APPLIED ELASTICITY

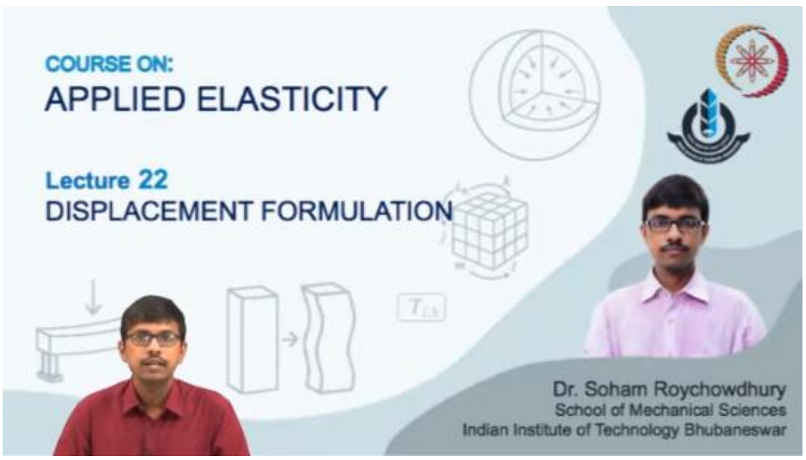
Dr. Soham Roychowdhury

School of Mechanical Sciences

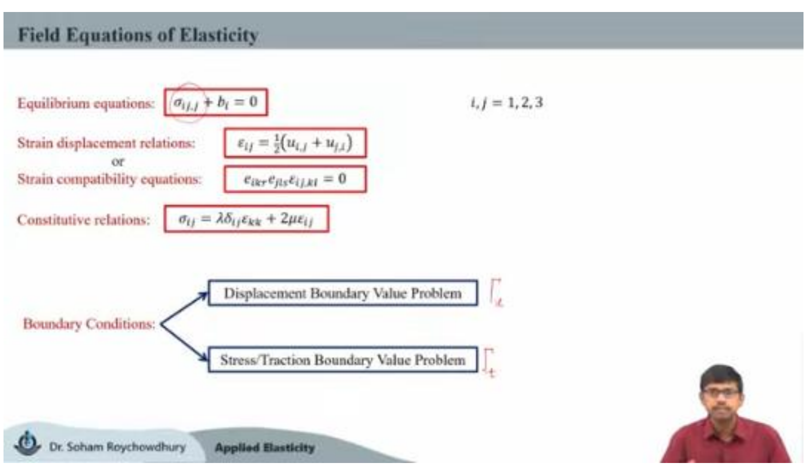
IIT Bhubaneswar

Week 5

Lecture 22: Displacement Formulation



Welcome back to the course on Applied Elasticity. In today's lecture, we are going to talk about the displacement formulation problems of elasticity.

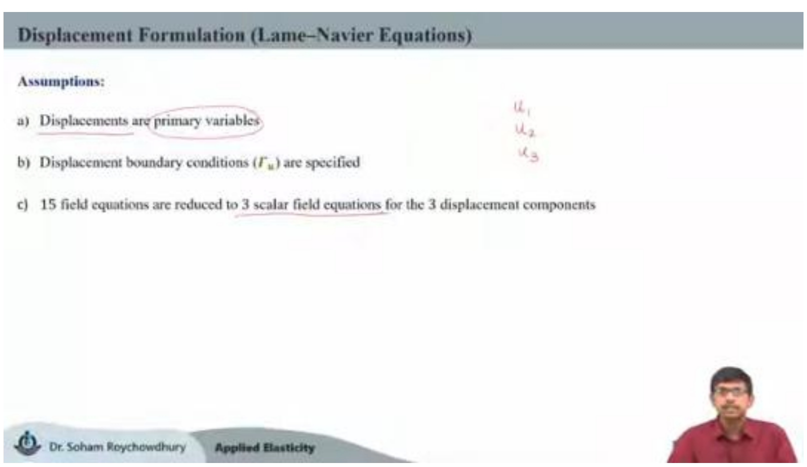


So, in the previous lecture, we introduced the concept of field equations of elasticity. And we checked that there exist 15 field equations of elasticity. So, we are starting with the equilibrium equations - three in number - which are $\sigma_{ij,j} + b_i = 0$, followed by 6 strain-displacement relations: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, or sometimes for the case of known strain fields, and for finding a unique displacement field instead of strain - displacement relations, we use six strain compatibility relations given by $e_{ikr}e_{jls}\varepsilon_{ij,kl}$.

Then, moving forward to the constitutive relation, with the assumption of the material being a linear elastic isotropic solid, we have $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$, where λ and μ are known as the two Lame constants.

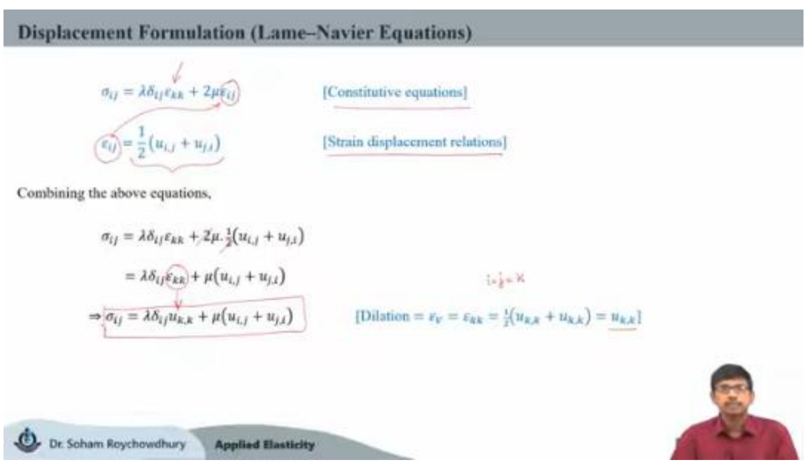
Now, we need boundary conditions for solving the equilibrium equation, which involves one derivative term. So, these being differential equations, we need to have boundary conditions. Now, boundary conditions can be of two possible types: one is called displacement boundary conditions, specified over the displacement boundaries Γ_u and for such cases, the problem is known as a displacement boundary value problem. Whereas, if the boundary conditions are specified on the surface tractions or stresses, then we call those to be surface traction boundary conditions, written as Γ_t and the problem is known as a stress boundary value problem or, alternately, a traction boundary value problem.

Now, the solution approach for a stress boundary value problem is different from that of the displacement boundary value problem. We will be discussing these two formulations - displacement-based formulation and stress-based formulation - and this is required to be chosen, that which formulation is to be used depends on the available boundary condition type.



So, in this lecture, we will talk about the displacement boundary value problem, also known as the displacement formulation of the theory of elasticity. The equations which we are going to derive are named the Lame-Navier equations of elasticity, valid for displacement boundary value problems. So, what are the assumptions? For this particular problem, the displacements are the primary variables. Equations should be written in terms of three displacement components.

The displacement boundary conditions Γ_u are specified. We do not have surface traction boundary conditions and the field equations — 15 field equations we had for any elasticity problem — would be reduced to only 3 scalar equations for 3 displacement components. Since displacements are our primary variables, we can have only 3 primary variables. u_1, u_2, and u_3 — 3 displacement components — and thus, there would be 3 scalar equations, known as field equations on displacement components. Those are known as Lame-Navier equations, and we are trying to derive them now.

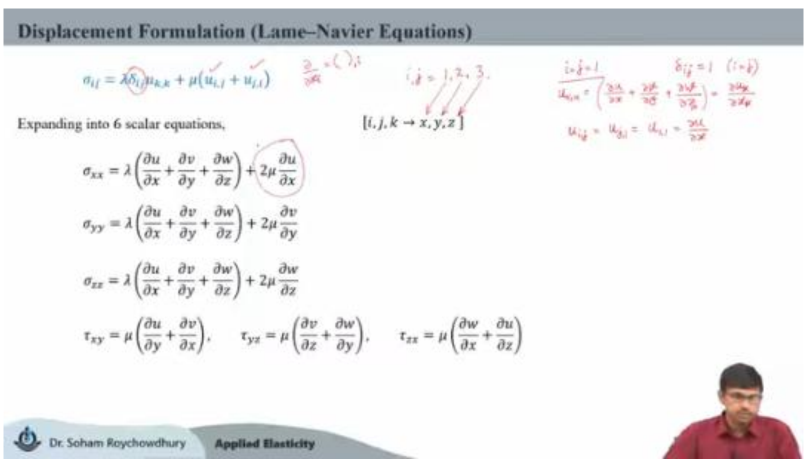


So, looking back at the constitutive and strain-displacement relations, constitutive equations relate σ_{ij} (stress components) with ε_{ij} (strain components), where ε_{kk} is nothing but the trace of epsilon, or also known as dilation (volumetric strain). λ and μ are two constants known as Lame constants.

Now, ε_{ij} can be related with u_i (displacement components) with the help of strain-displacement relations, as $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. These are known to us; we have already discussed them. Now, let us combine these two equations. So, here in the constitutive equation, we have ε_{ij}, which would be replaced with the help of the strain-displacement relations. So, combining that, we can express σ_{ij} as $\lambda\delta_{ij}\varepsilon_{kk} + 2\mu.\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$. Here, this 2 and half would get cancelled, and we can write $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$.

Now, using the definition of dilation, dilation is ε_V, volumetric strain, deformed volume by undeformed volume, defined as stress of strain tensor $\tilde{\tilde{\varepsilon}}$, or ε_{kk}. So, using the strain displacement relation with i and j both being equals to k, substituting i and j with k, you can get ε_{kk} from the strain displacement relation and that would be $u_{k,k}$. So, this ε_{kk} is replaced with $u_{k,k}$ which is nothing but dilation. This relation σ_{ij} becomes $\lambda\delta_{ij}u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right)$. Note that this equation relates stress component σ_{ij} with the displacement component u_i. This is also known as stress-displacement relation.

So, we are removing or omitting strains and directly relating stress with the displacement components. This is the stress-displacement relation obtained.

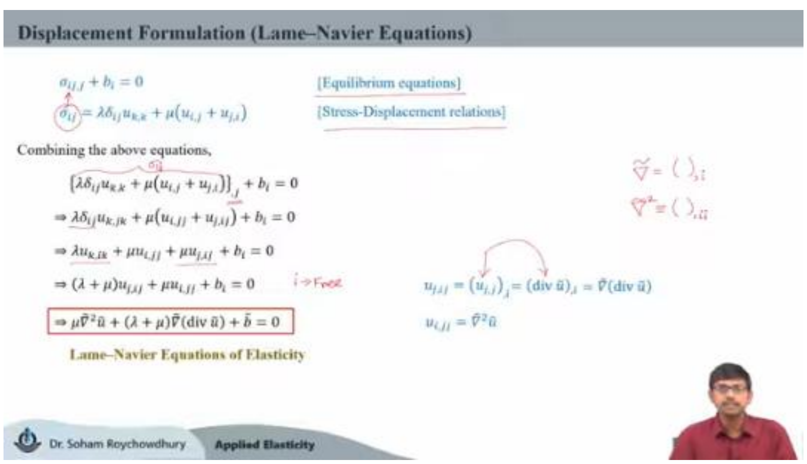


Now, we are expanding it into 6 scalar equations because i and j can take values of 1, 2, 3 and also σ_{ij} is symmetric. So, with that we are expanding it. Now, here as I am writing the expression in the rectangular Cartesian coordinate system, 1 refers to x-axis, 2 refers to y-axis, 3 refers to z-axis. So, $\frac{\partial(\ \ )}{\partial x_i}$, or something $(\ \ )_{,i}$, refers to the partial derivative with x_i component.

Now, here x_1 component is equivalent to x, x_2 component is equivalent to y, and x_3 component is equivalent to z. With that, the first scalar equation with $i = 1, j = 1$, we can obtain σ_{xx} as $\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}$. So, this is obtained with i and j both being equal to 1, and left hand side would be σ_{xx}.

Now, i and j both being 1, δ_{ij} would be 1, because i and j are equal. So, for that case, $u_{k,k}$, what is that? That is nothing but $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$. This is $\frac{\partial u_k}{\partial x_k}$, and k being a dummy index, we should have a summation sign here summed over 1, 2, 3 or x, y, z, and thus the first term becomes λ times this entire term $u_{k,k}$. Whereas in the second term, we have μ, now $u_{i,j}$ and $u_{j,i}$, both are equal to $u_{1,1}$, which is $\frac{\partial u}{\partial x}$ for the present problem where i and j both are equal to 1. So, both of these two terms, this one and this one being the same, the second term would be $2\mu\frac{\partial u}{\partial x}$.

Similarly, taking different combinations of i, j, and k, you can get these 6 scalar equations relating σ, the stress components, with the corresponding displacement components. So, these are the expanded forms of this stress-displacement relation.



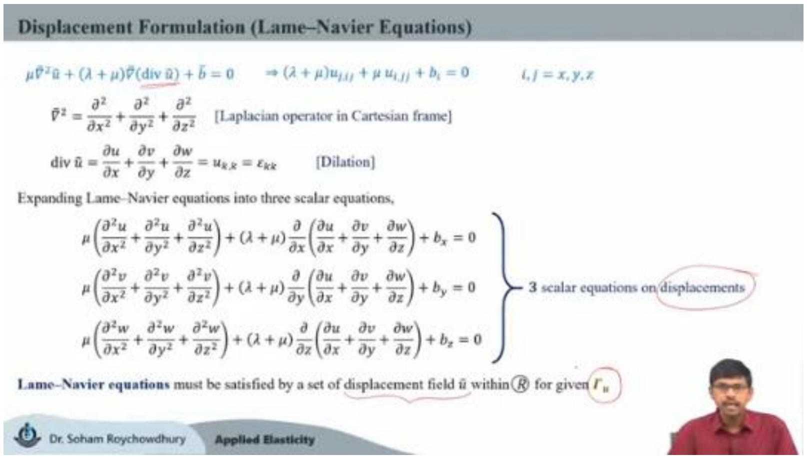
Moving forward, we are now trying to combine the stress-displacement relation with the equilibrium equation. The equilibrium equation is $\sigma_{ij,j}+b_i$, where σ_{ij} will be replaced as a function of displacement using the stress-displacement relations. So, replacing σ_{ij} from the stress-displacement relations, this quantity within the curly bracket is σ_{ij} that, comma j plus b_i equals to 0, is our equilibrium equation.

Now, this comma j, we are going to expand by taking this derivative with respect to x_j for all the terms separately. So, it would be $\lambda\delta_{ij}u_{k,jk}$, and λ and δ_{ij}, both constant, are taken out of the derivative. Whereas the second term is $\mu u_{i,jj} + u_{j,ij}$, and the last term is b_i. Expanding both terms from this μ and opening the bracket, we would be getting these four terms.

Now, if you carefully compare the first term $\lambda u_{k,ik}$ and the third term $\mu u_{j,ij}$, j and k are both dummy indices and can be changed to the same name. So, these two terms are identical, just the coefficients are different. For the first case, it is λ; for the third case, it is μ. So, these two can be combined and written as $(\lambda + \mu)u_{j,ij}$, plus we have the other two terms left: $\mu u_{i,jj} + b_i$. Now, $u_{j,ij}$, this particular term, we are trying to rewrite as $u_{j,j,i}$.

Now, $u_{j,j}$ is the same as the divergence of u, and anything $(\quad)_{,i}$ refers to the gradient operator. So, $u_{j,ij}$ is the gradient of the divergence of u. We know that this gradient operator is the quantity_,i, the derivative with respect to x_i. So, this particular term is nothing but $\tilde{\nabla}(\operatorname{div}\tilde{u})$. Now, considering this term $u_{i,jj}$, we know that the Laplacian operator is something_,jj, meaning the double derivative with respect to x_j.

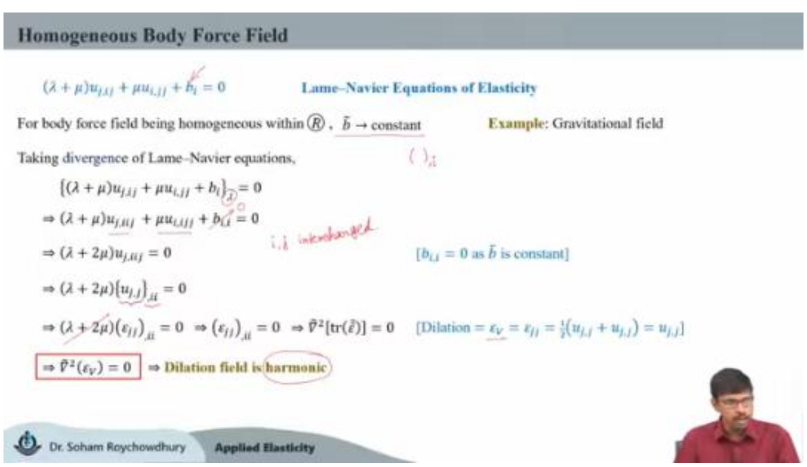
So, $u_{i,jj}$ is the Laplacian of u. Writing this, expressing $u_{j,ij}$ as $\tilde{\nabla}(\operatorname{div}\tilde{u})$ and $u_{i,jj}$ as $\tilde{\nabla}^2\tilde{u}$, we get this equation to be $\mu\tilde{\nabla}^2\tilde{u} + (\lambda + \mu)\tilde{\nabla}(\operatorname{div}\tilde{u}) + \tilde{b}$. This equation is known as the Lame-Navier equations of elasticity. You can see, if you look at the previous indicial expression, i is the only free index present, j is a dummy index. i can take three values: $1, 2, 3$. Thus, 3 scalar equations would come from this Lame-Navier equations of elasticity, which is used to solve for the displacement components in displacement boundary value problems.



Now, if you explicitly write the Laplacian operator in the rectangular Cartesian frame as $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and the divergence of u as $u_{k,k}$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ in the Cartesian frame. This is also known as dilation or volumetric strain ε_{kk} or ε_V.

Expanding those 3 displacement terms — 3 scalar equations in terms of 3 displacement field variables u, v, and w — we get these 3 forms. So, these are 3 scalar equations on the displacement components known as the Lame-Navier equations, and for solving this, we must have the boundary conditions specified over the displacement field u on Γ_u.

So, with the help of the given Γ_u, we can solve these 3 scalar equations involving the 3 displacement components u, v, w and get the displacement field. This gives us the complete solution of any displacement BVP, a displacement boundary value problem.



Now, we will move forward to the body force term. So, if you look at the Lame-Navier equations of elasticity for a displacement boundary value problem, this we had derived as $(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + b_i$, where b_i is the body force term.

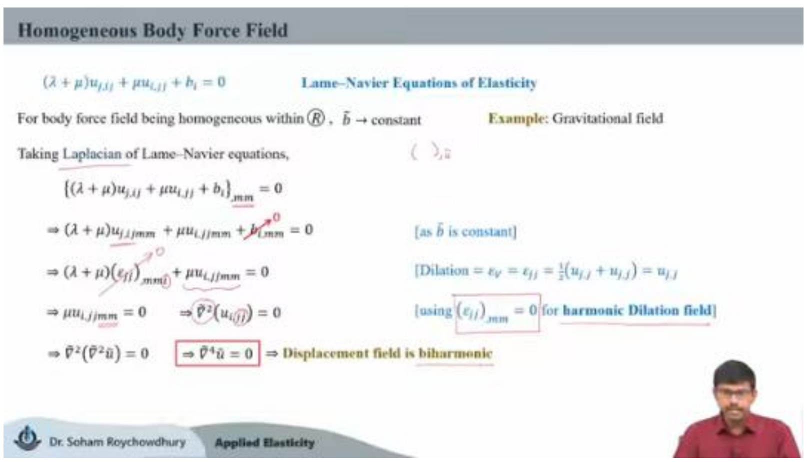
Now, body force may be of different types; it may be there, or it may not be there, depending on the problem. So, let us consider a special case when the body force is nonzero but homogeneous. Homogeneous means it is constant over the entire volume, independent of the material coordinate location. One example of such a field is a gravitational field. So, if a body is under gravity, the gravity field will apply the body force on the body, or the material continuum.

However, this amount of gravity field, that is, basically $\tilde{g}$, the gravitational acceleration, that effect will be felt by all the material points in the same fashion, independent of the location of the material point within the continuum. All of them will face or feel the same body force field due to gravity. So, gravity is one example of a homogeneous body force field. So, considering the body force field to be homogeneous, the $\tilde{b}$ vector would be a constant vector, and we will try to simplify the equation further for such a homogeneous body force assumption. So, we are taking the divergence of the Lame-Navier equation. Divergence means taking the derivative with respect to x_i or x_j once. So, this entire equation, the Lame-Navier equation, is written with $(\quad)_{,i}$. So, we are taking the derivative with respect to x_i, which is taking the divergence of the Lame-Navier equation.

The right-hand side is kept as 0; the divergence of 0 is 0 only. Now, expanding it, we are getting $(\lambda + \mu)u_{j,iij} + \mu u_{i,ijj} + b_{i,i}$. So, this comma i operator is distributed for all three terms, as written in all three terms. Now, $\tilde{b}$ being a constant vector, b_i (all the components of body forces) are constant; thus, $b_{i,i} = 0$. This last term would go to 0, and the other two terms can be combined into a single term by interchanging i and j. These two dummy indices are interchanged for any one of the terms, and with that, we can write $(\lambda + 2\mu)u_{j,iij} = 1$, and $u_{j,iij}$ is once again rewritten as $u_{j,j_{ii}}$ (which is basically $\text{div}(\tilde{u})_{,ii}$).

This comma ii refers to the Laplacian operator. Now, $u_{j,j}$ we know that is nothing but dilation or ε_{jj}, as we had already proved. So, replacing this as ε_{jj} within brackets, and as the right-hand side is 0, we can cancel this $(\lambda + 2\mu)$ term. Thus, $\varepsilon_{jj,ii} = 0$. This means the Laplacian of the trace of $\tilde{\tilde{\varepsilon}}$ is equal to 0. So, the Laplacian of ε_V, the Laplacian of the dilation field, is 0, and if the Laplacian of any function is 0, we call that function a harmonic function.

Thus, the dilation field ε_V or ε_{kk} must be a harmonic field for the case of a homogeneous body force field of the Lame-Navier equations. Now, this we had derived by taking the divergence of this Lame-Navier equations.



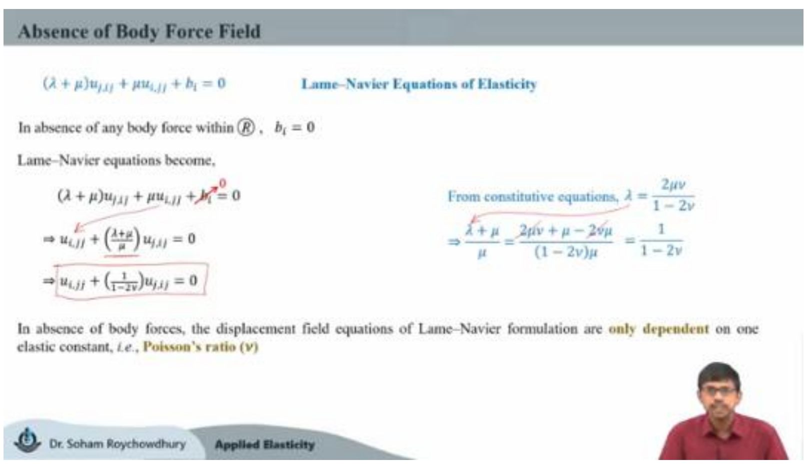
Now, we are taking the Laplacian of the Lame-Navier equations. So, that is taking the derivative with respect to i twice, i.e., with respect to x_i, we are taking the derivative twice here, and the dummy index is named m. So, with respect to the same x_i or x_m, we are taking the derivative of the Lame-Navier equations twice, which is nothing but using the Laplacian operator over the Lame-Navier equations. Expanding, this term would be: $(\lambda + \mu)u_{j,ijmm} + \mu u_{i,jjmm} + b_{i,mm}$. $\tilde{b}$ being constant, the $b_{i,mm}$ term goes to 0. So, the other two terms are left. Now, writing the first term, this $u_{j,j}$ as ε_{jj}, this would be $(\lambda + \mu)\varepsilon_{jj,mmi} + \mu u_{i,jjmm}$.

Now, if you consider the dilation field to be harmonic, which we had just proved in the previous slide. For the case of a homogeneous body force field, a harmonic dilation field is there, and thus, $\varepsilon_{jj,mm} = 0$. So, this particular term would go to 0. This contains

$\varepsilon_{jj,mm}$. We have one extra i, meaning another additional derivative is there. But still, it would go to 0 because the dilation force field is harmonic. So, only last term would remain that is $\mu u_{i,jjmm}$.

So, these two m refers to one Laplacian operator. So, $\widetilde{\nabla}^2\left(u_{i,jj}\right) = 0$. Now, this comma jj refers to another Laplacian operator. Any index being repeated twice means that is leading to addition of one Laplacian operator. So, this comma jj refers to another Laplacian and thus we are having two Laplacian operators. These can be combined to a biharmonic operator.

So, $\widetilde{\nabla}^4\tilde{u} = 0$ means displacement field is biharmonic, where $\widetilde{\nabla}^4$ means product of two Laplacian operators. So, if we are having homogeneous body force field, the Lame-Navier equations for displacement boundary value problem gets reduced to a single equation, which is nothing but the biharmonic nature of the displacement field variable $\tilde{u}$.



For the case of zero body force, for the displacement boundary value problem without any body force, we are going to check what would be the form of the Lame-Navier equations. So, considering no body forces acting on the system $\tilde{b}$ vector goes to 0 or $b_i = 0$. So, for that particular case the last term of the Lame-Navier equation goes to 0 and thus this becomes $(\lambda + \mu)u_{j,ij} + \mu u_{i,jj}$.

Now, dividing the two non-zero terms of left hand side with the constant μ and writing this second term at the beginning, we can get $u_{i,jj} + \left(\frac{\lambda+\mu}{\mu}\right) u_{j,ij}$. Now, if you recall about the constitutive equations for isotropic homogeneous linear elastic solids, the Lame constants λ and μ can be related with the Poisson's ratio ν through this particular form: $\lambda = \frac{2\mu\nu}{1-2\nu}$.

Now, using this form of λ, we will try to simplify this particular expression $\frac{\lambda+\mu}{\mu}$. So, if you simplify this by substituting this expression of λ here, this would be $\frac{2\mu\nu+\mu-2\mu\nu}{(1-2\nu)\mu}$. Now, these two terms would cancel each other, and then by canceling μ in the numerator and denominator, we would get $\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu}$. Substituting this back into the Lame-Navier equation, it would be $u_{i,jj} + \left(\frac{1}{1-2\nu}\right) u_{j,ij} = 0$. This is the form of the Lame-Navier equation in the absence of any body force.

Now, you can check that here only one material constant is present, which is the Poisson's ratio. So, if we have zero body force present for any displacement boundary value problem, there are three displacement field equations for this Lame-Navier formulation, which depends only on a single material constant, and that is nothing but the Poisson's ratio.

So, in this particular lecture, we discussed the displacement formulation for elasticity where only displacement boundary conditions are specified. We derived the Lame-Navier equations and then considered two special cases: homogeneous body force and zero body force acting on a displacement boundary value problem. Thank you.