APPLIED ELASTICITY

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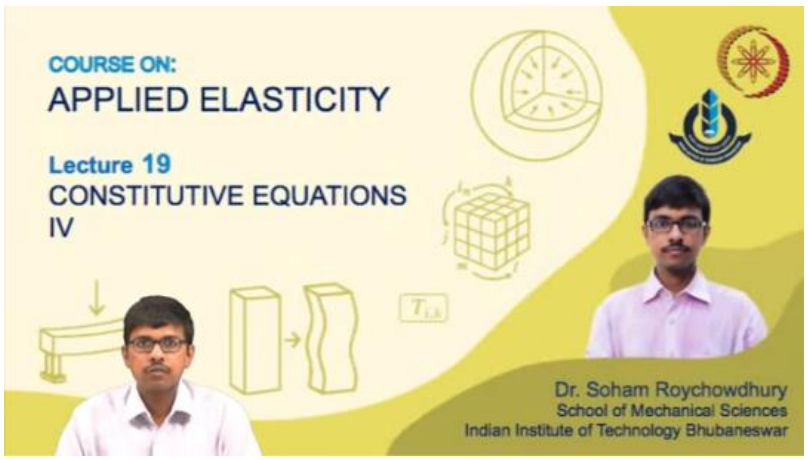
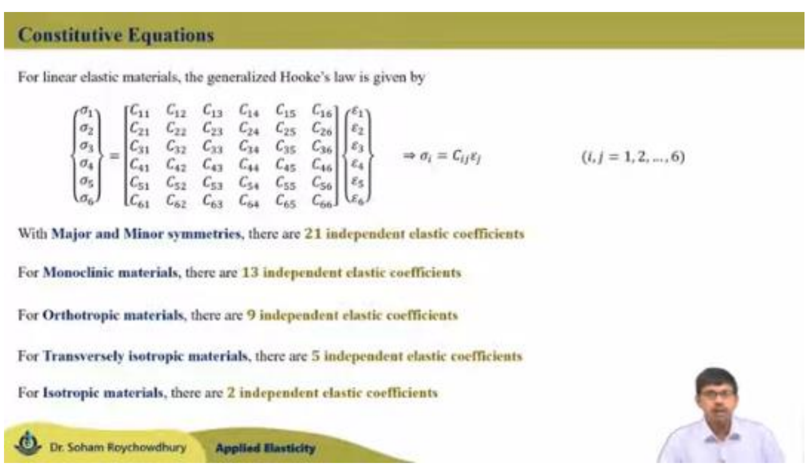
WEEK: 04
Lecture- 19



Welcome back to the course on applied elasticity. In the previous few lectures, we discussed the constitutive equations. In this lecture, we will continue with the same topic of constitutive equations. We were interested in linear elastic solids, for which the constitutive equations—which are nothing but the generalized Hooke's law—were derived to be like this.

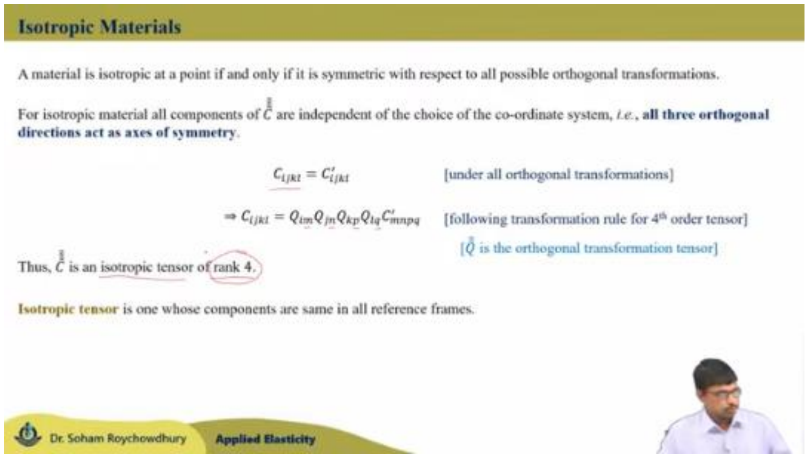


Through these constitutive equations for linear elastic materials, we relate 6 stress components (σ_1 to σ_6) with 6 strain components (ε_1 to ε_6). While deriving this equation, we assumed that the strain components (or deformations) are small and the stress-strain behavior is linear in nature.

Thus, stress (σ) and strain (ε) are related with the help of the C matrix, which we call the material stiffness matrix or material stiffness tensor. In this particular equation, the generalized Hooke's law is represented using engineering notation where only one subscript is associated with σ, one with epsilon, and two with the material stiffness constant C. Here, i and j can take values from

1 to 6, so $\sigma_i = C_{ij}\varepsilon_j$. By imposing major and minor symmetries on the C matrix, instead of 36, we can have only 21 independent non-zero elastic constants or coefficients. After this, we considered different types of materials with varying symmetries—such as one or multiple planes of symmetry, or one or multiple axes of symmetry—and the number of independent non-zero elastic constants in C kept reducing. For example, if you choose a monoclinic material with any one plane of symmetry, and if the material properties are symmetric with respect to that plane, we call it a monoclinic material. For such materials, the number of non-zero independent elastic constants reduces from 21 to 13.

Further, if we impose three orthogonal planes of symmetry in the material, it becomes an orthotropic material, for which the number of independent non-zero elastic constants reduces from 13 to 9. Now, after imposing the planes of symmetry, if you consider the axis of symmetry, for a material having one axis of symmetry or being symmetric, With respect to any arbitrary angle rotation about that particular axis of symmetry, we call that material a transversely isotropic material, and for that, the number of independent elastic coefficients in C becomes 5. And instead of 1, if you impose any three mutually orthogonal axes as axes of symmetry within a material, then that material is known as an isotropic material, and thus the number of independent elastic constants is shown to be only 2.



So, starting from 81, 81 components in C_{ijkl}, the fourth-order elastic stiffness tensor, first imposing major and minor symmetry and introducing the Voigt-Kelvin notation (engineering notations), we had reduced the number of independent constants to 21. By taking further symmetries of either planes of symmetry or axes of symmetry, we had reduced the number of independent elastic constants to 2, which is the case for linear elastic isotropic homogeneous materials.

Now, we are going to examine the constitutive equation of isotropic material in the form involving two constants, λ and μ, which are known as Lamé constants. So, first, How is an isotropic material defined? An isotropic material is a material in which, at any point, it is symmetric with respect to all possible orthogonal transformations.

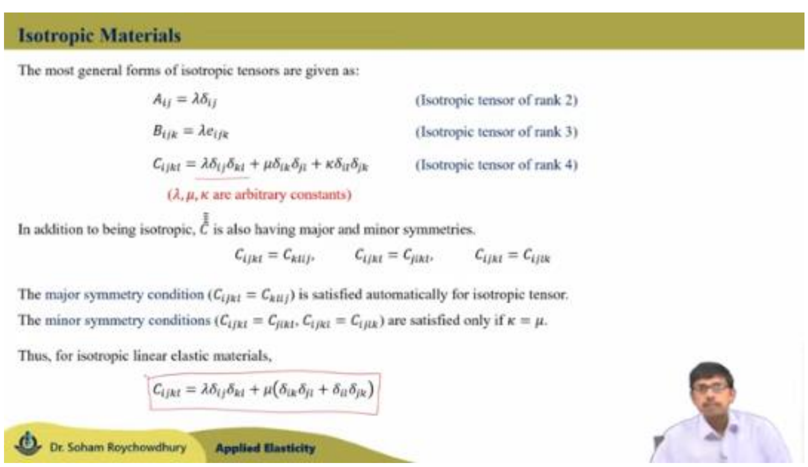
So, whatever orthogonal transformation we can think of at a material point, if the material properties or behavior are symmetric with respect to any of those, then we call that material an isotropic material. Orthogonal transformation is characterized by Q, the orthogonal transformation tensor, and for any form of Q, if the material is symmetric, then we call that material an isotropic material. For an isotropic material, the components of C, the elastic stiffness tensor, are independent of the choice of coordinate system, so the components of C become frame-invariant. They are independent of the choice of reference frame or e_1, e_2, e_3 axes. For such cases, any three orthogonal choices of e_1, e_2, e_3 can act as three orthogonal axes of symmetry. So, whatever right-handed orthogonal triad we choose, with that we can proceed with the conclusion that all those three chosen e_1, e_2, e_3 are three axes of symmetry for the isotropic materials.

Now, for the isotropic material as C is independent to choice of reference frame. So, C_{ijkl} component of C before transformation should be equals to C'_{ijkl} that is component of C after transformation and this transformation is characterized by any Q, Q can have any form. Now, if we use the property of transformation or law of transformation for a fourth order tensor, as you know,

C is a fourth order tensor involving four indices, four subscripts i, j, k, l through which we are relating two second order tensor, stress tensor and strain tensor. So, following the transformation law for any fourth order tensor, we can write the right hand side C'_{ijkl} as $Q_{im}Q_{jn}Q_{kp}Q_{lq}$ into C'_{mnpq} where this newly introduced m n p and q small q these are the 4 dummy indices on the right hand side whereas i j k and l are the free indices which are appearing on all the terms of both the sides.

Now as it is a fourth order tensor these can be related, these can be transformed from one frame to another frame with the help of 4 orthogonal transformation tensor components. So, 4 capital Q terms are appearing in this equation. Now, for such cases C is known as an isotropic tensor of rank 4. why isotropic tensor and why rank 4 as we have seen it is having four indices C_{ijkl} it is a fourth order tensor that is why we call it to be a rank 4 tensor

and isotropic tensors are defined as the tensor quantities whose components are same in all frames of reference so thus This property as it is satisfied by C, we can call C to be one isotropic tensor and this being a fourth order tensor, it is isotropic tensor of rank 4. Now, there are standard forms for mathematically expressing any isotropic tensor of rank 2, 3, 4 and more. So, we are going to look into the definition of isotropic tensor.



The general form of any isotropic tensor which is of rank 2, the second order is given like this. So, capital $A_{ij} = \lambda\delta_{ij}$. If any tensor of rank 2 can be expressed in this form, $\lambda\delta_{ij}$, where δ is the Kronecker delta, then we call that tensor an isotropic tensor of rank 2. Similarly, if B is an isotropic tensor of rank 3, and we can express B_{ijk}, the components of an isotropic tensor of rank 3, in this fashion, that is, λe_{ijk}, then B is an isotropic tensor of rank 3.

And this right-hand side, small e_{ijk}, is nothing but the permutation symbol. Now, coming to the isotropic tensor of rank 4, C_{ijkl}, any general isotropic tensor of fourth order or rank 4 can be expressed like this: $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \kappa\delta_{il}\delta_{jk}$. So, all these deltas are Kronecker deltas having different indices, and in three terms, we are associating three different constants: λ, μ, and κ.

Those are arbitrary constants associated with this fourth-order isotropic tensor. So, this particular expression is valid for any fourth-order isotropic tensor, not specifically for this material stiffness tensor. Now, we will try to impose the properties of the material stiffness tensor C, which we already know, in this form of a fourth-order general form of a fourth-order isotropic tensor or isotropic tensor of rank 4. Now, in addition to being an isotropic tensor of rank 4, C also has the properties of major symmetry and minor symmetry, which we had discussed. So, how were these defined, major symmetry and minor symmetry? $C_{ijkl} = C_{klij}$. C_{ijkl} should also be equal to C_{jikl}, and $C_{ijkl} = C_{ijlk}$. These three are the mathematical descriptions of major and minor symmetries of C, or the material stiffness matrix, or the material stiffness tensor.

Now, the first one, $C_{ijkl} = C_{klij}$, is the major symmetry condition. Now, for the isotropic tensor given by this particular equation. This particular major symmetry condition is automatically satisfied. So, we can check or verify that C_{ijkl} is already written here as $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \kappa\delta_{il}\delta_{jk}$. Let us try to write C_{klij}.

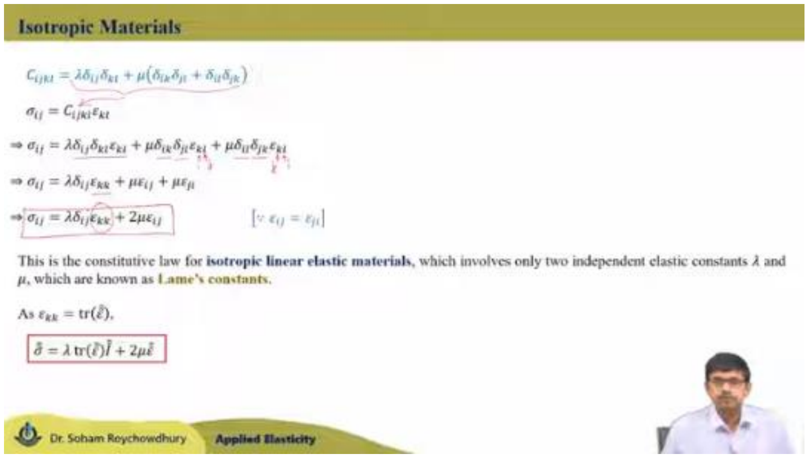
That is another isotropic tensor, so we can write this as λ times the first two subscripts $\delta_{ij}\delta_{kl}$ plus μ times δ (first and third subscripts) ki, another delta (second and fourth subscripts) lj, plus κ. δ (first and fourth subscripts). Those are k and j. And then δ (second and third subscripts).

That is l and i. Now, let us compare this with the given form. So, if you look at the first term here and the first term here. Both of them are the same. $\lambda\delta_{ij}\delta_{kl}$. Now, look at the second term.

$\mu\delta_{ik}\delta_{jl}$ here, and we also have $\mu\delta_{ik}\delta_{jl}$. So, if we are changing the order of the two indices in the Kronecker delta, no changes occur; they are identical. So, the second term of C_{ijkl} is the same as the second term of C_{klij}. Similarly, if you look at the third term of C_{klij}, that is $\kappa\delta_{il}\delta_{jk}$, and here we have $\kappa\delta_{kj}\delta_{li}$—just the subscript order of the indices in the Kronecker delta is changed. So, the third term is also the same. Thus, we can say that $C_{ijkl} = C_{klij}$. This major symmetry property is automatically satisfied for the fourth-order isotropic tensor C. No additional constraints are required to satisfy this.

Now, let us proceed to the minor symmetry condition. Now, there are two minor symmetry conditions. One is $C_{ijkl} = C_{jikl}$, arising from the symmetry of the stress tensor. The first two indices are flipped, whereas, due to the symmetry of the epsilon strain tensor, the second minor symmetry condition is $C_{ijkl} = C_{ijlk}$, where the last two indices (third and fourth) are flipped. Now, we will try to see whether these two conditions are automatically satisfied or if we need additional constraints to satisfy them. So, let us start with C_{jikl}. And our objective is to show that this is equals to C_{ijkl}. If I am writing this with the help of the definition of isotropic tensor of rank 4, $\lambda\delta_{ji}\delta_{kl}$, this is first term plus μ times δ first and third, these two index that is j and k. then δ_{il} that is second and fourth plus κ times delta first and last. So, jl and then finally, first and third I guess no second and third that is ik. So, this would be C_{jikl} that is isotropic tensor of rank 4. Now, if we start comparing with the given form first term $\delta_{ji}\delta_{kl}$ λ this is same as this because δ_{ij} and δ_{ji} are same. Now, if I compare the second term $\mu\delta_{jk}\delta_{il}$ here δ_{il} and δ_{jk} term is sitting here in the third term with associated constant coefficient to be κ whereas that is sitting here in the second term with associated constant μ. So, thus these two terms will be same only if we have this additional condition of $\kappa = \mu$.

Similarly, the last term $\delta_{jl}\delta_{ik}$ is having κ as coefficient and in C_{ijkl} here that term is having μ as its coefficient. So, we need to have this additional constraint $\kappa = \mu$ and with that only we can satisfy all the minor symmetry condition. Here if you impose $\kappa = \mu$ then we will have $C_{ijkl} = C_{jikl}$. In the similar fashion, you can expand this C_{ijkl}, write this expression, compare with C_{ijkl} and can also show that the second minor symmetry condition is also satisfied with the same constraint κ is equals to μ. So, I am not doing it explicitly, it can also be proved in a similar fashion. So, thus for isotropic linear elastic material imposing κ to be μ C_{ijkl} is having this particular form. C_{ijkl} is equal to $\lambda\delta_{ij}\delta_{kl}$. The first term from here remains as it is, and κ, being the same as μ, is taken to be a common factor outside, that is multiplied with $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. So, this is the form of the isotropic fourth-order material elastic stiffness tensor, which is an isotropic tensor of rank 4 and which also satisfies the properties of major and minor symmetries.

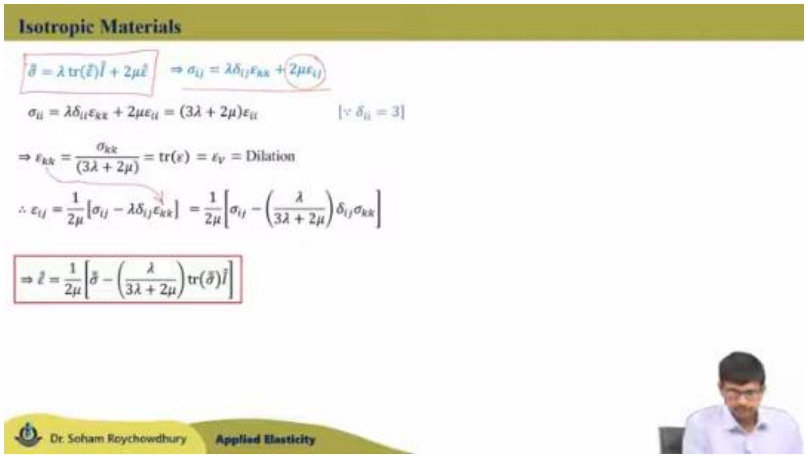


Now, moving forward, starting with this expression, we will substitute it in the constitutive equation, which is nothing but $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. Now, substituting C_{ijkl} in this particular form here. And multiplying all the terms with ε_{kl}, we would get σ_{ij} to be this.

Now, you can see in the first, second, and third terms, ε_{kl} is multiplied with two different Kronecker delta functions. Now, simplifying it further, If you look at this term, multiplying ε_{kl} with δ_{kl}, k and l become the same index. Thus, the ε subscripts become the same, ε_{kk}, and it is pre-multiplied with λ and δ_{ij}. Coming to the second term, this δ_{ik} will cause the first subscript of ε to be i, and this δ_{jl} will cause the second subscript of ε to be j. Thus, this term becomes $\mu\varepsilon_{ij}$. Similarly, in the last term, δ_{il} will cause k to be i, δ_{jk} will cause—no, δ_{il} will cause l to be i, and δ_{jk} would cause k to be j. So, this last term would become $\mu\varepsilon_{ji}$. So, $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + \mu\varepsilon_{ji}$.

Now, as ε_{ij} and ε_{ji} are the same. Due to the symmetry of the strain tensor, we can write σ_{ij} as $\lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$. This particular expression is called the is known as the constitutive equation, constitutive law for homogeneous linear elastic isotropic solids, which involves only two material constants.

One is lambda, another one is mu, and these two are known as Lame constants. So, these are the two commonly used constants for writing, for expressing the constitutive law for isotropic linear elastic solids. Now, if you look at this ε_{kk} term here, this one. That we can write as the $\mathrm{tr}(\tilde{\tilde{\varepsilon}})$, and thus in the matrix form, in the tensor form, we can write the $\tilde{\tilde{\sigma}} = \lambda\,\mathrm{tr}(\tilde{\tilde{\varepsilon}})\tilde{\tilde{I}}$, which is the identity tensor, $+2\mu\tilde{\tilde{\varepsilon}}$. So, this is the constitutive equation in the tensor form, where lambda and mu are two constants known as Lame constants.



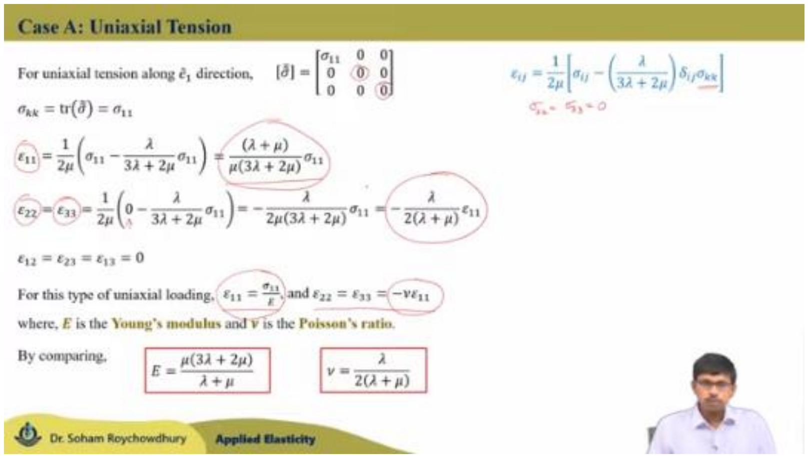
Now, substituting i equals to j in this equation, $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$. Substituting j to be equal to i, we can write the right-hand side as $\lambda\delta_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ii}$ and δ_{ii} being equal to $\delta_{11} + \delta_{22} + \delta_{33}$, summing up with repeated index i, this would be just 3 summation of 3 Kronecker delta, and thus the first term would become 3 $\lambda\varepsilon_{ii}$. σ_{ii} would be $(3\lambda + 2\mu)\varepsilon_{ii}$, or alternately we can write ε_{ii} or ε_{kk}. k or i is the name of the dummy index, which we can change anytime.

So, ε_{kk} would be $\frac{\sigma_{kk}}{(3\lambda+2\mu)}$. This is the $\mathrm{tr}(\varepsilon)$, which is nothing but the volumetric strain ε_V, also known as dilation. Now, instead of writing σ in terms of ε, which was written here, we will try to write epsilon in terms of the sigma component. So, taking this epsilon last term on one side and sending the rest of the terms to the other side.

And dividing it by 2μ, we can write ε_{ij} as $\frac{1}{2\mu}\left[\sigma_{ij} - \lambda\delta_{ij}\varepsilon_{kk}\right]$. Now, this term ε_{kk}, whatever we got, we will substitute here as $\frac{\sigma_{kk}}{(3\lambda+2\mu)}$.

So, substituting that here, ε_{ij} can be written in this particular form. And in the matrix form, in the tensor form, this $\tilde{\tilde{\varepsilon}}$, the strain tensor, can be written as $\frac{1}{2\mu}\left[\tilde{\tilde{\sigma}} - \left(\frac{\lambda}{3\lambda+2\mu}\right)\mathrm{tr}\left(\tilde{\tilde{\sigma}}\right)\right]$, $\mathrm{tr}\left(\tilde{\tilde{\sigma}}\right)\tilde{\tilde{I}}$, the identity tensor. So, this is the alternate form of the constitutive law.

This was the first form of the constitutive equation for an isotropic solid where sigma is on the left-hand side, and this is the alternate form of the constitutive equation where epsilon strain is on the left-hand side, and strain is written as a function of stress. Now, we will take three different case studies. The first case is the case of uniaxial tension. The constitutive equation is written here where the strain epsilon ij is written in terms of sigma for the isotropic solid.

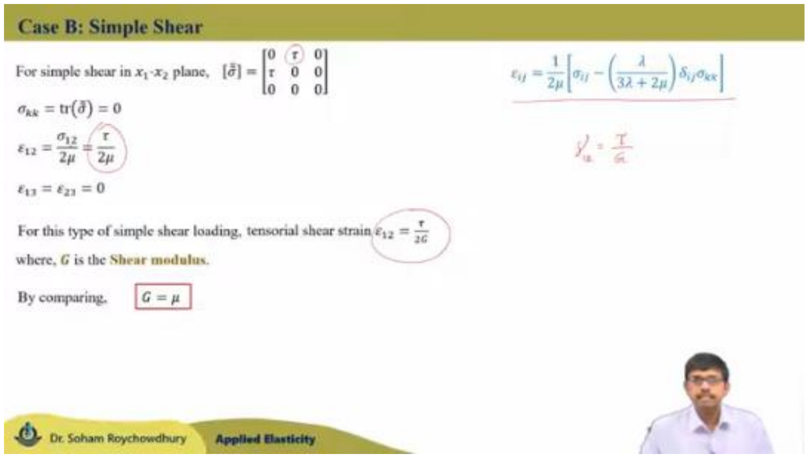


Now, let us consider a case of uniaxial tension along the $\tilde{e}_1$ direction, for which the stress matrix, the stress tensor, would be like this: σ_{11} 0 0 and zeros in all the locations of the second and third row. Now, we will try to find out ε_{ij} for this particular given stress tensor. So, for that, this σ_{kk} term is required. So, σ_{kk} is the stress of σ summation of three diagonal terms, which is simply σ_{11} for the present problem. Thus, by using this ε_{ij} relation with i equals to 1 and j equals to 1, we can obtain ε_{11} as $\frac{1}{2\mu}\left(\sigma_{11} - \frac{\lambda}{3\lambda+2\mu}\sigma_{11}\right)$. And simplifying it further, it would end up as $\frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}\sigma_{11}$. Similarly, using i and j both to be 2 and then both to be 3, ε_{22} and ε_{33} can be obtained like this. So, σ_{22} and σ_{33}, both of them are 0 here. As you can see, at this location, we do not have any non-zero value of stress. So, while finding ε_{22} or ε_{33}, the first term would come out to be 0, and simplifying it, we will get $-\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{11}$.

And it would be further simplified by using this expression as $-\frac{\lambda}{2(\lambda+\mu)}\varepsilon_{11}$. And all other remaining shear strain components would come out to be 0. Thus, for this type of uniaxial loading, we have 3 non-zero strain components: ε_{11}, 2 2, and 3 3. And, as you know from simple solid mechanics, ε_{11} is σ_{11} by E, ε_{22} and 3 3.

Lateral strains are $-\nu\varepsilon_{11}$ for the case of E being Young's modulus and ν being Poisson's ratio for uniaxial tension along direction 1. Now, comparing these ε_{11} equation with this and this equation with this, we can write that $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$.

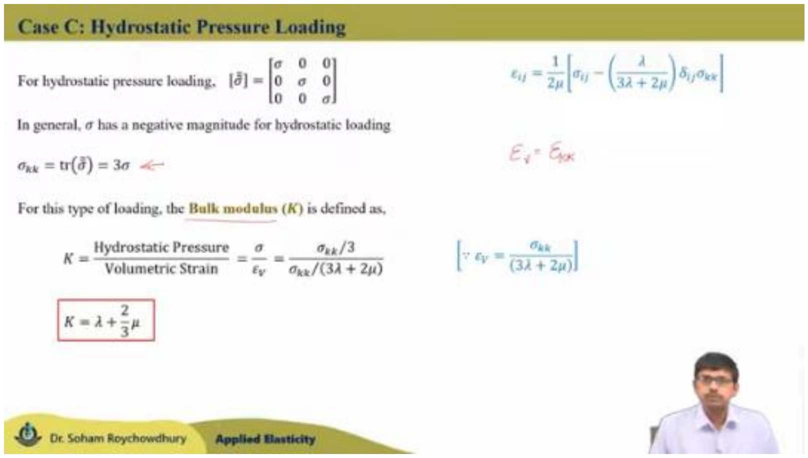
So, with this, we are able to relate two commonly available material constant Poisson's ratio and Young's modulus with the two lame constant lambda and mu.



Now, proceeding to the second case study that is case of simple shear, we are considering simple shear stress on x_1-x_2 plane. So, at 1, 2 and 2, 1 location, we are having the simple shear stress τ present. Now, for this case, all the diagonal terms being 0, σ_{kk} would end up with 0.

Now, we are interested to find out the shear strains because for the shear stress problems, the non-zero shear strain is supposed to be generated. We do not expect any normal strain. So, finding the only non-zero shear strain which would be at this location, 1, 2 location, σ_{12} with the help of this equation, it would come out to be $\frac{\sigma_{12}}{2\mu}$ or $\frac{\tau}{2\mu}$.

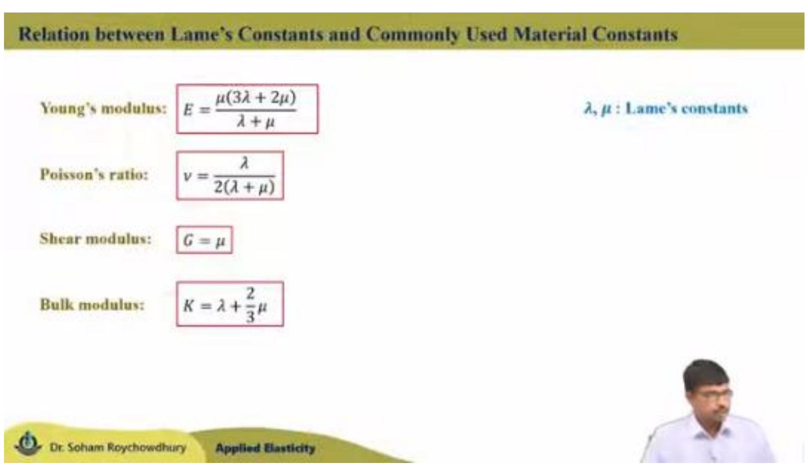
Other two shear strains ε_{13} and ε_{23} would come out to be 0. Now, for this type of simple shear loading, we know that the tensorial shear strain $\varepsilon_{12} = \frac{\tau}{2G}$. The engineering shear strain $\gamma_{12} = \frac{\tau}{G}$ and by using the relation of tensorial and engineering shear strain we can write $\varepsilon_{12} = \frac{\tau}{2G}$ where G is nothing but the shear modulus modulus of rigidity of the material. Now comparing this form of ε_{12} and this form of ε_{12} we can conclude that G is nothing but μ so one of the lame constant is nothing but the shear modulus of the material.



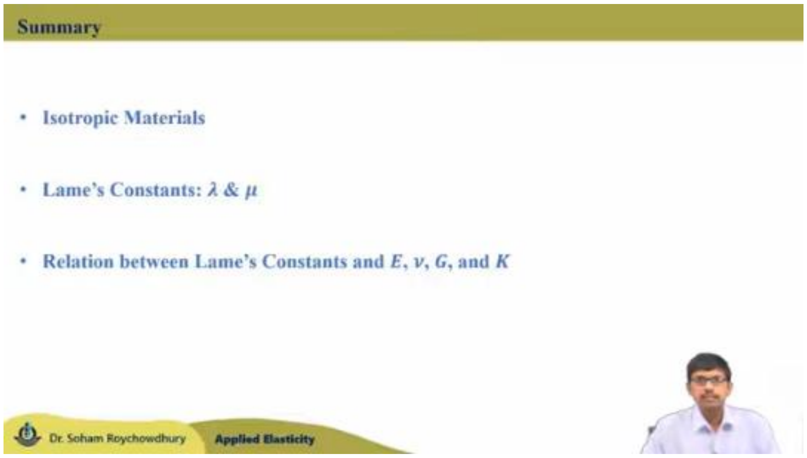
And finally, coming to the last case of hydrostatic pressure loading where all three diagonal terms are $\tilde{\tilde{\sigma}}$ and non-diagonal terms are 0 and normally $\tilde{\tilde{\sigma}}$ is having a negative magnitude for the hydrostatic pressure loading.

So, here $\mathrm{tr}(\tilde{\tilde{\sigma}})$ would be σ_{kk} equals to 3σ summation of three diagonal terms which are all σ here. Now, for this type of loading, the volume is supposed to change and the deformed and undeformed volume, these two quantities are related through a material property called bulk modulus denoted by capital K. It is defined as K equals to hydrostatic pressure divided by volumetric strain.

So, hydrostatic pressure acting on the on the system is nothing but σ and volumetric strain is nothing but ε_V which we can write as ε_{kk}. The expression for that we had already achieved. So, $K = \frac{\sigma}{\varepsilon_V}$, and ε_V is replaced as $\frac{\sigma_{kk}/3}{\sigma_{kk}/(3\lambda+2\mu)}$, and using this we write $\sigma_{kk}/3$ as our sigma hydrostatic pressure is $\sigma_{kk}/3$, and by canceling σ_{kk}, we can write $K = \lambda + \frac{2}{3}\mu$. So, this is the relation between the K and the two Lame constants, λ and μ.



Thus, with λ and μ being Lame constants, we have related all four material properties—Young's modulus E, Poisson ratio ν, shear modulus G, and bulk modulus K—to λ and μ by considering those three simple case studies of uniaxial tension, simple planar shear, and hydrostatic pressure loading. So, instead of using E, ν, G, or K, we can use λ and μ, the two Lame constants, for writing the constitutive equations for isotropic linear elastic solids.



In this particular lecture, we discussed isotropic materials, wrote the general isotropic tensor equation of rank 4, introduced the concepts of the two Lame constants lambda and mu, and then obtained the relation between the Lame constants and other commonly available material constants such as E, nu, K, and G. Thank you.