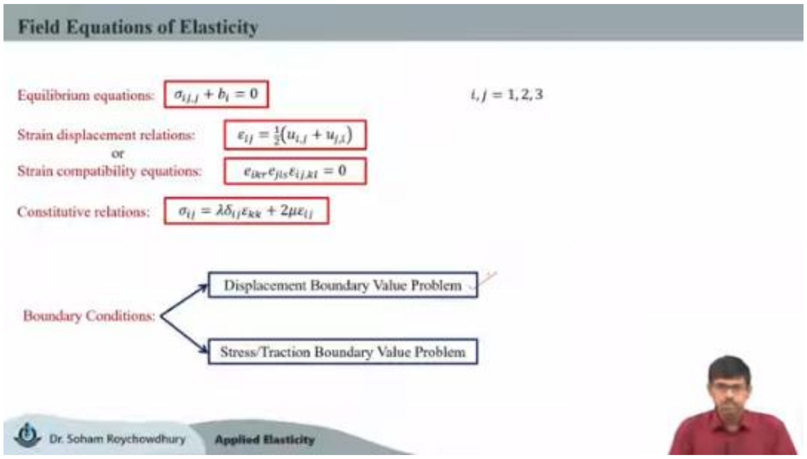
Week 5

Lecture 23: Stress Formulation



Welcome back to the course on Applied Elasticity. In today's lecture, we are going to talk about the stress formulation of the theory of elasticity. In the previous lecture, we talked about the displacement formulation of elasticity, which is applicable for displacement boundary value problems. Today, we are going to discuss the solution method for stress boundary value problems in elasticity.



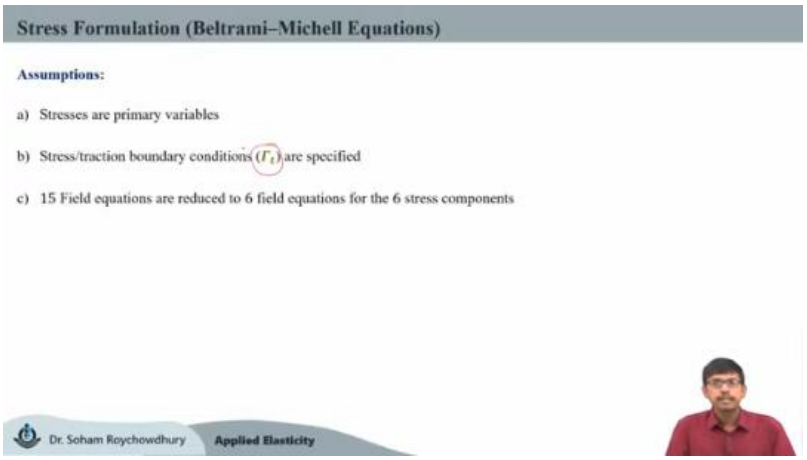
Just to have a quick recap of the field equations of elasticity, we have a total of 15 field equations in elasticity. The first three are equilibrium equations: $\sigma_{ij,j} + b_i = 0$, with i, j, k having values of 1, 2, and 3, and b_i are the components of the body force. Next is the strain-displacement equation, which relates 6 strain components to 3 displacement components and is given by $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$. There are 6 strain-displacement relations.

For some cases, to find the unique displacement fields from given known strain fields, we need to use strain compatibility equations, which are given as $e_{ikr}e_{jls}\varepsilon_{ij,kl} = 0$. Finally, we have six constitutive equations relating stress components and strain components. These are given as $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$, where λ and μ are known as Lame constants. This particular constitutive equation is valid for linear elastic isotropic solids only.

Now, with all these 15 equations of elasticity, we can solve for the 15 unknowns: 6 stress components, 6 strain components, and 3 displacement components. But since the equilibrium equations are differential equations, we need boundary conditions to solve them. Based on the boundary conditions, elasticity problems can be classified into two types of boundary value problems.

If the boundary conditions are prescribed over the displacements, we call those problems displacement boundary value problems, with boundary conditions Γ_u. The displacements are prescribed over the surface boundary Γ. The second type of boundary value problem is the stress boundary value problem, also known as the traction boundary value problem, where on the boundary surface Gamma, the known traction components ($\tilde{t}$) are specified. These are called Γ_t type of boundary conditions.

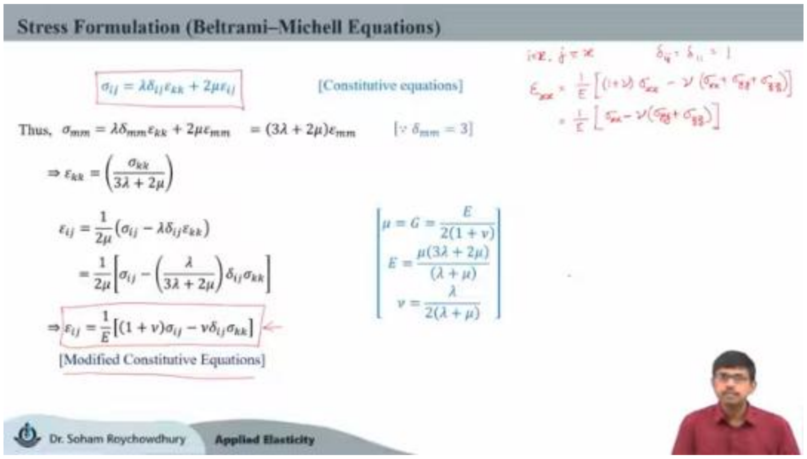
So, in the previous lecture, we had already discussed the solution method for the displacement boundary value problem, which is known as the Lame-Navier formulation or Lame-Navier equations of elasticity. Now, in today's lecture, we are going to talk about the solution method for the stress boundary value problem.



First, starting with the assumptions of the stress boundary value problem formulation. Here, the stress components are the primary variables. For Cauchy stress resultant being symmetric, we have 6 independent stress components in $\tilde{\tilde{\sigma}}$ or Cauchy stress tensor. So, we have 6 primary variables.

Then, all the boundary conditions must be prescribed over the stress or tractions on the boundary surface Γ. So, Γ_t type boundary conditions or traction boundary conditions must be specified over the external boundary. And the 15 field equations of elasticity for the stress formulation would be reduced to only 6 field equations involving 6 independent stress components.

So, with these assumptions, we will move forward to derive the equations for solving the stress formulation problems. The name of this equation, which we are going to derive for the stress boundary value problem, is the Beltrami-Michell equations.



So, starting with the constitutive equation, $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$. This is the constitutive relation relating stress component σ_{ij} with strain component ε_{ij}, where ε_{kk} is dilation.

Now, replacing i with j, both the indices are taken to be the same in $\tilde{\tilde{\sigma}}$, so let us replace i and j with m. Both the indices being the same, we can write σ_{mm} as, using the constitutive equation, $\lambda\delta_{mm}\varepsilon_{kk} + 2\mu\varepsilon_{mm}$. Now, ε_{kk} and ε_{mm} are the same. Same index is repeated; both are dummy indices and can be named with the same name. The value of this δ_{mm} is equal to 3, that is, summation over m, δ_{mm} would be 3. Thus, this σ_{mm} can be written as $(3\lambda + 2\mu)\varepsilon_{mm}$. Then, dividing both the sides with $(3\lambda + 2\mu)$, we can get ε_{mm} as $\frac{\sigma_{mm}}{3\lambda+2\mu}$.

So, the reason for doing this exercise is to write this ε_{kk}, i.e., $\tilde{\tilde{\varepsilon}}$ with both the indices to be the same, that is what we are trying to write in terms of the corresponding stress. So, using this, we can write ε_{kk} as $\frac{\sigma_{kk}}{3\lambda+2\mu}$, and then replacing this expression of ε_{kk} back in the constitutive equation, we can write ε_{ij}, taking this term ε_{ij} of the constitutive equation towards the left, and sending all the rest of the terms towards the right. We can write ε_{ij} as $\frac{1}{2\mu}\left(\sigma_{ij} - \lambda\delta_{ij}\varepsilon_{kk}\right)$, and then the derived form of ε_{kk}, this expression would be replaced here. Thus, ε_{ij} would become $\frac{1}{2\mu}\left[\sigma_{ij} - \left(\frac{\lambda}{3\lambda+2\mu}\right)\delta_{ij}\sigma_{kk}\right]$.

Now, using the constitutive relations, interrelation between different types of material constants E, that is Young's modulus, μ that is one of the Lame constants, λ, another Lame constant, G, shear modulus, and ν, the Poisson's ratio, which were derived like this.

This we had already discussed. If you use all these relations between different material constants here and replace λ and μ, our objective is to replace λ and μ in terms of other known material constants such as Poisson's ratio, Young's modulus, and modulus of rigidity. If you do so, this expression of ε_{ij} can be modified as $\frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$.

This is called the modified constitutive equation, where we are expressing the strain components in terms of the stress components. So, if you look at the initial constitutive equation, whatever was written at the top with which we started, that was relating $\tilde{\tilde{\sigma}}$ with $\tilde{\tilde{\varepsilon}}$. $\tilde{\tilde{\sigma}}$ was on the left-hand side, and from that, we ended up with this modified constitutive equation, where $\tilde{\tilde{\varepsilon}}$, the strain components, are written in terms of stress components. As it is a stress formulation, every quantity is required to be written in terms of stresses, whether it is strain or displacement.

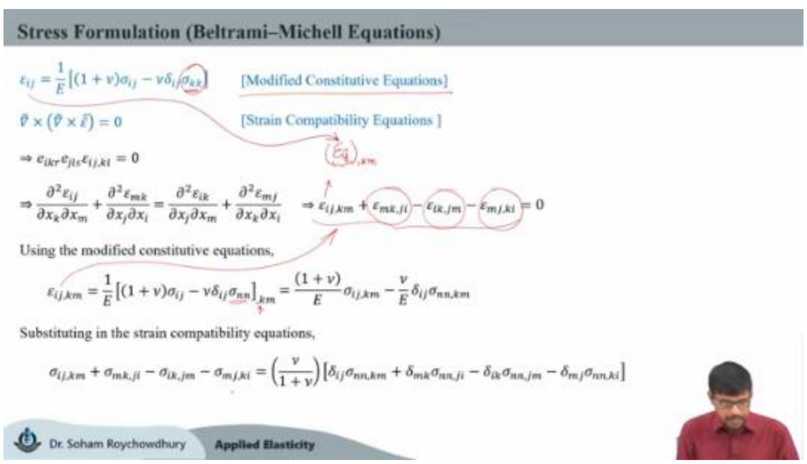
We have to express all of them in terms of stresses. So, first, we are getting $\tilde{\tilde{\varepsilon}}$ strain in terms of stresses, and then from this strain, we would try to find out the displacement. And as you know, for finding displacement from the strain fields and to ensure the unique displacement field, we need to use the strain compatibility equation. So, for the stress formulation of elasticity, the strain compatibility equations are required to be used, whereas for the displacement formulation, we have not used the strain compatibility relation; we had just used the strain-displacement relation because, for that particular case, we were trying to express everything in terms of displacement quantities, but here it is just the opposite: we need to express strain and displacement in terms of stresses, and for that, the use of strain compatibility equations is essential.

Now, from this equation, we will just try to write a few of the $\tilde{\tilde{\varepsilon}}$ components. So, let us choose $i = x$ and $j = y$ in the rectangular Cartesian coordinate system. With that, first, let us start with both i and j being the same, and that is x. That would give us the normal strain ε_{xx} along the x direction. So, what would it be? This would be $\frac{1}{E}\left[(1+\nu)\sigma_{xx} - \nu\delta_{ij}\sigma_{kk}\right]$. Now, for this case, $\delta_{ij} = \delta_{xx} = \delta_{11} = 1$.

So, in the second term, δ_{ij} is 1 and then we have σ_{kk}. σ_{kk} means the summation of σ_{11}, σ_{22}, σ_{33}, which is $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ in the rectangular Cartesian coordinate frame. From this, if you cancel this $\nu\sigma_{xx}$ term, this would become $\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})$.

So, the first normal strain along x direction can be written as $\frac{\sigma_{xx}}{E}$, i.e., the corresponding normal stress by E, minus ν, the Poisson's ratio, times summation of other two normal stresses $\frac{(\sigma_{yy}+\sigma_{zz})}{E}$, where E is the Young's modulus. This is the well-known form of stress-strain relations for the case of linear elastic isotropic solids. This is normally there in the undergraduate solid mechanics books.

With the help of modified constitutive equation, if you expand, you would be getting this commonly available form or relation between $\tilde{\tilde{\varepsilon}}$ components with the $\tilde{\tilde{\sigma}}$ components in terms of Poisson's ratio ν and Young's modulus E. So, I am not explicitly writing the remaining one. Using the similar approach, all rest can be written.



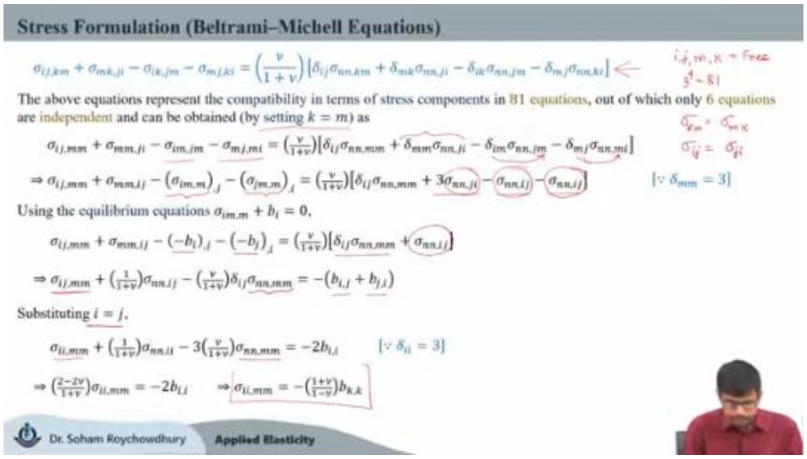
Moving forward, we will use these modified constitutive relations and the strain compatibility relations to ensure the presence of unique displacement field. As you know, the strain compatibility relation $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}) = 0$ can be written as $e_{ikr}e_{jls}\varepsilon_{ij,kl} = 0$ in tensorial notation or expanding it further, we know that we also derived this. This is the alternate form of the strain compatibility relation. So, in the indicial notation, this big expression is written as $\varepsilon_{ij,km} + \varepsilon_{mk,ji} - \varepsilon_{ik,jm} - \varepsilon_{mj,ki} = 0$.

Now, we would be using this modified constitutive equation to simplify all the epsilon terms on the left-hand side of the strain compatibility equation. So, if you consider this first term as ε_{ij}, then its derivative with respect to x_k and x_m is why $(\quad)_{,km}$ — two derivative terms are present. Now, whatever is inside the bracket, ε_{ij} would be replaced from the modified constitutive equation. This has to be done for all four terms. I am doing it only for the first term.

$\varepsilon_{ij,km}$, by using the modified constitutive equation, can be written as $\frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{nn}\right]_{,km}$. You note that here the dummy index in the last term was σ_{kk}, but while using it in the compatibility equation, I have changed the name of that dummy index to n because k is already existing as another free index. So, using k in σ_{kk} would give rise to confusion; to avoid that, I have renamed it, and since k was a dummy index in this σ_{kk} term, it can be named with any other variable, here I have named it as n.

Now, expanding this, we would get $\frac{1+\nu}{E}\sigma_{ij,km} - \frac{\nu}{E}\delta_{ij}\sigma_{nn,km}$. This is the expression for the first term of the constitutive equation. Similarly, substituting the modified constitutive equation in all the other terms of the compatibility expression in a similar fashion, and then putting all of them back together in the strain compatibility equation you would be getting this big equation, where, on the left hand side, we have four terms, which are all second order derivatives of different stress components, whereas, right hand side, $\frac{\nu}{1+\nu}$ times four more terms, where all these four terms contain δ, the Kronecker delta.

Now, these equations represent the compatibility equations in terms of stress components. Compatibility equations are normally defined in terms of strains, but as it is a stress formulation, we have replaced strain components ε_{ij} in terms of stress components σ_{ij}.



Thus, the compatibility equation in form of stress components for stress formulation can be written like the equation on the top in the indicial notation. Now i, j, k, m, all 4 are free indices. We have 4 free indices i, j, m and k, and all 4 being free indices, total number of equations would be $3^4 = 81$. Now, as σ_{ij} and ε_{ij} are symmetric due to symmetry of σ_{ij}, these 81 equations would result only 6 independent equation, all rest would be same. That can be obtained by setting $k = m$, because $\tilde{\tilde{\sigma}}$ or stress tensor is symmetric. So, by setting $k = m$, the above equation looks like this.

Now, we can simplify many of the terms on the right hand side. The first term of right hand side remain as it is whereas in the second term $\delta_{mm}\sigma_{nn,ji}$, we can write δ_{mm} as 3 by using the summation over dummy index m and δ, being Kronecker delta, δ_{mm} would be 3. So, this term becomes $3\sigma_{nn,ji}$.

Now, third term: $\delta_{im}\sigma_{nn,jm}$. So, due to presence of this Kronecker delta, δ_{im}, we are replacing m with i and thus, this becomes $\sigma_{nn,ij}$ with a negative sign. Similarly, in the last term, due to presence of Kronecker delta, δ_{mj}, replacing m with j, we get minus $\sigma_{nn,ij}$ or $\sigma_{nn,ji}$, as partial derivative orders can be flipped at any point of time. And on the left hand side, if you see third and fourth term, I am writing those as: third term was $\sigma_{im,jm}$ - I am writing this as $\left(\sigma_{im,m}\right)_{,j}$. So, we are taking one partial derivative after another. Same is done for the fourth term as well.

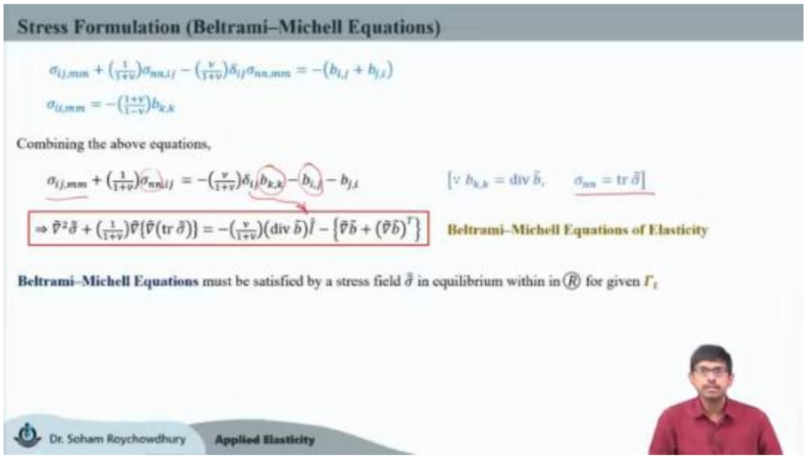
Now, using the equilibrium equation, if you recall the equilibrium equation that was $\sigma_{im,m} + b_i = 0$, or $\sigma_{ij,j} + b_i = 0$ where j or m is the name of the dummy index. Now, replacing this term here, $\sigma_{im,m}$ as $-b_i$, and $\sigma_{jm,m}$ as $-b_j$, on third and fourth terms of the left hand side using the equations of equilibrium. And simplifying the other terms on the right-hand side, we would get this equation.

So, $\sigma_{im,m}$ is written as $-b_i$, $\sigma_{jm,m}$ is written as $-b_j$. And on right-hand side, these three terms are basically having identical σ term: $\sigma_{nn,ij}$, $\sigma_{nn,ij}$, $\sigma_{nn,ij}$. So, $3 - 1 - 1$. So, it would be just a single term of $\sigma_{nn,ij}$. Rest of the terms, first and second term of left hand side, and first term of right hand side remain same.

Now, if you take this particular term on the left-hand side, and take the body force terms on the right-hand side, we are rewriting it in such a fashion that all the stress terms are sent to one side, and the body force terms are sent to the other side. Then it would look like this.

Now, substituting $i = j$, once again from symmetry of $\tilde{\tilde{\sigma}}$. So, first we set $k = m$. Why? Because σ_{km} should be the same as σ_{mk}. Now we are setting $i = j$ because we also know $\sigma_{ij} = \sigma_{ji}$. i, j, m, k, all being free indices, this should be valid. Now, if you look at the last term, $\sigma_{nn,mm}$, to simplify this, we also need to go for this substitution of $i = j$. So, if you substitute $i = j$, this first term would be the same as this particular term $\sigma_{nn,mm}$. Two indices before the comma are the same, and two indices after the comma are also the same.

So, substituting $i = j$, this equation would look like this, where the first term and third term are the same, but with two different coefficients. Also, on the right-hand side, $b_{i,j}$ and $b_{j,i}$, both would become $b_{i,i}$. Thus, the right-hand side with $i = j$ becomes minus $2b_{i,i}$. Simplifying this further, combining the terms on the left-hand side, we would get $\sigma_{ii,mm} = -\left(\frac{1+\nu}{1-\nu}\right) b_{k,k}$. So, this form of $\sigma_{ii,mm}$ can be used to substitute here, and with that, we can further simplify this particular equation.



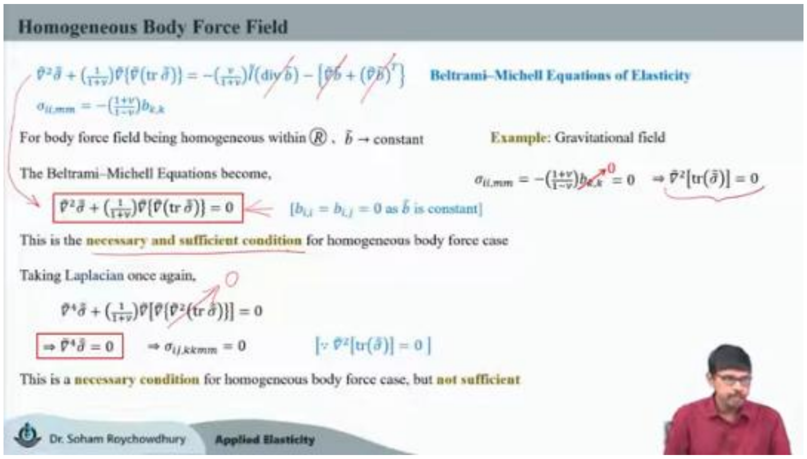
Using the expressions derived in the previous slide for $\sigma_{ii,mm}$, and the equation derived using the compatibility equation in terms of stresses, we can combine them and write the compatibility equation in terms of stress as this: $\sigma_{ij,mm} + \left(\frac{1}{1+\nu}\right)\sigma_{nn,ij} = -\left(\frac{\nu}{1+\nu}\right)\delta_{ij}b_{k,k} - b_{i,j} - b_{j,i}$. $b_{k,k}$ is nothing but $\text{div}(\tilde{b})$, and σ_{nn} is $\text{tr}(\tilde{\tilde{\sigma}})$. Replacing this $b_{k,k}$ as $\text{div}(\tilde{b})$, $b_{i,j}$ as $\tilde{\nabla}\tilde{b}$, $b_{j,i}$ as $\left(\tilde{\nabla}\tilde{b}\right)^T$, and here, $\sigma_{ij,mm}$ as $\tilde{\nabla}^2\tilde{\tilde{\sigma}}$, and in the second term, σ_{nn} as $\text{tr}(\tilde{\tilde{\sigma}})$.

With all these, this particular equation can be written as:

$$\tilde{\nabla}^2\tilde{\tilde{\sigma}} + \left(\frac{1}{1+\nu}\right)\tilde{\nabla}\{\tilde{\nabla}(\text{tr}\,\tilde{\tilde{\sigma}})\} = -\left(\frac{\nu}{1+\nu}\right)(\text{div}\,\tilde{b})\tilde{\tilde{I}} - \left\{\tilde{\nabla}\tilde{b} + \left(\tilde{\nabla}\tilde{b}\right)^T\right\}$$

Here, this identity tensor $\tilde{\tilde{I}}$ comes from δ_{ij}, the Kronecker delta.

This equation is known as the Beltrami-Michell equations of elasticity, which is used for solving the stress-based formulation or stress boundary value problems of elasticity. With the solution of this equation, we can get the stress fields within the overall region when the stress boundary conditions are prescribed over Γ_t.



Now, moving forward, we are considering the body force field to be homogeneous. So, I have written the Beltrami-Michell equations of elasticity and, along with that, the expression of $\sigma_{ii,mm}$, both I have rewritten here.

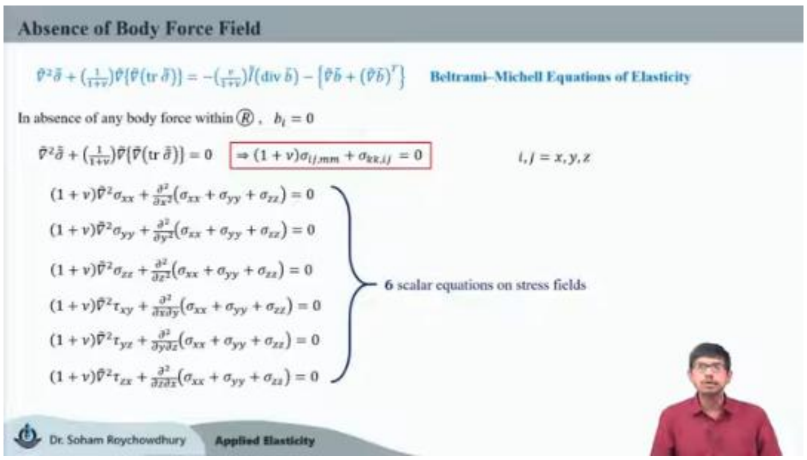
Now, for the body force field being homogeneous within the total volume of the material, that is, $\tilde{b}$ is a constant vector, which is the case of gravitational force field acting on the body. For such cases, $\sigma_{ii,mm}$ can be simplified where $b_{k,k}$ would go to 0 because $\tilde{b}$ being a constant, derivative of components of $\tilde{b}$ with respect to x_k would go to 0, and thus, $\sigma_{ii,mm}$ will go to 0. Now, the left-hand side $\sigma_{ii,mm}$ can be written as $\tilde{\nabla}^2\left[\text{tr}\left(\tilde{\tilde{\sigma}}\right)\right]$, and that is equal to 0 for the stress formulation with a homogeneous body force field.

Now, simplifying the Beltrami-Michell equation, these can be rewritten in this form with the assumption of a homogeneous body force field, with $\tilde{b}$ being a constant vector; $b_{i,i}$ and $b_{i,j}$, $\text{div}\left(\tilde{b}\right)$, that is $b_{i,i}$, and $\tilde{\nabla}\tilde{b}$, that is $b_{i,j}$, both terms would go to 0 for constant $\tilde{b}$. So, all these terms would go to 0 on the right-hand side, and this equation would become $\tilde{\nabla}^2\tilde{\tilde{\sigma}} + \left(\frac{1}{1+\nu}\right)\tilde{\nabla}\left[\tilde{\nabla}\left(\text{tr}\,\tilde{\tilde{\sigma}}\right)\right] = 0$. This is the necessary and sufficient condition for solving the stress formulation to be satisfied by the stress field $\tilde{\tilde{\sigma}}$ for the homogeneous body force case.

Now, if you take the Laplacian of this equation once again, the right-hand side would remain 0 only, and the left-hand side would have the biharmonic operator of the $\tilde{\tilde{\sigma}}$, the stress tensor, plus $\left(\frac{1}{1+\nu}\right)\tilde{\nabla}\left[\tilde{\nabla}\left\{\tilde{\nabla}^2\left(\text{tr}\,\tilde{\tilde{\sigma}}\right)\right\}\right]$, and $\tilde{\nabla}^2\left(\text{tr}\,\tilde{\tilde{\sigma}}\right)$ from here is already shown to be

0. This term will go to 0, and we would be left with only $\tilde{\nabla}^4 \tilde{\tilde{\sigma}} = 0$. So, the stress field must be biharmonic: $\sigma_{ij,kkmm} = 0$. This is a necessary condition for a homogeneous body force field, but not a sufficient one.

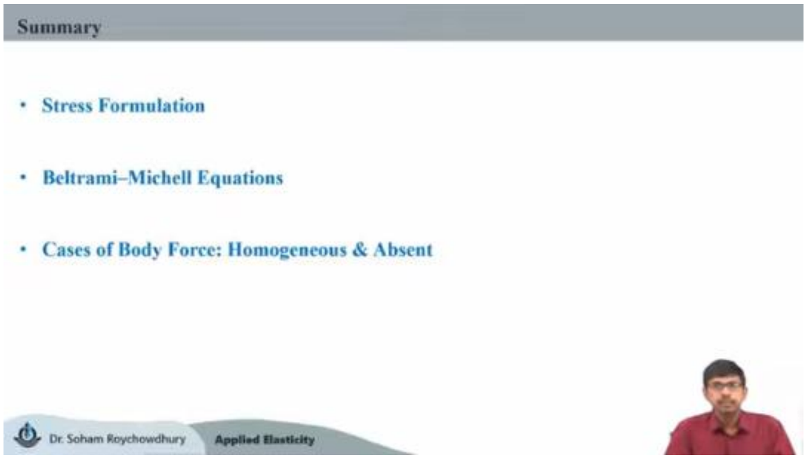
So, this particular equation gives us the necessary and sufficient condition for the stress formulation with a homogeneous body force field, whereas this one, $\tilde{\nabla}^4 \tilde{\tilde{\sigma}} = 0$, is only a necessary condition but not sufficient for the same body force field.



Now, we are going to consider the case of zero body forces. In the absence of any body force, $\tilde{b}$ goes to 0; all b_i components are 0, and with that, both the right-hand side terms of the Beltrami-Michell equation would go to 0, and you would have the $\tilde{\nabla}^2 \tilde{\tilde{\sigma}} + \left(\frac{1}{1+\nu}\right) \tilde{\nabla}\left[\tilde{\nabla}\left(\text{tr}\,\tilde{\tilde{\sigma}}\right)\right] = 0$. So, $(1+\nu)\sigma_{ij,mm} + \sigma_{kk,ij} = 0$ is the Beltrami-Michell equation in the indicial notation in the absence of any body forces.

Explicitly expanding this equation for the rectangular Cartesian coordinate system where i and j can take x, y, z values, we get 6 equations as this. The first equation would be $(1+\nu)\tilde{\nabla}^2\sigma_{xx} + \frac{\partial^2}{\partial x^2}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = 0$.

Similarly, you can expand the other 5 equations, and in total, we would get these 6 scalar equations on the stress fields, which are required to be solved for the stress-based formulation or stress boundary value problem for prescribed Γ_t with zero body forces.



So, in this lecture, we discussed the stress-based formulation of elasticity, derived the Beltrami-Michel equations, and considered two types of body forces: one is a homogeneous body force field, and another is a zero body force field.

Thank you.