**APPLIED ELASTICITY**


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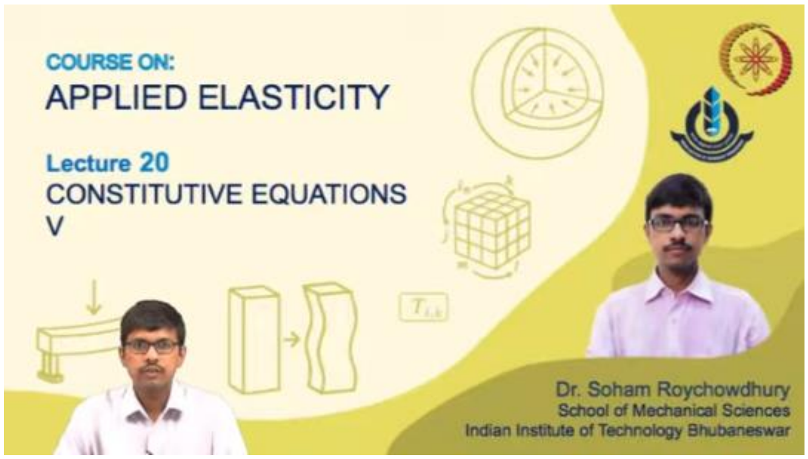
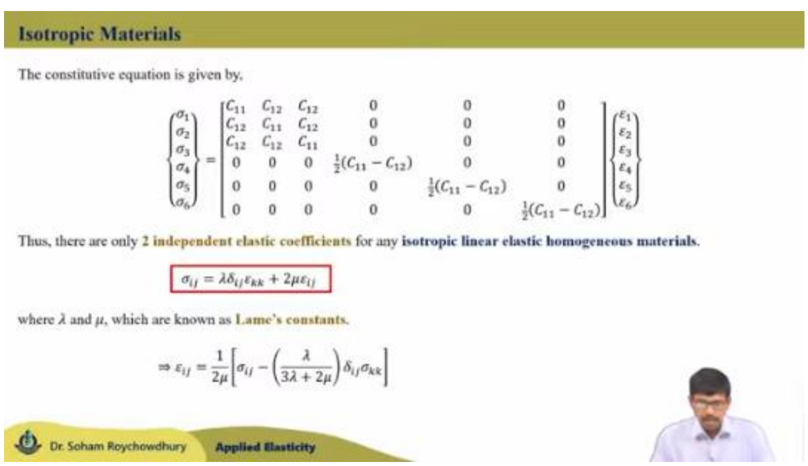
**WEEK: 04**
**Lecture- 20**

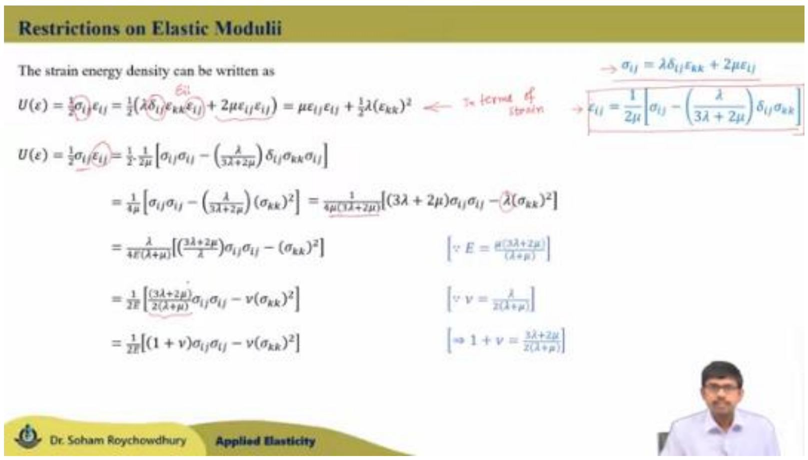



Welcome back to the course on applied elasticity. We will continue our discussion on the constitutive equations, which we have been discussing in the past few lectures. So, first, starting with the isotropic material for a quick recap.




The constitutive equation for the isotropic material was given like this, which relates the six stress components with the six strain components in the engineering notation, and you can see this involves two independent elastic constants, $C_{11}$ and $C_{12}$.

In the alternate form, we can also write the constitutive equation for isotropic, homogeneous, linear elastic solid material as $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$, where $\lambda$ and $\mu$ are known as the two Lame constants. Now, instead of writing $\sigma_{ij}$ as a function of $\varepsilon_{ij}$ or writing stress as a function of strain, we can take strain to the left-hand side, and this equation can be modified to write the strain component $\varepsilon_{ij}$. of stress component as $\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \left(\frac{\lambda}{3\lambda+2\mu}\right)\delta_{ij}\sigma_{kk}\right]$, where this term $\sigma_{kk}$ is nothing but the trace of the stress tensor $\sigma$, and $\lambda$ and $\mu$ are two constants known as Lame constants.




Now, moving forward, I have here written the Constitutive equation for isotropic solid in terms of Lame constant in two possible forms. One in terms of $\sigma$, $\varepsilon$ is expressed in another one, $\sigma$ is expressed in terms of $\varepsilon$. Now, strain energy density, which was defined as strain energy stored within the body per unit volume. That we can write as $\frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ by definition of the strain energy density.

This is defined as half of the inner product of sigma tensor, stress tensor, and strain tensor epsilon. So, $\frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ is our strain energy density function. Now, here in this equation, we are replacing this $\sigma_{ij}$. As this $\lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$. So, expanding this, multiplying this half and $\varepsilon_{ij}$ with all the terms, we can write $U(\varepsilon)$, the strain energy density, in this particular form. $\mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}\lambda(\varepsilon_{kk})^2$. So, how is it coming? If you look at this equation, this $\varepsilon_{ij}$ and $\delta_{ij}$, if you multiply, that would result in $\varepsilon_{ii}$.

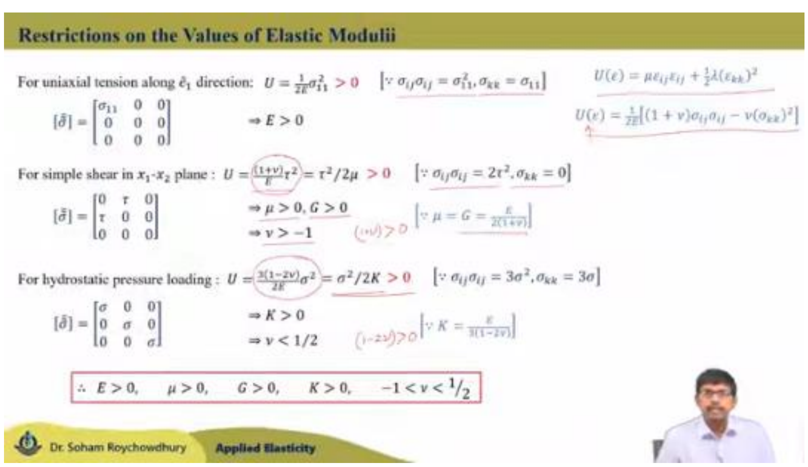
$j$ is being replaced with $i$ and $\varepsilon_{ii}$ and $\varepsilon_{kk}$; they are identical. In both cases, $i$ and $k$ are dummy indices; names can be changed. So, considering both to be $\varepsilon_{kk}$, we are getting that term, the $\lambda$ term, as $\frac{1}{2}\lambda(\varepsilon_{kk})^2$. And this term, this 2 and half will get cancelled, and thus it would become $\mu\varepsilon_{ij}\varepsilon_{ij}$. So, this is the expression of strain energy in terms of $\varepsilon$, in terms of strain.

So, we can write the strain energy density function $U(\varepsilon)$ in terms of $\varepsilon$ like this. $\mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}\lambda(\varepsilon_{kk})^2$, where $\varepsilon_{kk}$ is nothing but the trace of that strain tensor. Now, alternately, we will try to write $U$, the strain energy density, in terms of stress components as well, so let us try to do that. $U(\varepsilon) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$, and here we will substitute this $\varepsilon_{ij}$ as this using this equation. So, writing that, substituting $\varepsilon_{ij}$ as $\frac{1}{2\mu}\left[\sigma_{ij}\sigma_{ij} - \left(\frac{\lambda}{3\lambda+2\mu}\right)\delta_{ij}\sigma_{kk}\sigma_{ij}\right]$, and another half is outside.

So, this $\sigma_{ij}$ has been included and multiplied with this form of $\varepsilon_{ij}$, and thus this expression is looking like this. So, strain energy we can further simplify and write as $\frac{1}{4\mu}\left[\sigma_{ij}\sigma_{ij} - \left(\frac{\lambda}{3\lambda+2\mu}\right)(\sigma_{kk})^2\right]$. Now, taking this $3\lambda + 2\mu$ out as a common factor in the denominator. It would be $\frac{1}{4\mu(3\lambda+2\mu)}$, and then within bracket $(3\lambda + 2\mu)\sigma_{ij}\sigma_{ij} - \lambda(\sigma_{kk})^2$. Now, if you recall the conditions or the relations, whatever we had obtained, different elastic constants, one of which was $E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$. So, using this, we can rewrite this part $\mu(3\lambda + 2\mu)$ as $E(\lambda + \mu)$. This equation becomes 1 by 4 E divided $\frac{\lambda}{4E(\lambda+\mu)}\left[\left(\frac{3\lambda+2\mu}{\lambda}\right)\sigma_{ij}\sigma_{ij} - (\sigma_{kk})^2\right]$. So, these $\lambda$ coefficients of $\sigma_{kk}$ whole square are also taken common in the numerator outside the bracket.

Now, further, recalling the relation between Poisson's ratio $\nu$ with Lame constants $\lambda$ and $\mu$, $\nu$, Poisson's ratio was related with $\lambda$ and mu as $\nu = \frac{\lambda}{2(\lambda+\mu)}$. Now, using this and carefully rewriting the equation, we can write this as $\frac{1}{2E}\left[\frac{(3\lambda+2\mu)}{2(\lambda+\mu)}\sigma_{ij}\sigma_{ij} - \nu(\sigma_{kk})^2\right]$. So, our objective is to add this $\nu$ in the coefficient of $(\sigma_{kk})^2$.

And then, this term, the coefficient of $\sigma_{ij}\sigma_{ij}$, can be written as $1 + \nu$. So, if you find out this, simplify the expression of $1 + \nu$, that is nothing but this coefficient of $\sigma_{ij}\sigma_{ij}$. Thus, $U(\varepsilon)$, the strain energy density becomes $\frac{1}{2E}[(1 + \nu)\sigma_{ij}\sigma_{ij} - \nu(\sigma_{kk})^2]$. Now, we are writing both the expressions of strain energy density. The first one is in terms of strain, whereas the second one is in terms of stress, which we had just now derived.



Now, considering the three cases which we had discussed in the last lecture, the first case was uniaxial tension along the $e_1$ direction, then the case of simple shear, and then the case of hydrostatic pressure loading. So, first considering the case of uniaxial tension along the $e_1$ direction, the stress tensor $\tilde{\tilde{\sigma}}$ was given to be $\sigma$ 1100, 000, and 000.

With that, using this particular form of $U$, which was in terms of stress, we can get the total strain energy density function to be $\frac{1}{2E}\sigma_{11}^2$. For this particular state of stress, $\sigma_{kk}$. That is, the trace of sigma is nothing but $\sigma_{11}$, the summation of the three diagonal terms: $\sigma_{11}$ plus 0 plus 0.

So, $\sigma_{kk}$ is $\sigma_{11}$, and $\sigma_{ij}\sigma_{ij}$ has only one term: $\sigma_{11}^2$. So, substituting $\sigma_{ij}\sigma_{ij}$ and $\sigma_{kk}$ in the expression of $U$, we can write $U = \frac{1}{2E}\sigma_{11}^2$. Now, we all know that due to this uniaxial tension, the body is deforming. Some finite amount of strain energy is supposed to be stored within the body. And thus, this value of $U$ must be a positive quantity.

We must have some amount of positive strain energy stored within the body. Now, $U$ being positive, $\frac{1}{2E}\sigma_{11}^2$ should also be a positive quantity. Now, $\sigma_{11}^2$ is already a positive quantity. For this quantity to be positive, we must have $E$ to be positive. Thus, the Young's modulus $E$ must be a positive quantity.

This material constant cannot take a negative value in any case. Now, moving forward to the case of simple shear in the $x_1$, $x_2$ plane. $\sigma$, the stress tensor, is given as 0, $\tau$ 0, $\tau$ 0, 0 and 0, 0, 0. For this case, the trace of $\sigma$, $\sigma_{kk} = 0$, and $\sigma_{ij}\sigma_{ij} = 2\tau^2$. So, substituting these two in the expression of strain energy,

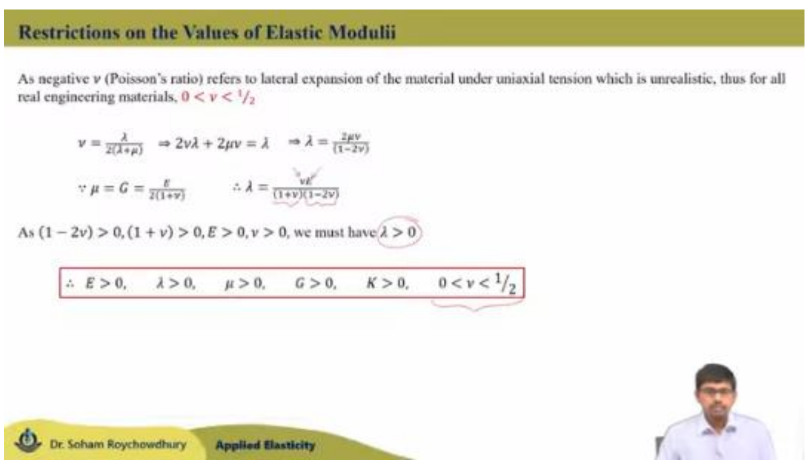
we can obtain the expression of strain energy density $U$ for the simple shear case to be $\frac{(1+\nu)}{E}\tau^2$ and using the relation between $G$, $\mu$, and $E$. We know that the modulus of rigidity $G$ can be related to the Young's modulus $E$ using this particular relation. So, $E$ equals $2G$ times $2(1+\nu)$, or $G = \frac{E}{2(1+\nu)}$. We had already shown that $\mu$, the second Lame constant, is nothing but the modulus of rigidity or $G$.

So, using that here, we can write this $\frac{(1+\nu)}{E}\tau^2$ as $\tau^2/2G$ or $\tau^2/2\mu$. Similar to the previous case, here also we must have a positive strain energy stored within the body due to the simple shear loading. Thus, $\tau^2/2\mu$ greater than or equal to 0 ensures $\mu$ or $G$ to be positive because $\tau^2$ is already a positive quantity. Coming to this first form of $U$, $\frac{(1+\nu)}{E}\tau^2$ must be positive, where we had already shown that $E$ is a positive quantity and $\tau^2$ is obviously a positive quantity. Thus, we must have this $(1+\nu)$,

$(1+\nu)$ quantity to be positive, and $\nu$ must be greater than minus 1. So, we are getting one restriction over the value of Poisson's ratio and one of the Lame constants, as well as the modulus of rigidity, both should be positive. Now, coming to the next case of hydrostatic pressure loading, where the $\tilde{\tilde{\sigma}}$ is given like this: all the diagonal terms are the same, equal to $\sigma$, and the non-diagonal terms are 0. For this particular case, $\sigma_{ij}\sigma_{ij} = 3\sigma^2$, and $\sigma_{kk}$, the trace of sigma, is $3\sigma$.

So, substituting those in the expression of $U$, the strain energy density function can be obtained as $\frac{3(1-2\nu)}{2E}\sigma^2$. And now, using the relation between the bulk modulus and Young's modulus in terms of Poisson's ratio, We know that $K = \frac{E}{3(1-2\nu)}$, and thus $U, U$ can be written for hydrostatic pressure loading as $\sigma^2/2K$. And to ensure positive strain energy stored within the body for hydrostatic loading, this quantity must be greater than 0. And hence, from this equation, $\sigma^2/2K$ greater than 0, we must have $K > 0$. And from this equation, $\frac{3(1-2\nu)}{2E}\sigma^2 > 0$, we should have $(1-2\nu) > 0$, which means $\nu$ should be less than half. So, combining all these conditions derived for different types of loading on the restrictions of the values of elastic moduli,

$$E > 0,\ \mu > 0,\ G > 0,\ \ K > 0,\ \ -1 < \nu < {}^{1}\!/_{2}.$$

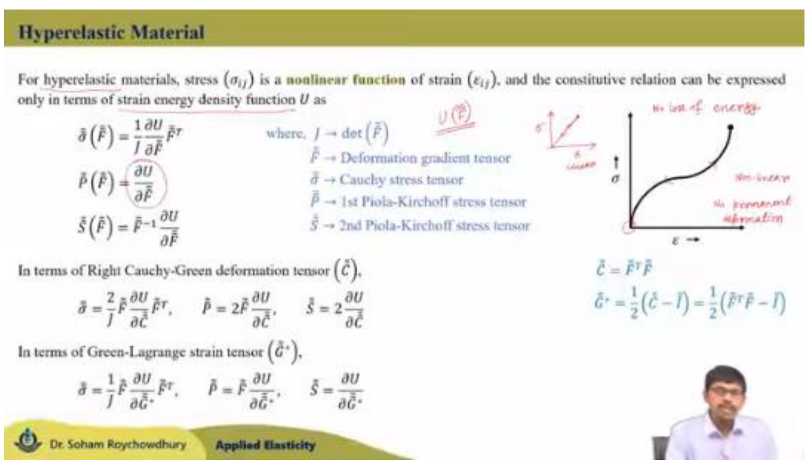



Now, moving forward, this $\nu$ range, which is derived to be between minus 1 to half. Now, a negative value of $\nu$ physically means if you apply a load on the body in one direction, in the other two directions, the two lateral directions, the body will also expand, which is unrealistic in nature, and cannot happen for any kind of engineering problem for commonly available materials unless they are specially designed materials, auxetic or metamaterials. So, we are not including such cases here.

So, in general, negative Poisson's ratio cases are not feasible in real-life applications, and thus the negative part of the range of Poisson's ratio ($\nu$) is dropped, and hence the range of $\nu$ is modified as 0 to half. So, the $\nu$ value must lie between 0 and half. We know that $\nu$ and $\lambda$ are related as $\nu = \frac{\lambda}{2(\lambda+\mu)}$, and from that, by simplifying this, we can write $\lambda = \frac{2\mu\nu}{(1-2\nu)}$. $\mu = G = \frac{E}{2(1+\nu)}$, and thus replacing mu, we can write $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$. Now, we have already shown that $1+\nu$, $1-2\nu$, $\nu$, and $E$, all these four quantities in this expression, all these four quantities are positive.

Thus, $\lambda$ must also be a positive constant, and hence we know that now both the Lamé constants, $\lambda$ and $\mu$, are positive. Three material constants—Young's modulus ($E$), modulus of rigidity ($G$), and bulk modulus ($K$)—must be positive, and the Poisson's ratio should lie between 0 and half. Mathematically, it can be proved that $\nu$ can lie between minus 1 and half.

But from real-life constraints, the negative value of the Poisson's ratio is dropped. Thus, it can lie between this range of 0 to half, which is only in the positive region. So, these are the restrictions on the values of different elastic constants for isotropic linear elastic solids.




Now, from linear elasticity, we will move forward to the constitutive equation of non-linear elastic solids. The non-linear elastic solids are normally named hyperelastic solids. And for such materials, the stress $(\sigma_{ij})$ is a nonlinear function of strain $(\varepsilon_{ij})$, and the constitutive equations become non-linear. Thus, For this kind of material, instead of writing constitutive equations directly, we define the constitutive equations in terms of $U$, that is, the strain energy density function.

So, here $\tilde{\tilde{\sigma}}$ versus $\tilde{\tilde{\varepsilon}}$, the typical stress-strain behavior for a hyperelastic or non-linear elastic solid is shown. Or the variation of $\tilde{\tilde{\sigma}}$ with $\tilde{\tilde{\varepsilon}}$ is non-linear, as you can clearly see here. For the linear elastic solid, this curve was a straight line till the elastic limit for the linear elastic solid.

Whereas, for the hyperelastic or nonlinear elastic solid, this curve looks like this. And note that both during loading and unloading, for linear elastic as well as hyperelastic solid, they will follow the same curve, and after unloading, as it comes back to the origin to the initial configuration, we do not have any permanent deformation for nonlinear solid as well.

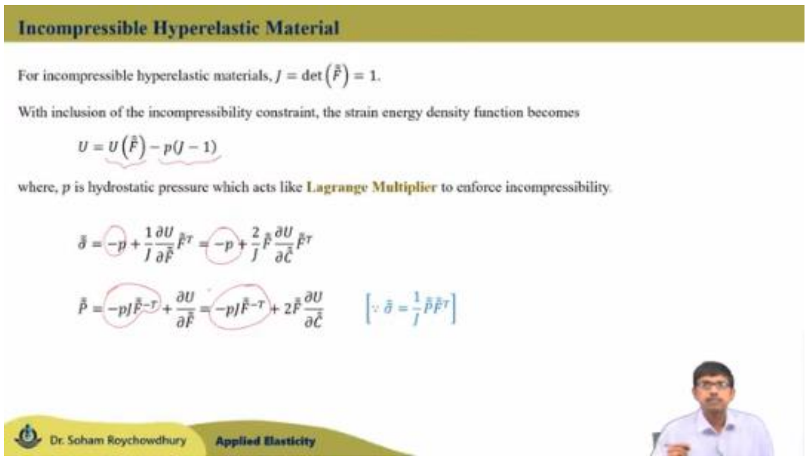
And as the loading and unloading parts are the same, there is no loss of energy as well. which happens for the case of viscoelastic material. For that case, no permanent deformation is there, but the loading path and unloading path are different, some amount of energy is dissipated or lost during the loading and unloading process of viscoelastic material.

So, that is not of our concern. Now, we are going to look into the nonlinear elastic or hyperelastic material constitutive equations. So, for the hyperelastic materials, the stress components are written as the derivative of the strain energy density function with respect to some deformation or strain tensor. So, we can write three different stress components,

$\tilde{\tilde{\sigma}}$, the Cauchy stress tensor; $\tilde{\tilde{P}}$, the first Piola-Kirchhoff stress tensor; and $\tilde{\tilde{S}}$, the second Piola-Kirchhoff stress tensor, like this. So, $\tilde{\tilde{\sigma}}$, which is a function of the deformation gradient tensor $\tilde{\tilde{F}}$, can be written as $\frac{1}{J}\frac{\partial U}{\partial \tilde{\tilde{F}}}\tilde{\tilde{F}}^T$, where $\tilde{\tilde{\sigma}}$ is the Cauchy stress tensor, $\tilde{\tilde{F}}$ is the deformation gradient tensor, and $J$ is the determinant of $\tilde{\tilde{F}}$, that is, the Jacobian of the system. The first Piola-Kirchhoff stress tensor, $\tilde{\tilde{P}}$, can be written as $\frac{\partial U}{\partial \tilde{\tilde{F}}}$, and $\tilde{\tilde{S}}$, the second Piola-Kirchhoff stress tensor, can be written as $\tilde{\tilde{F}}^{-1}\frac{\partial U}{\partial \tilde{\tilde{F}}}$. So, for hyperelastic material, it is extremely important to know the form of $U$ as a function of $\tilde{\tilde{F}}$, and using that, we can find out all these derivative quantities, $\frac{\partial U}{\partial \tilde{\tilde{F}}}$, and then putting it in this form, in this given expression, you can find the corresponding stress component. It may be any one of the Piola-Kirchhoff stresses, or it may be the Cauchy stress.

Now, alternately, instead of $\tilde{\tilde{F}}$, we can write the constitutive equation for hyperelastic solids in terms of $\tilde{\tilde{C}}$, which is the right Cauchy-Green deformation tensor. So, expressions in terms of $\tilde{\tilde{C}}$ become $\tilde{\tilde{\sigma}} = \frac{2}{J}\tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{C}}}\tilde{\tilde{F}}^T$, $\tilde{\tilde{P}} = 2\tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{C}}}$, and $\tilde{\tilde{S}} = 2\frac{\partial U}{\partial \tilde{\tilde{C}}}$, where $\tilde{\tilde{C}} = \tilde{\tilde{F}}^T\tilde{\tilde{F}}$, or the right Cauchy-Green deformation tensor. Similarly, in terms of the Green-Lagrange strain tensor, which is defined in the material coordinate,

With respect to that, we can write the constitutive equations as $\tilde{\tilde{\sigma}} = \frac{1}{J}\tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{G}}^*}\tilde{\tilde{F}}^T$ , $\tilde{\tilde{P}} = \tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{G}}^*}$ , $\tilde{\tilde{S}} = \frac{\partial U}{\partial \tilde{\tilde{G}}^*}$, where $\tilde{\tilde{G}}^*$ or the Green-Lagrange strain tensor is $\frac{1}{2}\left(\tilde{\tilde{C}} - \tilde{\tilde{I}}\right)$ or $\frac{1}{2}\left(\tilde{\tilde{F}}^T\tilde{\tilde{F}} - \tilde{\tilde{I}}\right)$. So, based on the chosen strain or deformation quantity, we can use any one of these three forms of stress definition for writing the constitutive equation of hyperelastic solids.

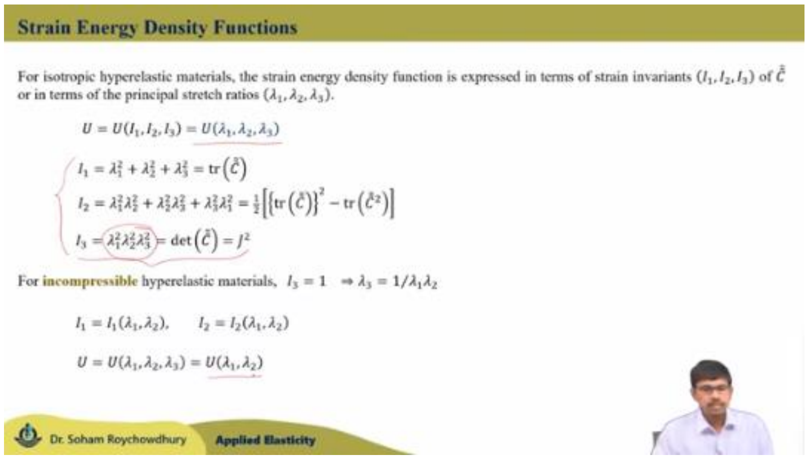



Now coming to the incompressible hyperelastic solids. So mostly the rubber or polymer kinds of materials show this nonlinear constitutive equation or hyperelastic behaviors, and they are mostly incompressible in nature for most cases. So, if you have a material to be incompressible, the Jacobian $J = \det\left(\tilde{\tilde{F}}\right) = 1$, which is deformed volume by undeformed volume.

So, we need to include this additional constraint of $J = 1$ in the strain energy density function, and thus the strain energy density function is modified as $U = U\left(\tilde{\tilde{F}}\right) - p(J-1)$, where $p$ is a hydrostatic pressure which acts like a Lagrange multiplier in this equation. So, this $U\left(\tilde{\tilde{F}}\right)$ was the form of strain energy density for the general compressible material. Along with that, to ensure incompressibility, we are adding this second term $p(J-1)$, where $p$ is any scalar known as a Lagrange multiplier.

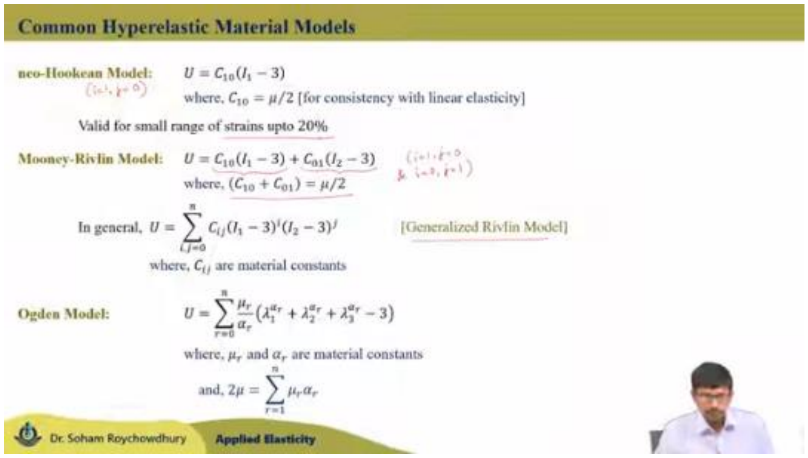
Now, for the incompressible material, we can write the stress components as $\tilde{\tilde{\sigma}}$ equals the Cauchy stress tensor equals $-p + \frac{1}{J}\frac{\partial U}{\partial \tilde{\tilde{F}}}\tilde{\tilde{F}}^T$. In terms of $\tilde{\tilde{C}}$, it can be written as $-p + \frac{2}{J}\tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{C}}}\tilde{\tilde{F}}^T$. And, the first Piola-Kirchhoff stress tensor $\tilde{\tilde{P}}$ can be written as $-pJ\tilde{\tilde{F}}^{-T} + \frac{\partial U}{\partial \tilde{\tilde{F}}}$ or $-pJ\tilde{\tilde{F}}^{-T} + 2\tilde{\tilde{F}}\frac{\partial U}{\partial \tilde{\tilde{C}}}$. So, you can see this first term is the additional term. coming due to the addition of this incompressibility.

The second term in this stress expression was already there for the compressible hyperelastic solids. So, one extra term is coming in the stress equation, which is taking care of the Lagrange multiplier hydrostatic pressure term, which is arising due to the incompressibility of the material.




Now, if you look at the form of the strain energy density function, For the isotropic hyperelastic materials, the strain energy density function is normally expressed in terms of principal strain invariants $I_1, I_2, I_3$ of $\tilde{\tilde{C}}$, the right Cauchy-Green deformation tensor, and they can also be written in terms of $\lambda_1, \lambda_2, \lambda_3$, the principal stretch ratios. So, we know that the $I_1$ first strain invariant is the $\text{tr}\left(\tilde{\tilde{C}}\right)$, the second strain invariant $I_2$ is $\frac{1}{2}\left[\left\{\text{tr}\left(\tilde{\tilde{C}}\right)\right\}^2 - \text{tr}\left(\tilde{\tilde{C}}^2\right)\right]$, then $I_3$ is the $\det\left(\tilde{\tilde{C}}\right)$, which is nothing but the square of the Jacobian of the system. Now, as you can see through these relations, $I_1, I_2, I_3$ are related to $\lambda_1, \lambda_2, \lambda_3$. Those are nothing but the eigenvalues or square of the eigenvalues for $\tilde{\tilde{C}}$, the right Cauchy-Green deformation tensor. Thus, $U$ can be written as a function of $\lambda_1, \lambda_2, \lambda_3$, the three principal stretches of the system. Now, if the material is incompressible, the last one $I_3$, which is equal to $J^2$, would be equal to 1, and $I_3 = 1$, we can write $\lambda_1^2\lambda_2^2\lambda_3^2 = 1$, thereby reducing $\lambda_3 = 1/\lambda_1\lambda_2$. So, in this particular case, all three stretches are not independent.

One of the stretches in the third direction, $\lambda_3$, is related to the other two stretches as $1/\lambda_1\lambda_2$, and hence, $I_1, I_2$ would be functions of only two stretches, $\lambda_1, \lambda_2$, and thus, U would also be a function of $\lambda_1$ and $\lambda_2$, the two principal stretches, and another one is written in terms of these two principal stretches. Now, we would be looking into the typically available forms of common hyperelastic materials.



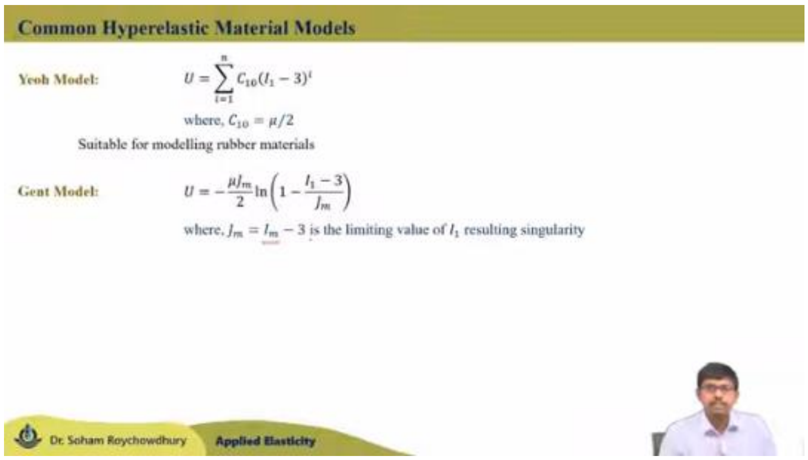

So, several models or constitutive equations are proposed for hyperelastic materials, and we are basically proposing the form of the strain energy density function. These are proposed by different people, and based on them—their names—these models are also named. The first one is called the Neo-Hookean model. where $U = C_{10}(I_1 - 3)$. $I_1$ is the first strain invariant of $\tilde{\tilde{C}}$, and this $C_{10}$ is a material constant.

Now, to have consistency with linear elasticity, this $C_{10}$ is chosen as $\mu/2$. $\mu$, we already know, is one of the Lame constants or $G$, the shear modulus. Now, this particular model, the Neo-Hookean hyperelastic model, is valid for a small range of strain up to 20 percent, not more than that. Till then, it would be giving more or less accurate results. Beyond that, the results will deviate from the actual or experimental values.

Now, moving forward, the next model is the Mooney-Rivlin model, which has two terms. The first term is $C_{10}(I_1 - 3)$, and the second term is $C_{01}(I_2 - 3)$. The first term of the Mooney-Rivlin model is the same as the Neo-Hookean model, whereas the second term is added here, which is the extra term. So, here these two material constants $C_{10}$ and $C_{01}$ are related to $\mu$ using this expression: $(C_{10} + C_{01}) = \mu/2$. So, the Neo-Hookean model had only one material constant involved, whereas the Mooney-Rivlin model has two material constants involved. In general, we can define something called the generalized Rivlin model. which has $n$ number of terms, where $U$ is defined as $\sum_{i,j=0}^{n} C_{ij}(I_1 - 3)^i (I_2 - 3)^j$. $i$ and $j$ can take different values.

Now, taking $i$ to be 1 and $j$ to be 0 from this generalized Rivlin model, the Neo-Hookean was defined. So, $i$ to be 1 and $j$ to be 0 will give Neo-Hookean, and $i$ =1 and $j = 0$; $i$ =1 and $j = 0$ combination of these would give us the two-term Mooney-Rivlin model. $C_{ij}$ are the material constants. You can take additional terms; as you increase the number of terms, the results will be more accurate for higher strain values. Now, the next model is the Ogden model, which is also a series solution or series form of $U$.

$U$ is a series $r = 0$ to $n$, $\sum_{r=0}^{n} \frac{\mu_r}{\alpha_r}\left(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3\right)$, where, $\mu_r$ and $\alpha_r$ are material constants, which are related to shear modulus through this expression: $2\mu = \sum_{r=1}^{n} \mu_r \alpha_r$. Here also, as you increase the number of terms, for higher values of strain, we can better predict the material behavior.




The next model is the Yeoh model, where $U = \sum_{i=1}^{n} C_{10}(I_1 - 3)^i$, where $C_{10} = \mu/2$. This model is suitable for modeling various natural rubbers. Different rubbers or polymers can be modeled correctly with the help of this model. And coming to the last one, that is the Gent model. In this, the strain energy $U = -\frac{\mu J_m}{2} \ln\left(1 - \frac{I_1 - 3}{J_m}\right)$, where $J_m = I_m - 3$, which is the limiting value of $I_1$ resulting in singularity. So here, it is assumed that $I_1$ can go to a highest value of $I_m$, which is the maximum allowable value of $I_1$, and $J_m = I_m - 3$, $I_m - 3$, beyond which the singularity of the material has been reached. So, strain values cannot be increased further after $J_m$ reaches this particular value. $I_1$ is reaching the value of $I_m$, so the phenomenon of stretch locking is seen in some cases of polymers or rubbers, which can be predicted with the help of this particular Gent model. So these are a few models; there are many more, such as Arruda-Boyce and many others. With the help of this, we can predict the nonlinear constitutive behavior of hyperelastic models, and for all of them, a material strain energy density function is required to be involved. So, in this particular lecture, we first discussed the restrictions on the values of different elastic moduli of the material for the case of isotropic solids, then discussed nonlinear elasticity or hyperelasticity, and then looked into the expressions of strain energy density functions for different commonly available hyperelastic material models. Thank you.