**APPLIED ELASTICITY**

**Dr. SOHAM ROYCHOWDHURY**

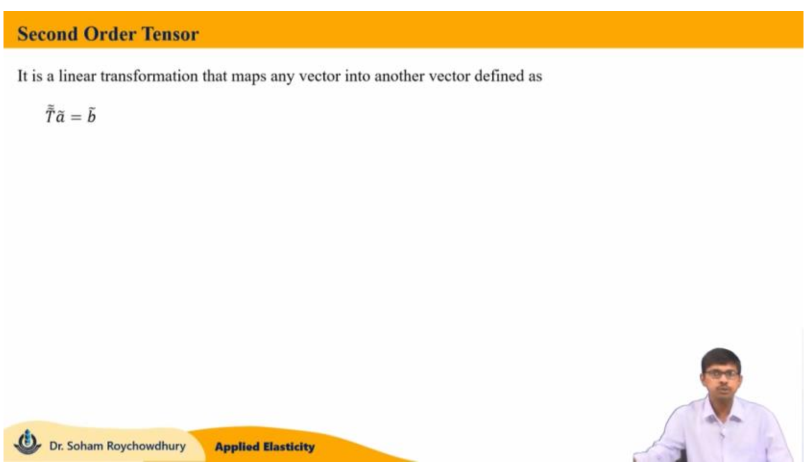
**SCHOOL OF MECHANICAL SCIENCES**

**INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR**

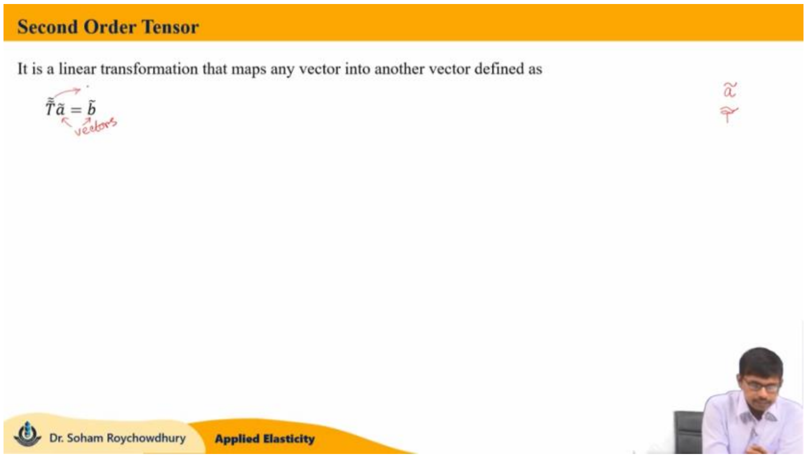
**WEEK: 01**
**Lecture- 02**

Welcome back to the course of applied elasticity. In the previous lecture, we were talking about the introduction to tensor algebra. We will continue with the same topic, tensor algebra, in this particular lecture as well. So in the previous lecture we had discussed about the summation conventions followed by an introduction to the Kronecker delta and permutation symbol.

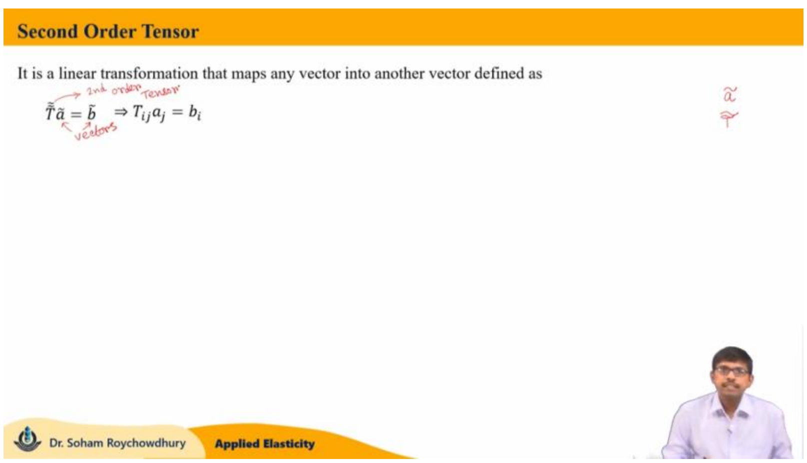
Now, we are going to give the definition of the second-order tensor. So, second-order tensors are basically defined as linear transformations that map one vector into another vector with the help of this particular transformation law. So, here $\tilde{\tilde{T}}$ is a second-order tensor which is acting over a vector a and that results in a new vector $\tilde{b}$. So both $\tilde{a}$ and $\tilde{b}$ are two vectors and here we were defining the vectors with on curl on top. This refers to a vector, whereas second-order tensors

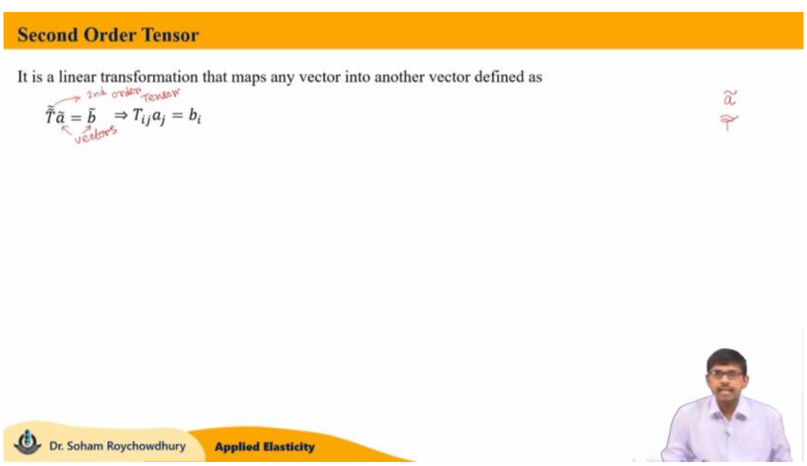


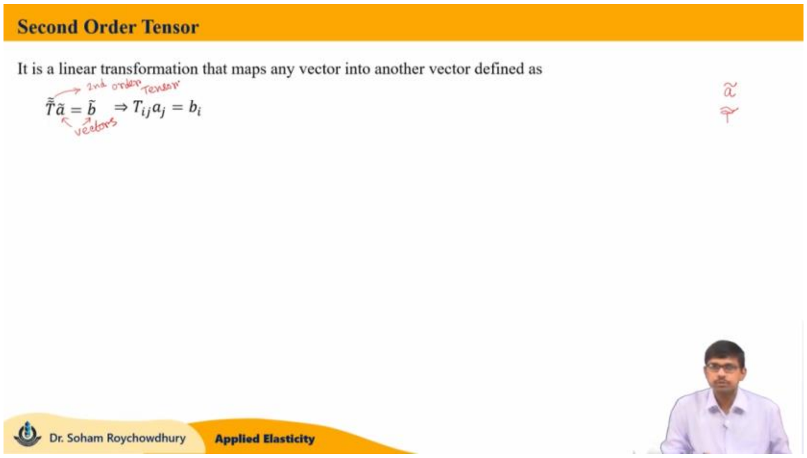

are referred with two curves on top. So, this $\tilde{\tilde{T}}$ is a second-order tensor. So, the definition of a second-order tensor is given by this particular linear transformation $\tilde{\tilde{T}}$ acting on a vector $\tilde{a}$, resulting in another vector $\tilde{b}$. So, you should understand that $\tilde{\tilde{T}}$ is kind of an operator, but it is a linear operator.



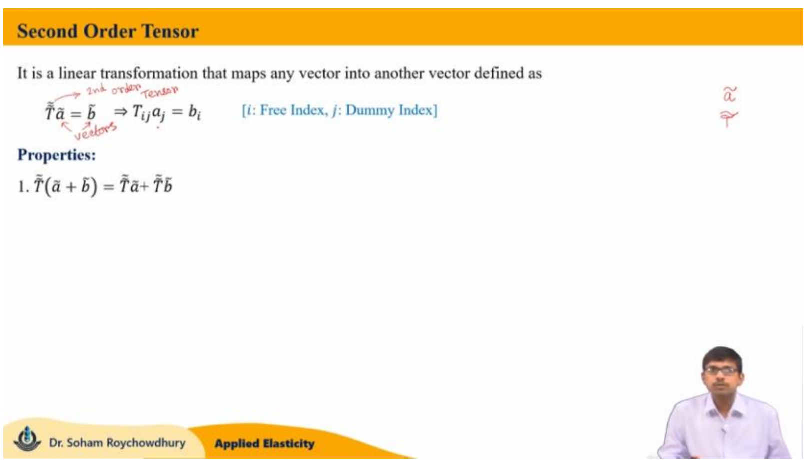

Now, in the With the help of summation convention, this can be written as $T_{ij}a_j$ equals $b_i$, where i is the free index and j is the dummy index. i appears once in all the terms, and j appears twice on the left-hand side term. So, the summation is assumed over j for the left-hand side term, and i is the free index here.

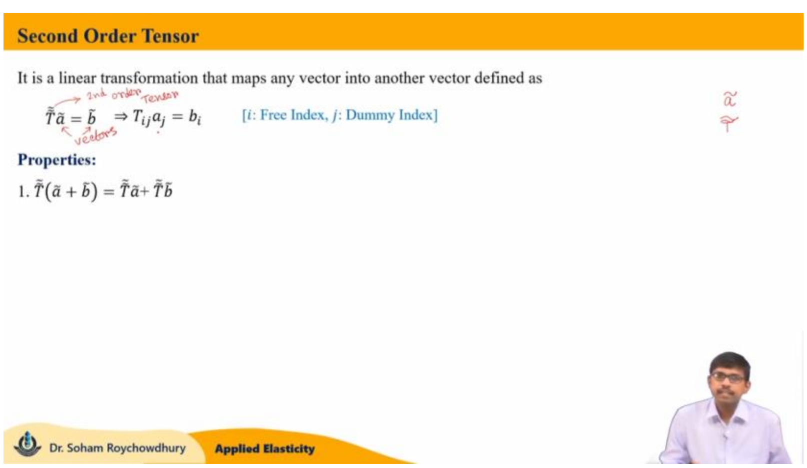


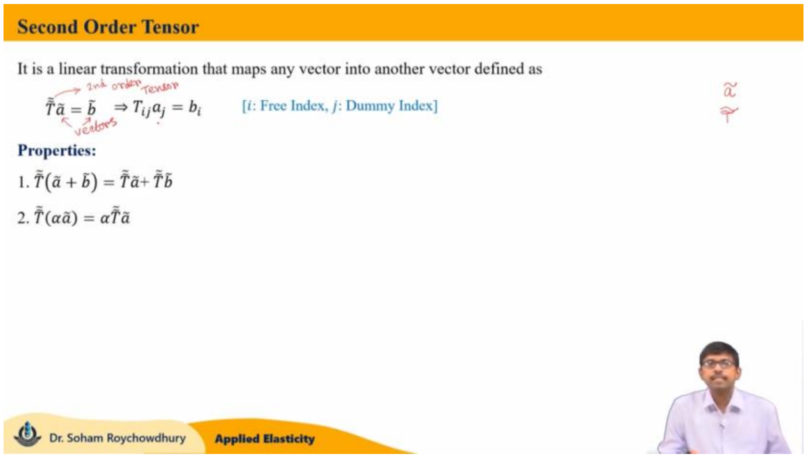






Coming to the properties of this linear transformation or the second-order tensor. As it is a linear transformation, this follows the laws of linear superposition. So, if you consider two vectors $\tilde{a}$ and $\tilde{b}$ and then this tensor $\tilde{\tilde{T}}$ is acting on $\tilde{a} + \tilde{b}$, that will be equal to the summation of $\tilde{\tilde{T}}$ acting over $\tilde{a}$ plus $\tilde{\tilde{T}}$ acting over B, the individual operators. This is the first property of linear superposition,







and the second property is $\tilde{\tilde{T}}$ acting over $\alpha$ times $\tilde{a}$ (where α is a scalar) equals $\alpha$ multiplied by $\tilde{\tilde{T}}$ acting over the original vector $\tilde{a}$ only. So, these two properties are satisfied for the case of second-order tensors, which is basically a linear transformation that transforms any vector into another vector. Now, coming to the components.

So in this equation. These $T_{ij}$ are the components of a tensor $\tilde{\tilde{T}}$. i and j can take values 1, 2, and 3. So, we define the components of a tensor for any chosen base vectors $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$, with the help of this kind of equation, where this tensor $\tilde{\tilde{T}}$ is operating on the unit vectors $\tilde{e}_1$, $\tilde{e}_2$, and $\tilde{e}_3$.

## Second Order Tensor

It is a linear transformation that maps any vector into another vector defined as

$\tilde{\tilde{T}}\tilde{a} = \tilde{b} \Rightarrow T_{ij}a_j = b_i$ [$i$: Free Index, $j$: Dummy Index]

**Properties:**

1. $\tilde{\tilde{T}}(\tilde{a} + \tilde{b}) = \tilde{\tilde{T}}\tilde{a} + \tilde{\tilde{T}}\tilde{b}$

2. $\tilde{\tilde{T}}(\alpha\tilde{a}) = \alpha\tilde{\tilde{T}}\tilde{a}$

**Components of a Tensor:**



## Second Order Tensor

It is a linear transformation that maps any vector into another vector defined as

$\tilde{\tilde{T}}\tilde{a} = \tilde{b} \Rightarrow T_{ij}a_j = b_i$ [$i$: Free Index, $j$: Dummy Index]

**Properties:**

1. $\tilde{\tilde{T}}(\tilde{a} + \tilde{b}) = \tilde{\tilde{T}}\tilde{a} + \tilde{\tilde{T}}\tilde{b}$

2. $\tilde{\tilde{T}}(\alpha\tilde{a}) = \alpha\tilde{\tilde{T}}\tilde{a}$

**Components of a Tensor:**



## Second Order Tensor

It is a linear transformation that maps any vector into another vector defined as

$\tilde{\tilde{T}}\tilde{a} = \tilde{b} \Rightarrow T_{ij}a_j = b_i$ [$i$: Free Index, $j$: Dummy Index]

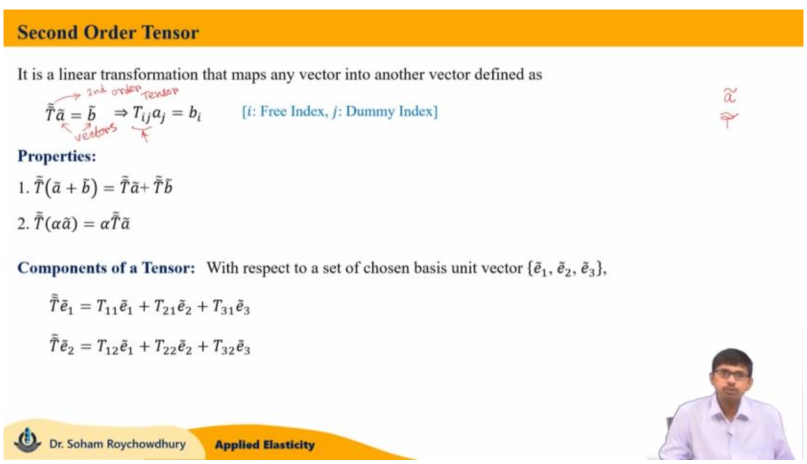
**Properties:**

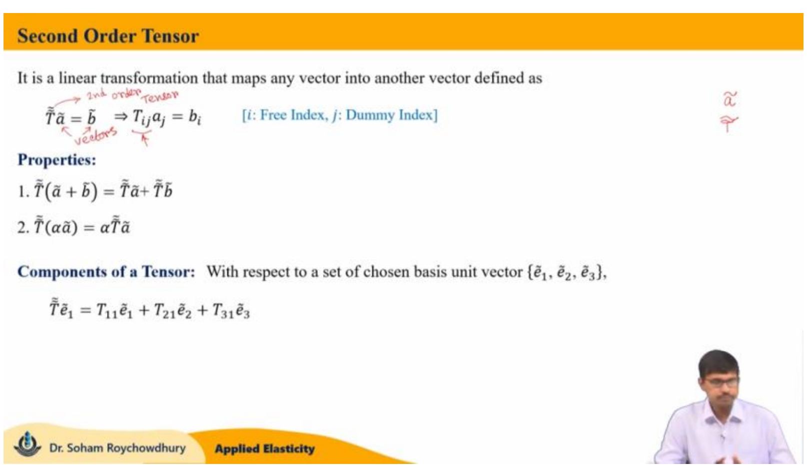
1. $\tilde{\tilde{T}}(\tilde{a} + \tilde{b}) = \tilde{\tilde{T}}\tilde{a} + \tilde{\tilde{T}}\tilde{b}$

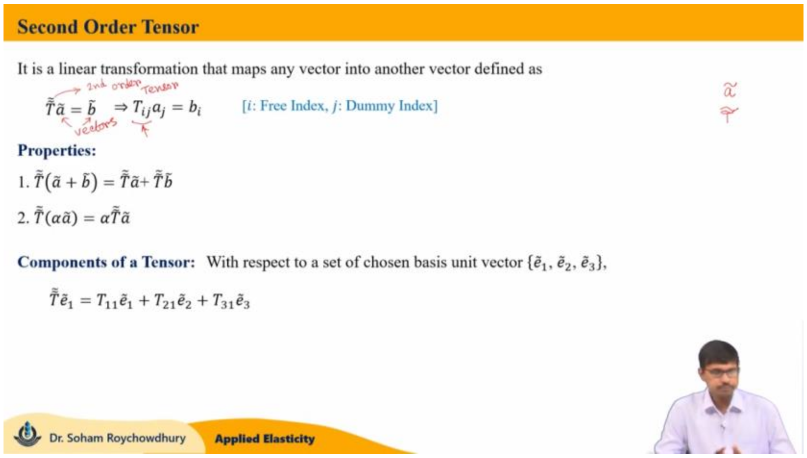
2. $\tilde{\tilde{T}}(\alpha\tilde{a}) = \alpha\tilde{\tilde{T}}\tilde{a}$

**Components of a Tensor:**

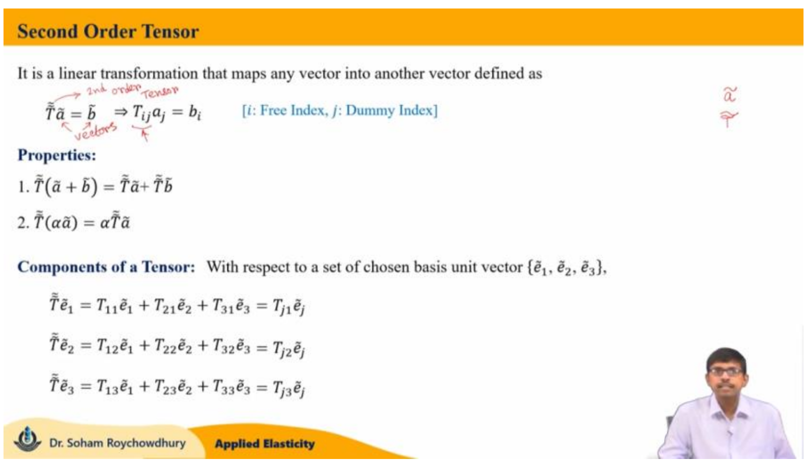

So if $\tilde{\tilde{T}}$ is acting over the unit vector or base vector $\tilde{e}_1$, that results in three terms: $T_{11}\tilde{e}_1 + T_{21}\tilde{e}_2 + T_{31}\tilde{e}_3$. Similarly, if $\tilde{\tilde{T}}$ is operated over the second base vector or unit vector $\tilde{e}_2$, that will result in three extra components: $T_{12}\tilde{e}_1 + T_{22}\tilde{e}_2 + T_{32}\tilde{e}_3$. Similarly, if we act $\tilde{\tilde{T}}$ on $\tilde{e}_3$, the third base vector, then that would result in $T_{13}\tilde{e}_1 + T_{23}\tilde{e}_2 + T_{33}\tilde{e}_3$. So, with the help of these three transformations,

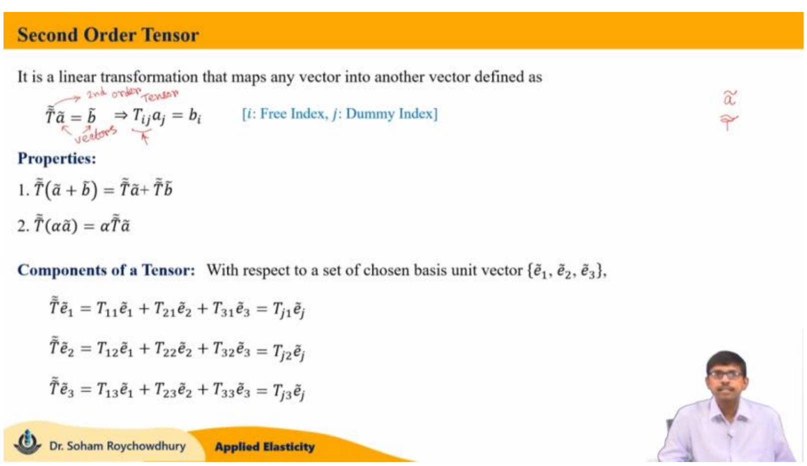


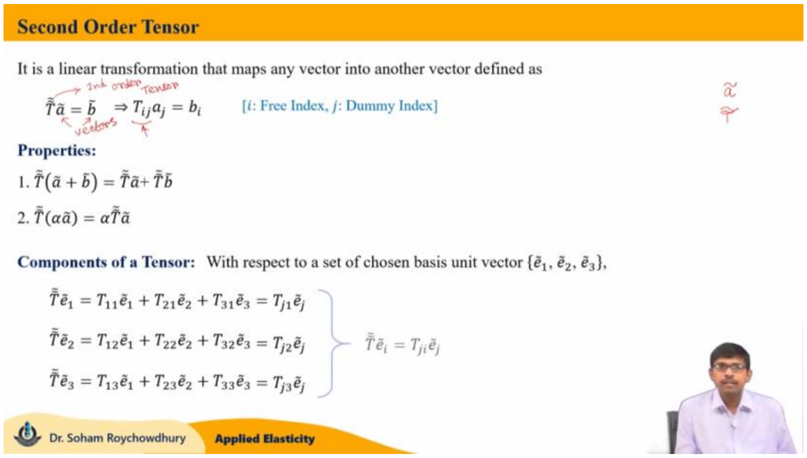





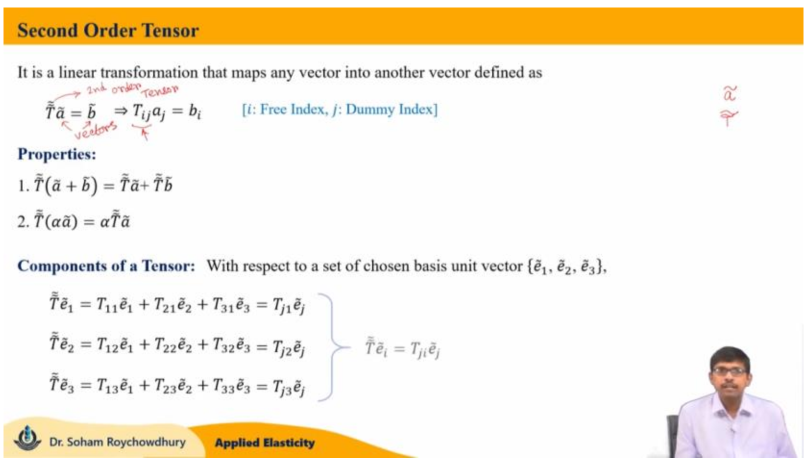

when the tensor is acting over the three base or unit vectors, with the help of that, we can define the tensor components. So, the first equation's three terms can be combined or condensed to a single term with the help of summation convention as $T_{j1}\tilde{e}_j$, whereas the second one is $T_{j2}\tilde{e}_j$, and the third one is $T_{j3}\tilde{e}_j$. And all three of them can be combined into a single tensorial equation as $\tilde{\tilde{T}}$ acting over $\tilde{e}_i$ equals $T_{ji}\tilde{e}_j$.

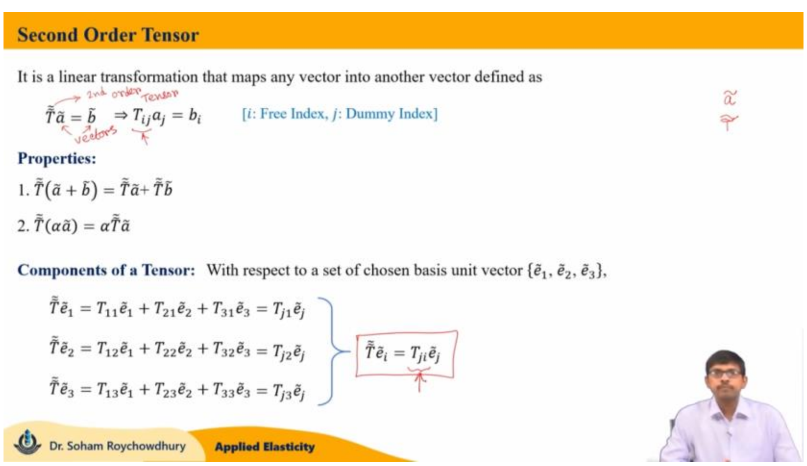


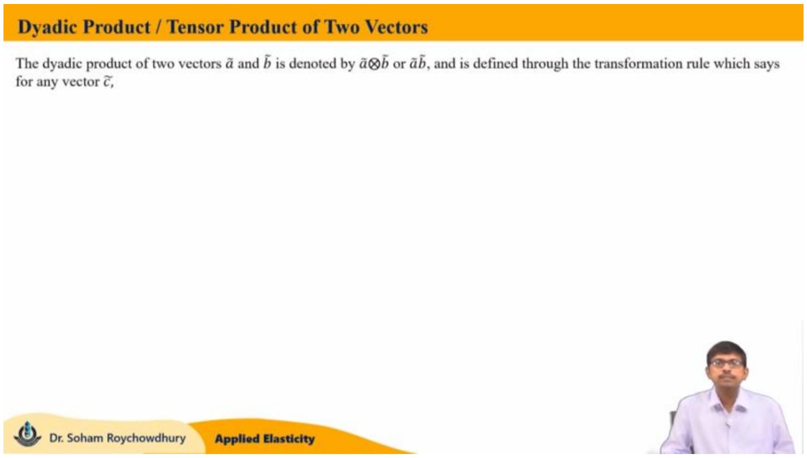


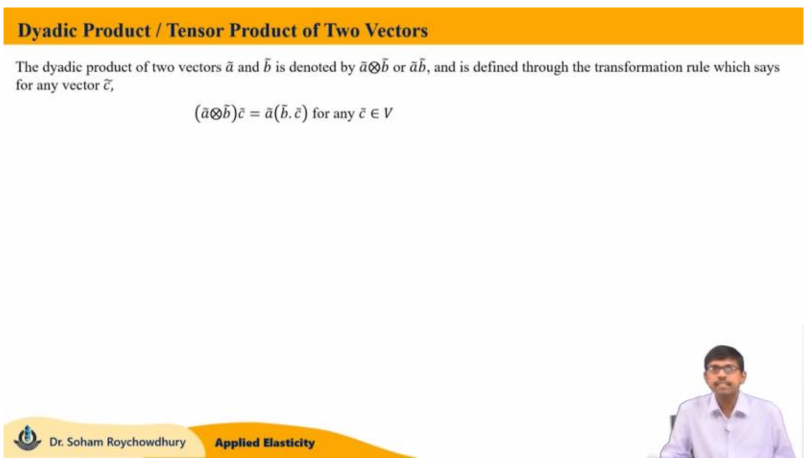

So, with the help of this particular equation, we can define the tensor component $T_{ji}$. So, $\tilde{\tilde{T}}$ acting over unit vector $\tilde{e}_i$ equals to $T_{ji}$ times $\tilde{e}_j$. Now, coming to the definition of dyadic product or tensor product of two vectors.

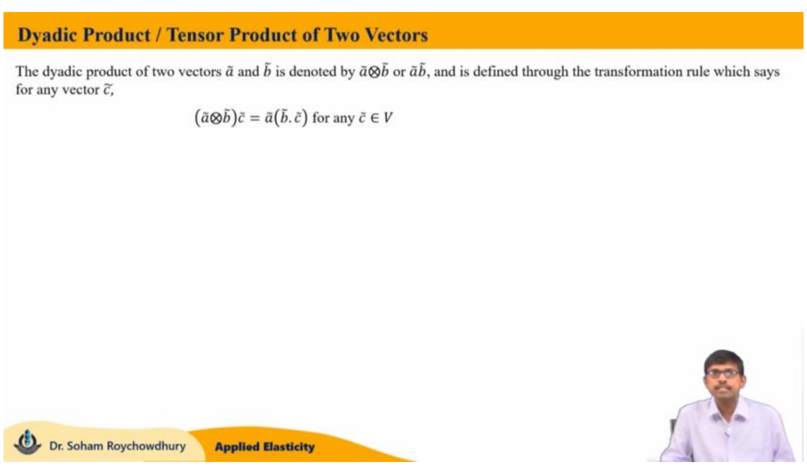


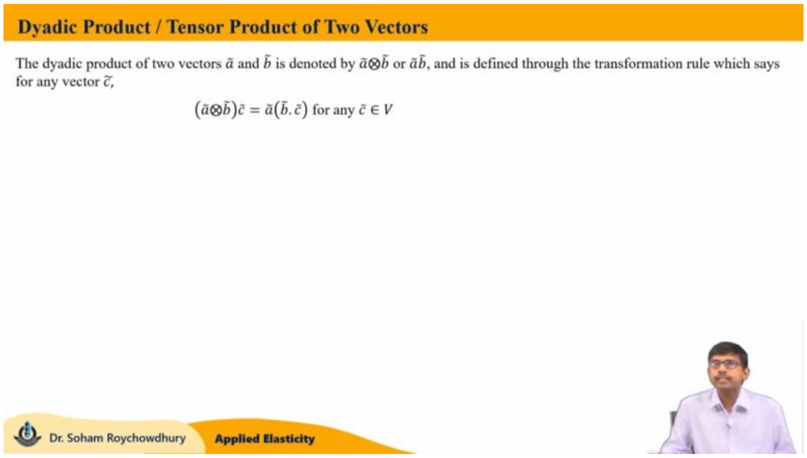
Consider two vectors $\tilde{a}$ and $\tilde{b}$. Their dyadic product or tensor product is defined through this transformation rule for any other vector $\tilde{c}$ defined in the vector space as $(\tilde{a}\otimes\tilde{b})$ acting over $\tilde{c}$ is equals to $\tilde{a}$ times $(\tilde{b}.\tilde{c})$.

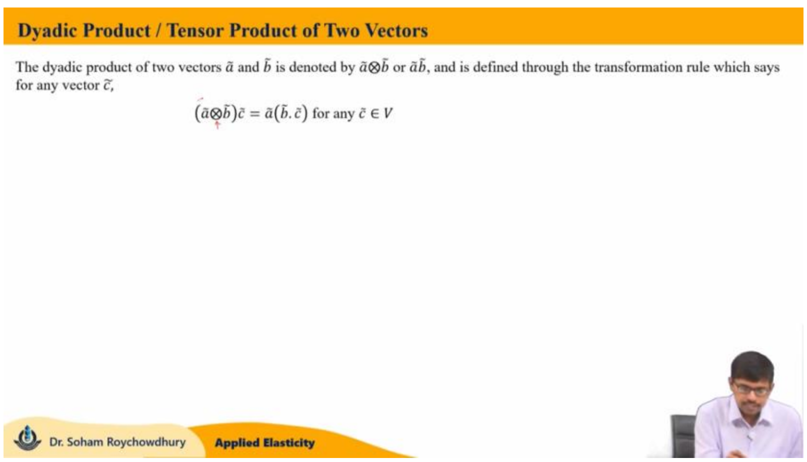


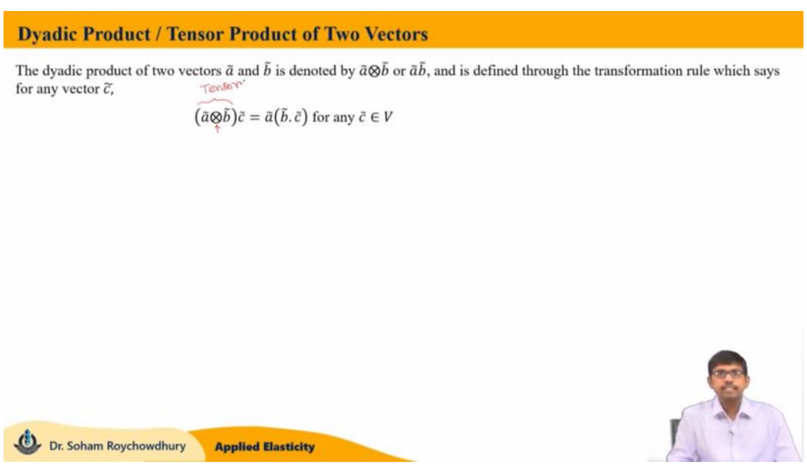



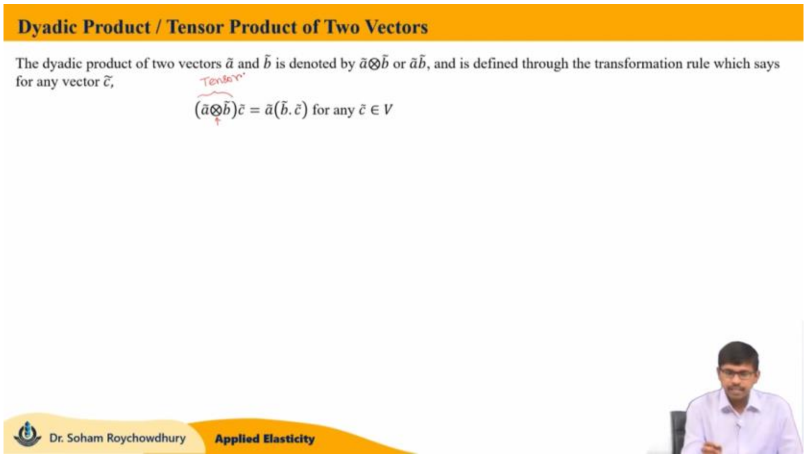
Here $\tilde{c}$ is any arbitrary vector in the vector space and this particular sign, this is called the dyadic product sign. So, $(\tilde{a}\otimes\tilde{b})$ acting over $\tilde{c}$. So, this $(\tilde{a}\otimes\tilde{b})$ is basically a tensor. This is resulting a second order tensor which is acting on any arbitrary vector $\tilde{c}$



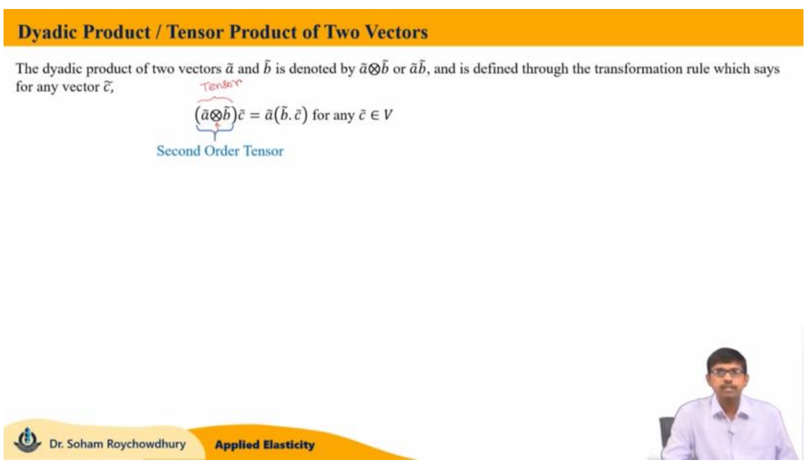



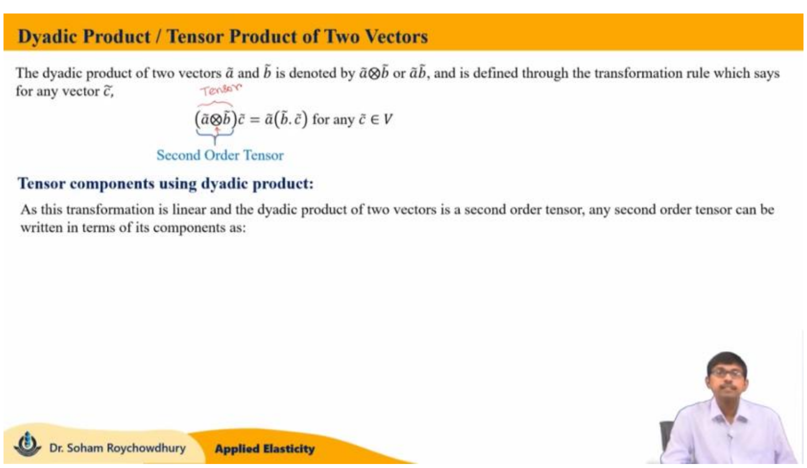




and that would result $\tilde{a}$ times $(\tilde{b}.\tilde{c})$. So, $(\tilde{b}.\tilde{c})$ is a scalar. So, right hand side is a vector. On the left, one second order tensor is acting over a vector. So, left hand side is also a vector. So, dyadic product or tensor product is defined with the help of this particular transformation law,

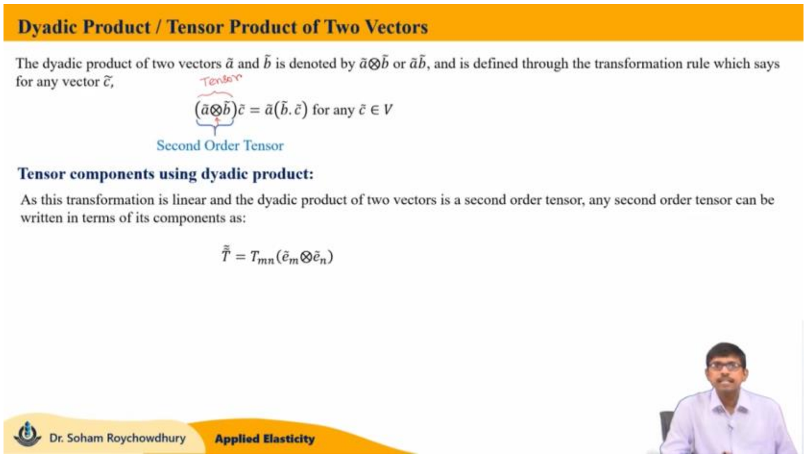


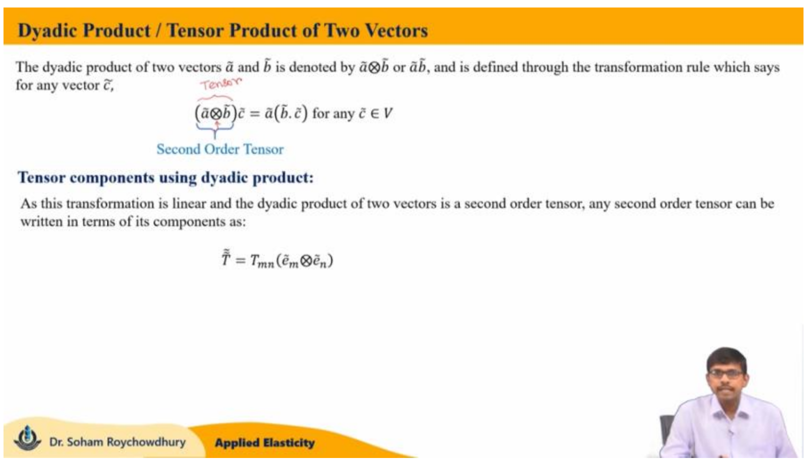

for any two vector $\tilde{a}$ and $\tilde{b}$. Now, with the help of this dyadic product, we can define the components of a tensor. So, as this transformation is linear and the dyadic product of these two vectors $\tilde{a}$ and $\tilde{b}$ is a second order tensor.



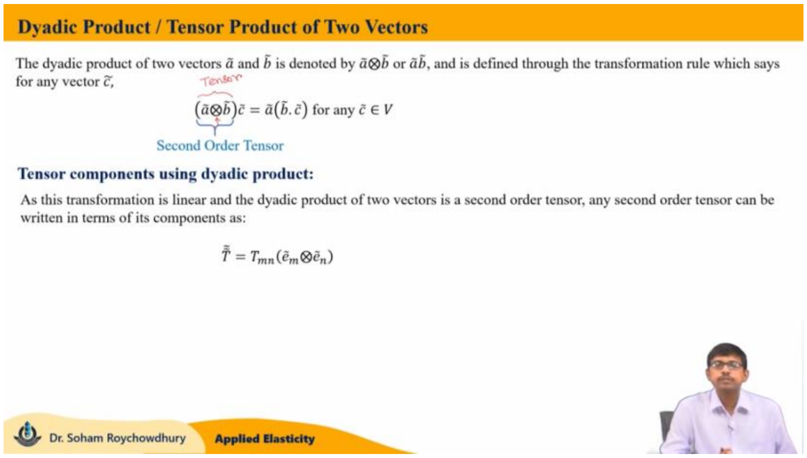



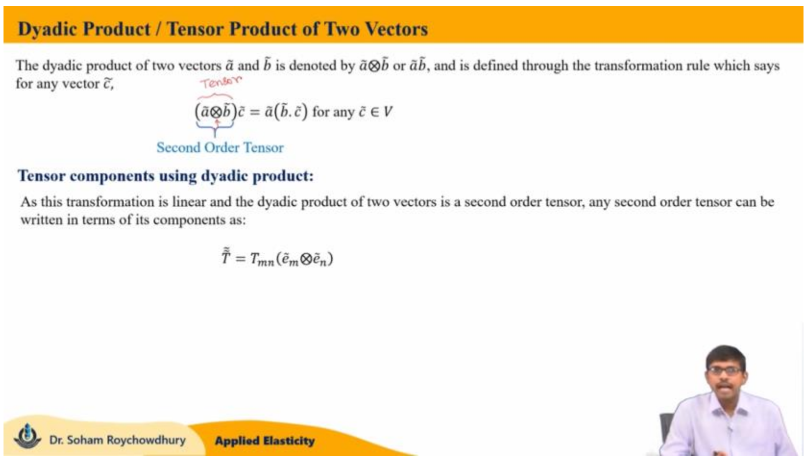
Any second order tensor can be written in terms of its components with the help of dyadic product as $\tilde{\tilde{T}}$ tensor equals to $T_{mn}$ $(\tilde{e}_m \otimes \tilde{e}_n)$ so $\tilde{e}_m$ and $\tilde{e}_n$ are two basis vectors we are taking the dyadic product of two base vector $\tilde{e}_m$ $\tilde{e}_n$

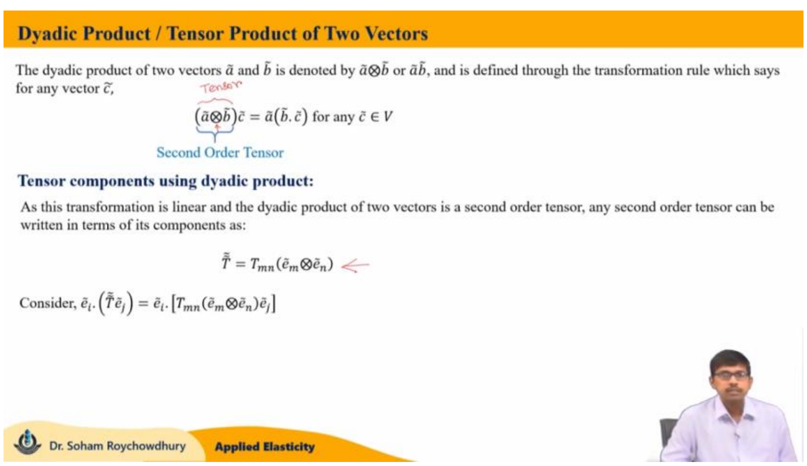



that multiplied with the corresponding tensor component $T_{mn}$, this would results the second order tensor $\tilde{\tilde{T}}$. So, definition of tensor component with the help of dyadic product is given by this particular expression. Now, we will try to prove this with the help of the definition of the dyadic product. So, $\tilde{e}_i$ dot




$\tilde{\tilde{T}}$ acting over $\tilde{e}_j$, we are starting with this expression. Now, $\tilde{\tilde{T}}$ acting over $\tilde{e}_j$ is simplified and for that this particular dyadic product definition of $\tilde{\tilde{T}}$ is replaced. So, $\tilde{\tilde{T}}$ in this equation is written as $T_{mn}$ $(\tilde{e}_m\otimes\tilde{e}_n)$ that is acting over $\tilde{e}_j$ and outside we have dot product of $\tilde{e}_i$. Now, this part can be simplified with the help of definition of the dyadic product. So, that would result $\tilde{e}_i$ dot $T_{mn}$ $\tilde{e}_m$ $\tilde{e}_n$ dot $\tilde{e}_j$.

## Dyadic Product / Tensor Product of Two Vectors

The dyadic product of two vectors $\tilde{a}$ and $\tilde{b}$ is denoted by $\tilde{a}\otimes\tilde{b}$ or $\tilde{a}\tilde{b}$, and is defined through the transformation rule which says for any vector $\tilde{c}$,

Tensor

$$\underbrace{(\tilde{a}\otimes\tilde{b})}_{\text{Second Order Tensor}}\tilde{c} = \tilde{a}(\tilde{b}.\tilde{c}) \text{ for any } \tilde{c} \in V$$

### Tensor components using dyadic product:

As this transformation is linear and the dyadic product of two vectors is a second order tensor, any second order tensor can be written in terms of its components as:

$$\tilde{\tilde{T}} = T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)$$

Consider, $\tilde{e}_i.\left(\tilde{\tilde{T}}\tilde{e}_j\right) = \tilde{e}_i.\left[T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)\tilde{e}_j\right]$



## Dyadic Product / Tensor Product of Two Vectors

The dyadic product of two vectors $\tilde{a}$ and $\tilde{b}$ is denoted by $\tilde{a}\otimes\tilde{b}$ or $\tilde{a}\tilde{b}$, and is defined through the transformation rule which says for any vector $\tilde{c}$,

Tensor

$$\underbrace{(\tilde{a}\otimes\tilde{b})}_{\text{Second Order Tensor}}\tilde{c} = \tilde{a}(\tilde{b}.\tilde{c}) \text{ for any } \tilde{c} \in V$$

### Tensor components using dyadic product:

As this transformation is linear and the dyadic product of two vectors is a second order tensor, any second order tensor can be written in terms of its components as:

$$\tilde{\tilde{T}} = T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)$$

Consider, $\tilde{e}_i.\left(\tilde{\tilde{T}}\tilde{e}_j\right) = \tilde{e}_i.\left[T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)\tilde{e}_j\right]$



## Dyadic Product / Tensor Product of Two Vectors

The dyadic product of two vectors $\tilde{a}$ and $\tilde{b}$ is denoted by $\tilde{a}\otimes\tilde{b}$ or $\tilde{a}\tilde{b}$, and is defined through the transformation rule which says for any vector $\tilde{c}$,

Tensor

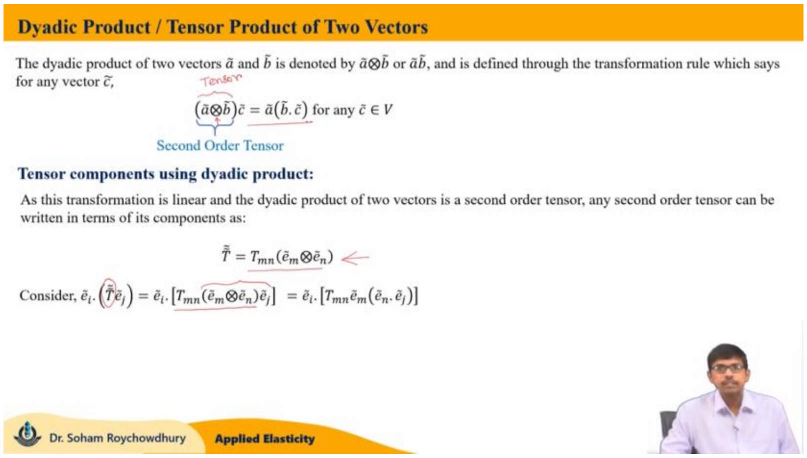
$$\underbrace{(\tilde{a}\otimes\tilde{b})}_{\text{Second Order Tensor}}\tilde{c} = \tilde{a}(\tilde{b}.\tilde{c}) \text{ for any } \tilde{c} \in V$$

### Tensor components using dyadic product:

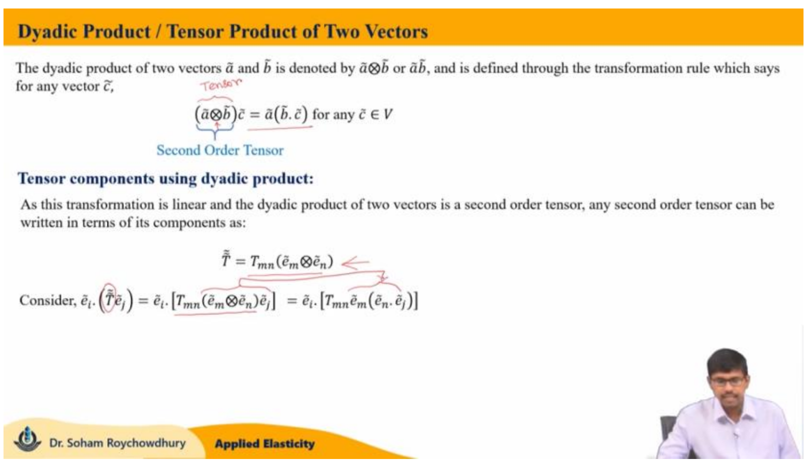
As this transformation is linear and the dyadic product of two vectors is a second order tensor, any second order tensor can be written in terms of its components as:

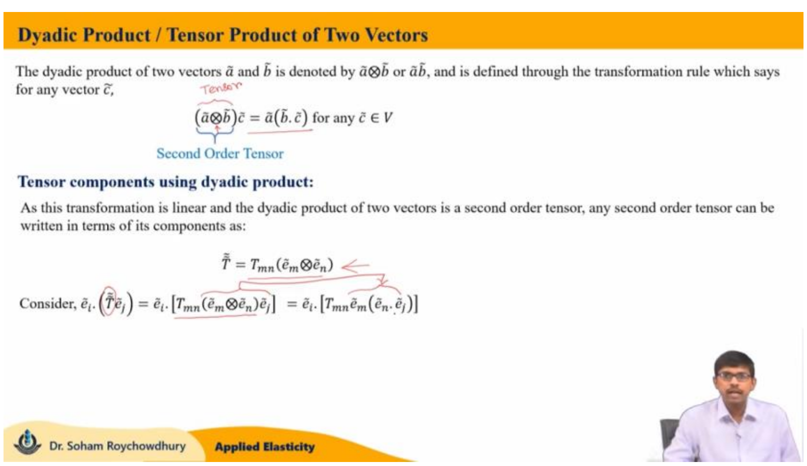
$$\tilde{\tilde{T}} = T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)$$

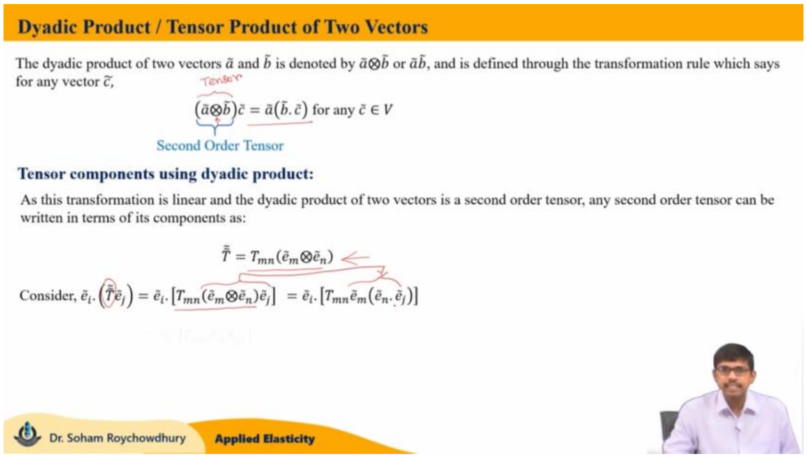
Consider, $\tilde{e}_i.\left(\tilde{\tilde{T}}\tilde{e}_j\right) = \tilde{e}_i.\left[T_{mn}(\tilde{e}_m\otimes\tilde{e}_n)\tilde{e}_j\right]$



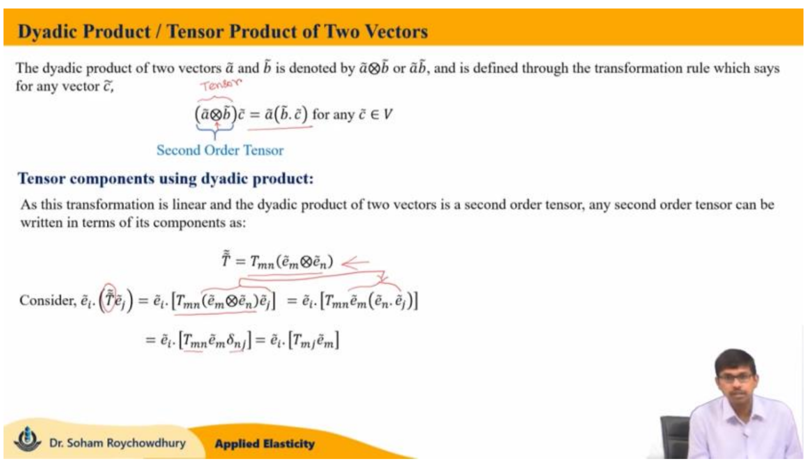

So, these two are equals. So, this part $(\tilde{e}_m\otimes\tilde{e}_n)$, $(\tilde{e}_m\otimes\tilde{e}_n)$ acting over $\tilde{e}_j$ is becoming $\tilde{e}_m$. into $\tilde{e}_n.\tilde{e}_j$ and $\tilde{e}_n.\tilde{e}_j$ dot product of two base vector $\tilde{e}_n$ and $\tilde{e}_j$ is nothing but $\delta_{nj}$ kronecker delta n j so this in total becomes $\tilde{e}_i.[T_{mn}\tilde{e}_m\delta_{nj}]$ now $T_{mn}$ and $\delta_{nj}$ both are scalar quantities and with the help of the property of Kronecker delta,

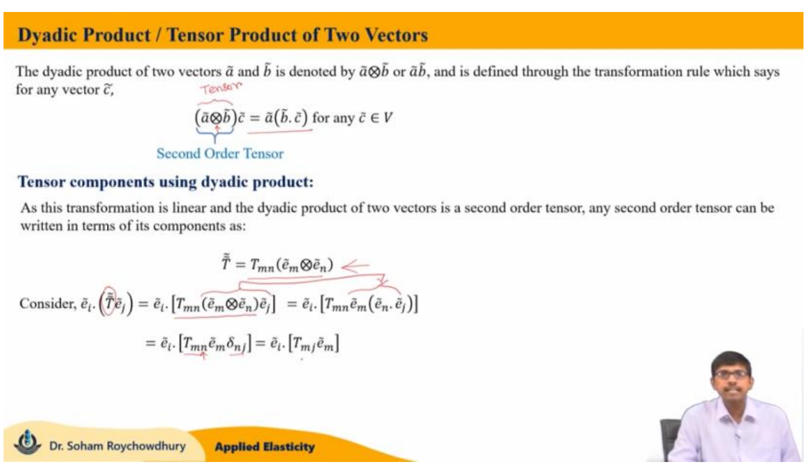


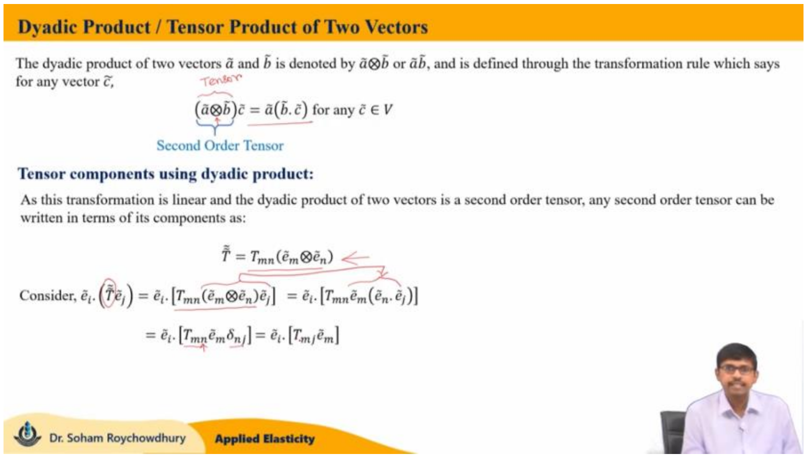



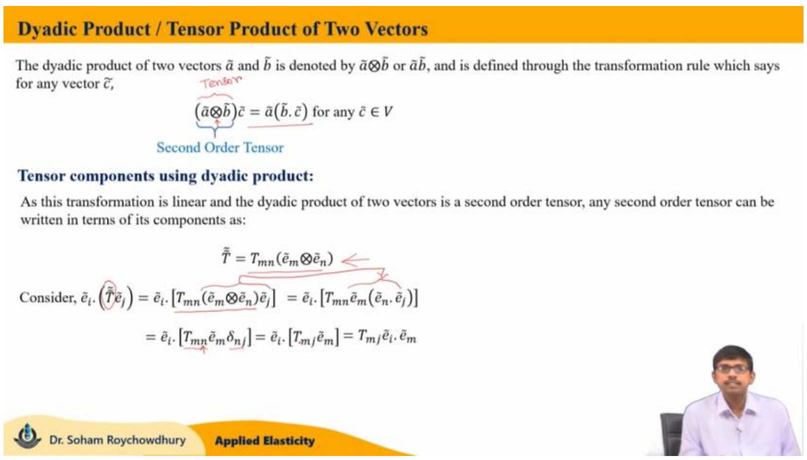


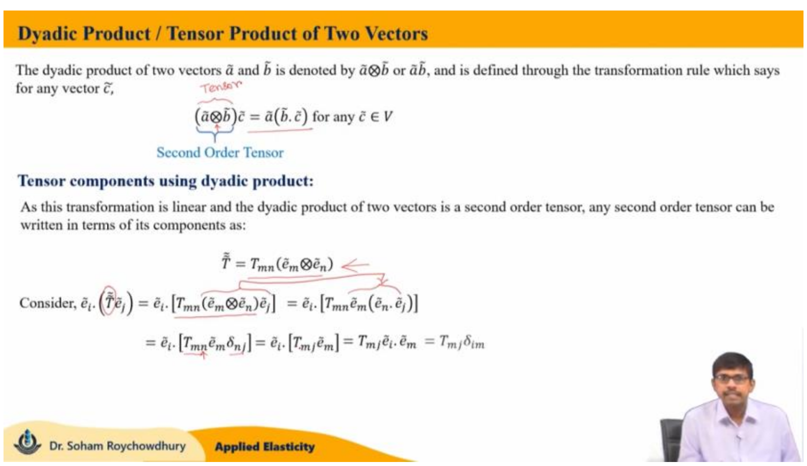

this two $T_{mn}$ and $\delta_{nj}$ can be combined to a single term of $T_{mj}$ $n$ is replaced with $j$ here as it is multiplied with the Kronecker delta. That overall expression becomes $\tilde{e}_i.[T_{mj}\tilde{e}_m]$. Once again, $T_{mj}$ is a constant that we can take out and $\tilde{e}_i.\tilde{e}_m$. This is nothing but $\delta_{im}$. So, $T_{mj}$ times $\delta_{im}$ once again using the property of Kronecker delta this is



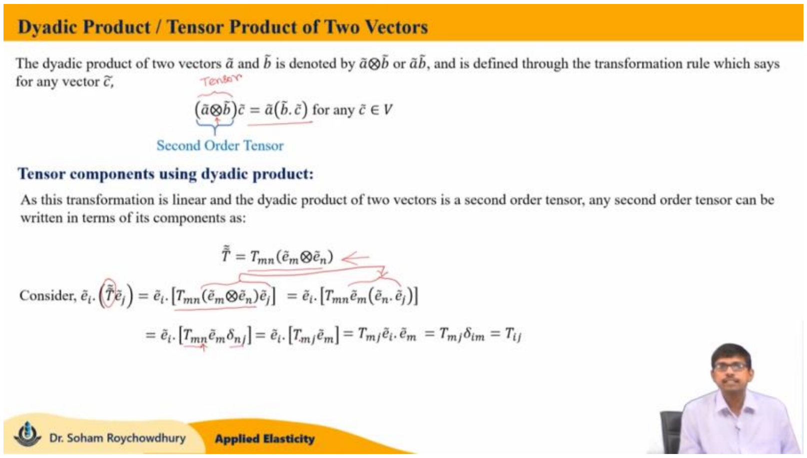






equals to $T_{ij}$. So, $\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_j$ this is equals to $T_{ij}$ for any tensor its component $T_{ij}$ can be written as unit vector $\tilde{e}_i$ dot the capital $\tilde{\tilde{T}}$. tensor acting over $\tilde{e}_j$. This expression defines the component of any tensor which can be which can be obtained with the help of dyadic product. Now, coming to the definition of identity tensor. So, identity tensor is a tensor which transforms a vector into the same vector and it is denoted with the help of this $\tilde{\tilde{I}}$. So, this $\tilde{\tilde{I}}$ is used to denote the identity tensors.




Now, by definition as the identity tensors transforms any vector into the same vector. So, let us consider a vector $\tilde{a}$, identity tensor acting over a vector $\tilde{a}$ would result the same vector $\tilde{a}$ as written here, $\tilde{\tilde{I}}$ acting over $\tilde{a}$ is equals to $\tilde{a}$. Now, by the definition of tensor components, $T_{ij}$ is $\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_j$. So, similarly, we can write $I_{ij}$, the $\tilde{e}_i$ $\tilde{e}_j$ component of identity tensor $\tilde{\tilde{I}}$ as $\tilde{e}_i$ unit vector dot capital $\tilde{\tilde{I}}$ identity tensor acting over $\tilde{e}_j$. Now, by definition of identity tensor, $\tilde{\tilde{I}}$ acting over $\tilde{e}_j$ would result only $\tilde{e}_j$.

So, this expression would be $\tilde{e}_i$ dot $\tilde{e}_j$, which is Kronecker's delta, $\delta_{ij}$. So, the components of the identity tensor capital $\tilde{\tilde{I}}$ are nothing but the components of the Kronecker delta, $\delta_{ij}$. Now, coming to different tensor operations. So, the first one is the summation of two tensors. Any two tensors having the same order can be summed up, and that would result in another tensor of the same order.

So, $\tilde{\tilde{T}}$ and $\tilde{\tilde{S}}$ being two second-order tensors, there summation should result in a second order tensor $\tilde{\tilde{T}} + \tilde{\tilde{S}} = \tilde{\tilde{W}}$ and in the indicial notation it can be written as $T_{ij} + S_{ij} = W_{ij}$ If this $\tilde{\tilde{T}} + \tilde{\tilde{S}}$, the summation of two tensors, is acting over any vector $\tilde{a}$, that would be equal to the two individual tensors $\tilde{\tilde{T}}$ and $\tilde{\tilde{S}}$ acting over $\tilde{a}$ individually and then summing them up. So, $\tilde{\tilde{T}} + \tilde{\tilde{S}}$ acting over $\tilde{a}$ equals $\tilde{\tilde{T}}$ acting over $\tilde{a}$ plus $\tilde{\tilde{S}}$ acting over $\tilde{a}$. And also, this can be written as $\tilde{\tilde{W}}$ acting over $\tilde{a}$.

This will result in one vector, and then $\tilde{\tilde{T}}$ will be acting on that vector. On the other hand, if $\tilde{\tilde{S}}$ appears first in the product, $\tilde{\tilde{S}}$ times $\tilde{\tilde{T}}$ acting over $\tilde{a}$ is $\tilde{\tilde{S}}$ acting over $\tilde{\tilde{T}}$ acting over $\tilde{a}$. So, out of these two tensors whose product we are taking, any one will be picked up; the second one will be acting on the vector first, and then that output will be subjected to the second tensor operation. Now, $\tilde{\tilde{T}}$ into $S_{ij}$, the component of the product of two tensors $\tilde{\tilde{T}}\tilde{\tilde{S}}$, to obtain this, we first use the basic definition of tensor components: $\tilde{e}_i.\left[\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)\tilde{e}_j\right]$. This particular part $\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)\tilde{e}_j$ is written as $\tilde{\tilde{S}}$ acting over $\tilde{e}_j$ and then $\tilde{\tilde{T}}$ operator is acting over that. Now, $\tilde{\tilde{S}}$ acting over $\tilde{e}_j$ can be written as $S_{kj}\tilde{e}_k$ by definition of the tensor components acting over the unit vectors. Now, $S_{kj}$ is a scalar here that we can take out and this $\tilde{\tilde{T}}$ operator will be acting over $\tilde{e}_k$. So, if we are taking out $S_{kj}$, then it would be $S_{kj}\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_k$.

Now, this total thing $\tilde{e}_i$ dot $\tilde{\tilde{T}}$ acting over $\tilde{e}_k$, this is nothing but the tensor component $T_{ik}$ by definition of the tensor components. $\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)_{ij}$ the components of the product of two tensor $\tilde{\tilde{T}}$ and $\tilde{\tilde{S}}$ this is equals to $T_{ik}S_{kj}$. So, the first index of the first tensor is same as the first index of the product. Second index of the last tensor is same as the second index of the product. Whereas, second index of the first tensor and first index of the second tensor,

these two are same and that is a dummy index. So, $\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)_{ij} = T_{ik}S_{kj}$. This is the form of product of two tensor with the help of the inditial notation or its components. Now coming to the transpose of a tensor. So transpose of any tensor $\tilde{\tilde{T}}$ which is denoted by $\tilde{\tilde{T}}$ transpose another capital $\tilde{\tilde{T}}$ written on the superscript. This is defined to be a tensor which satisfies the following identity for any two vectors $\tilde{a}$ and $\tilde{b}$. If $\tilde{a}$ dot $\tilde{\tilde{T}}$ acting over $\tilde{b}$,

can be written as $\tilde{b}$ dot $\tilde{\tilde{T}}$ transpose acting over $\tilde{a}$, then $\tilde{\tilde{T}}$ and $\tilde{\tilde{T}}^T$ transpose are the transposes of each other. Now, $\tilde{a}$ and $\tilde{b}$ being two arbitrary vectors, we can choose them to be anything. So, let us choose $\tilde{a}$ as $\tilde{e}_j$ and $\tilde{b}$ as $\tilde{e}_i$, two unit vectors. And replacing them in this particular identity, $\tilde{e}_i.(\tilde{\tilde{T}}^T\tilde{e}_j)$ and using the definition of tensor components, the left-hand side term can be written as $T_{ji}$. The right-hand term can be written as $T^T_{ij}$. $(\tilde{\tilde{T}}^T)_{ij}$ is equal to $(\tilde{\tilde{T}})_{ji}$; basically, here we are flipping the two indices. $i$ is going in the location of $j$,

$j$ is going in the location of $i$. And also, the transpose of the product of two tensors $\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)^T$ follows this particular identity: $\tilde{\tilde{S}}^T\tilde{\tilde{T}}^T$. So, $\left(\tilde{\tilde{T}}\tilde{\tilde{S}}\right)^T$ is equal to $\tilde{\tilde{S}}^T\tilde{\tilde{T}}^T$. So, the order in which these two tensors appear on the right-hand side is flipped. Now, coming to the definition of the inverse of a tensor. For any given tensor $\tilde{\tilde{T}}$,

if another tensor $\tilde{\tilde{S}}$ exists such that product of $\tilde{\tilde{S}}$ and $\tilde{\tilde{T}}$ is equals to an identity tensor I, then we call that $\tilde{\tilde{S}}$ to be inverse of $\tilde{\tilde{T}}$. So, $\tilde{\tilde{S}}$ times $\tilde{\tilde{T}}$ is equals to I. $\tilde{\tilde{S}}$ equals to $\tilde{\tilde{T}}$ inverse. If we multiply both the sides with $\tilde{\tilde{T}}$, the left hand side will become $\tilde{\tilde{S}}$ times $\tilde{\tilde{T}}$, whereas right hand side will become $\tilde{\tilde{T}}$ inverse $\tilde{\tilde{T}}$ that is equals to identity tensor $\tilde{\tilde{I}}$

.

So, this is the definition of tensor inverse and this exists only if determinant of this second order tensor $\tilde{\tilde{T}}$ whose inverse we are taking is not equals to 0. If determinant is equals to 0, then tensor inverse cannot be defined. Now, coming to the definition of an orthogonal tensor. An orthogonal tensor $\tilde{\tilde{Q}}$ is defined as a linear transformation.

for which the transform vectors preserve their length as well as the internal angle between them. Let us consider two vectors $\tilde{a}$ and $\tilde{b}$. Now by definition of this orthogonal tensor, if we are having $\tilde{\tilde{Q}}$ acting on $\tilde{a}$. So, a was the original vector magnitude of $\tilde{a}$ should be equals to magnitude of $\tilde{\tilde{Q}}$ acting over $\tilde{a}$. So, $\tilde{a}$ was the original vector.

$\tilde{\tilde{Q}}$ acting over $\tilde{a}$ is the transformed vector. The magnitude of both of them must be the same by the definition of an orthogonal tensor. So, the magnitude of $\tilde{\tilde{Q}}\tilde{a}$ is equal to the magnitude of $\tilde{a}$. Now, coming to the second point or second definition, which is that the angles between two vectors are also preserved. So, let us take two vectors $\tilde{a}$ and $\tilde{b}$, and their angle is basically defined as their dot product.

So, $\tilde{a}.\tilde{b}$ is equal to $\tilde{\tilde{Q}}$ times $\tilde{a}$ dot $\tilde{\tilde{Q}}$ times $\tilde{b}$. If these two transformations are satisfied, then $\tilde{\tilde{Q}}$ can be defined as an orthogonal tensor. Now, if you simplify the second expression, the second point. $\tilde{a}$ dot $\tilde{b}$ can be written as $a_i b_i$, whereas $\tilde{\tilde{Q}}$ acting over a dot $\tilde{\tilde{Q}}$ acting over $\tilde{b}$ can be simplified as $Q_{ij}a_j$ dot $Q_{ik}b_k$.

Now, here, writing $a_j$ at the beginning and $b_k$ at the end. It can be written as $a_j Q_{ij} Q_{ik} b_k$. Now, this particular term, $i$ and $j$ locations are now flipped by the definition of transpose, and we are writing this as $a_j {Q_{ji}}^T Q_{ik} b_k$. Now, the left-hand side of this equation $a_i b_i$ to this particular expression.

Now, right hand side is taken towards the left hand side. So, $a_i b_i - a_j {Q_{ji}}^T Q_{ik} b_k$ equals to 0. Now, in the first term, this aibi, this can be written as ajbj because $i$ and $j$ both are dummy index. This can also be written as $a_j \delta_{jk} b_k$ with the help of a Kronecker delta where $j$ and $k$ are identical. So, the first term is written as $a_j \delta_{kj} b_k$ minus second term remains as it is.

Now, taking $a_j$ and $b_k$ common and for any arbitrary vector $\tilde{a}$ and $\tilde{b}$ for this equation to be 0 we must have $\delta_{kj}$ equals to ${Q_{ji}}^T Q_{ik}$. That means, $\tilde{\tilde{Q}}^T \tilde{\tilde{Q}}$ is equals to identity tensor. This is the definition of orthogonal tensor. For any orthogonal tensor $\tilde{\tilde{Q}}$, this identity must be satisfied. $\tilde{\tilde{Q}}$ transpose $\tilde{\tilde{Q}}$ is equals to identity tensor.

and from that taking $\tilde{\tilde{Q}}$ on the right hand side keeping only $\tilde{\tilde{Q}}$ transpose on the left $\tilde{\tilde{Q}}$ transpose can be shown to be equals to $\tilde{\tilde{Q}}$ inverse. This is also another property of the orthogonal tensor. Now $\tilde{\tilde{Q}}\tilde{\tilde{Q}}^T$ that is equals to $\tilde{\tilde{Q}}\tilde{\tilde{Q}}^T$ is equals to $\tilde{\tilde{I}}$. If we take the determinant of this equation on both the sides. Left hand side will be product of two determinants, determinant of $\tilde{\tilde{Q}}$ and determinant of $\tilde{\tilde{Q}}^T$, whereas on the right hand side determinant of identity tensor $\tilde{\tilde{I}}$ equals to unity.

Now, the determinant of $\tilde{\tilde{Q}}$ is the same as the determinant of $\tilde{\tilde{Q}}$ transpose, and thus the left-hand side is the square of the determinant of $\tilde{\tilde{Q}}$, and that is equal to 1. So, the determinant of $\tilde{\tilde{Q}}$, the determinant of an orthogonal tensor, can be either plus 1 or minus 1. Now, if the determinant of $\tilde{\tilde{Q}}$ is plus 1, then physically $\tilde{\tilde{Q}}$ refers to a rotation tensor. If the determinant is equal to minus 1, then $\tilde{\tilde{Q}}$ refers to a reflection tensor. Now, after discussing orthogonal tensors, we are going to talk about symmetric and anti-symmetric tensors.

Any tensor $\tilde{\tilde{T}}$ can be called a symmetric tensor if $\tilde{\tilde{T}}$ is the same as its transpose. So, if $\tilde{\tilde{T}}$ equals $\tilde{\tilde{T}}$ transpose, we call $\tilde{\tilde{T}}$ to be a symmetric tensor, and in the indicial notation that can be expressed as $T_{ij}$ is equals to $T_{ji}$ $\tilde{\tilde{T}}$ can be called an anti-symmetric tensor if $\tilde{\tilde{T}}$ is equal to minus or negative of $\tilde{\tilde{T}}$ transpose, and thus the $T_{ij}$ component is equal to minus of $T_{ji}$. So, if $\tilde{\tilde{T}}$ and $\tilde{\tilde{T}}$ transpose are the same, then it is called a symmetric tensor.

If $\tilde{\tilde{T}}$ is equal to the negative of $\tilde{\tilde{T}}$ transpose, then it is called an anti-symmetric tensor. Now, any tensor $\tilde{\tilde{T}}$ can be written as a summation or superposition of one symmetric tensor and one anti-symmetric tensor. So, consider any tensor $\tilde{\tilde{T}}$, and that is the sum of two tensors: one is $\tilde{\tilde{T}}^S$, another is $\tilde{\tilde{T}}^A$. $\tilde{\tilde{T}}^S$ stands for a symmetric tensor, and $\tilde{\tilde{T}}^A$ stands for an anti-symmetric tensor. So, $\tilde{\tilde{T}}^S$ plus $\tilde{\tilde{T}}^A$ will give the total $\tilde{\tilde{T}}$. Any tensor can be written in terms of $\tilde{\tilde{T}}^S$ plus $\tilde{\tilde{T}}^A$, where

$\tilde{\tilde{T}}^S$ is the symmetric part, which is defined as $\tilde{\tilde{T}}$ plus $\tilde{\tilde{T}}$ transpose divided by 2, and $\tilde{\tilde{T}}^A$ is the antisymmetric part, which is defined as $\tilde{\tilde{T}}$ minus $\tilde{\tilde{T}}$ transpose divided by 2. Now, corresponding to all the anti-symmetric tensors, we can define a vector quantity known as the dual vector for that anti-symmetric tensor. So, let us say this small $t$ with superscript $a$, this is the dual vector for the anti-symmetric tensor capital $\tilde{\tilde{T}}$.

So, we are considering one anti-symmetric tensor capital $\tilde{\tilde{T}}$, which has a dual vector small $t$ with superscript $a$. Now, how is this dual vector defined? If small $\tilde{a}$ is any other vector, and if I operate the anti-symmetric tensor capital $\tilde{\tilde{T}}$ on any vector small $\tilde{a}$, that can be expressed as small $t$, the dual vector, cross small $\tilde{a}$, the arbitrary vector. If this relation is valid for such cases, small t can be named or called the dual vector of the anti-symmetric tensor capital $\tilde{\tilde{T}}$.

So, for any vector $\tilde{a}$, if this expression is valid, then small $t$ with superscript capital $A$ is the dual vector of the anti-symmetric tensor $\tilde{\tilde{T}}$. Now, if I expand this in matrix form, all these three equations if I write then component wise the antisymmetric tensor $\tilde{\tilde{T}}$ can be written like this, for the antisymmetric tensor $\tilde{\tilde{T}}$ must satisfy this property $T_{ij}$ is minus of $T_{ji}$. So for that particular case all the diagonal elements must go to 0. Then only, for the anti-symmetric tensor Tij equals to minus Tji, can be satisfied.

So, all diagonal elements for all antisymmetric tensors are 0, and non-diagonal elements these cross elements are equal with the opposite sign. So, these 0, $T_{12}$, $T_{13}$, - $T_{12}$, 0, $T_{23}$, - $T_{13}$,- $T_{23}$, 0 these refer to a typical antisymmetric tensor multiplied with this $a_1$, $a_2$, $a_3$, the components of the arbitrary vector small $\tilde{a}$ and that would be equal to the cross product of the dual vector and this small vector $\tilde{a}$.

So, the components of this cross product of these two vectors can be written as $t_2^A a_3 - t_3^A a_2$; the second component is $t_3^A a_1 - t_1^A a_3$, and the third component is $t_1^A a_2 - t_2^A a_1$. So, using this equation, the components of the antisymmetric tensor are related to the components of the corresponding dual vector. Now, this equation can be simplified on the left-hand side by taking the product of this $\tilde{\tilde{T}}$ matrix and the a vector, and then in the indicial notation, this can be written as $t_i^A$.

The $ith$ component of the dual vector is equal to minus half of the permutation symbol $e_{ijk}T_{jk}$. So, simplifying this expression, we get these three components of the dual vector. Now, if we expand this indicial notation with $i$ equals to 1, $T_{1a}$ would be minus half of $e_{123}T_{23}$ minus half of $e_{132}T_{32}$ and this would be minus $T_{32}$ by 2 minus $T_{23}$ by 2; in total, this is minus $T_{23}$.

Why? Because this fellow is equal to minus 1, $e_{132}$ is minus 1, $e_{123}$ is 1, and with the property of an anti-symmetric tensor. $T_{32}$ is minus $T_{32}$. So, this expression can be expanded, and with that, we can show that all the components of the anti-symmetric tensor can be represented with this equation. $t_i^A$ dual vector of an anti-symmetric tensor, is equal to minus. Now, coming to the eigenvalues of a tensor.

If a vector a transforms under a linear transformation $\tilde{\tilde{T}}$ into a vector parallel to itself, as $\tilde{\tilde{T}}$ acting over $\tilde{a}$ equals $\lambda$ times $\tilde{a}$, so the $\tilde{\tilde{T}}$ acting over $\tilde{a}$, the resultant vector is parallel to the actual vector a with a scaling factor λ. Then we call that λ to be the eigenvalue and a to be the eigenvector of the second-order tensor $\tilde{\tilde{T}}$. For definiteness, we assume all the eigenvectors to be of unit length, and n being the unit eigenvector,

this expression becomes $\tilde{\tilde{T}}$ acting over $\tilde{n}$ equals λ times $\tilde{n}$. Now, $\tilde{n}$ equals $\alpha_j \tilde{e}_j$. As $\tilde{n}$ is a unit eigenvector, we define this α with the help of this expression: $\alpha_j \tilde{e}_j$ α_j equals 1. Where $\tilde{n}$ is $\alpha_j \tilde{e}_j$, so $\alpha_j$ are the components of the unit normal unit vector eigenvector $\tilde{n}$. Now, with this, with the help of this $\tilde{n}$, as $a_j \tilde{e}_j$, we can write this expression $\tilde{\tilde{T}}$ acting over $\tilde{n}$ equals λ $\tilde{n}$ in the initial notation as this: $(T_{ij} - \lambda\delta_{ij})\ \alpha_j$ equals 0.

So, λ times $\tilde{n}$ is written as λ times the identity tensor I acting over $\tilde{n}$, and that I is here represented as $\delta_{ij}$. Now, if $\tilde{\tilde{I}}$ expand this and write all three equations separately, then it will be of this particular form. Now, these three equations can be condensed to a single determinant equation: determinant of $\tilde{\tilde{T}} - \lambda\tilde{\tilde{I}}$, where I is the identity tensor, equals 0. So, the determinant of $\tilde{\tilde{T}}$ minus λ$\tilde{\tilde{I}}$, is 0. This is known as the characteristic equation of the second-order tensor $\tilde{\tilde{T}}$, and by solving this,

we can obtain all three eigenvalues λ for this particular second-order tensor $\tilde{\tilde{T}}$. And then, by putting those λ values back into these three expressions, we can get the components of $\alpha_1$, $\alpha_2$, and $\alpha_3$, which would help us determine the eigenvectors $\tilde{n}$. So, in this particular lecture, we discussed the definition of a second-order tensor, followed by different tensor operations. We also discussed the orthogonal tensor, symmetric and antisymmetric tensors, and the eigenvalues of second-order tensors.

Thank you.