APPLIED ELASTICITY

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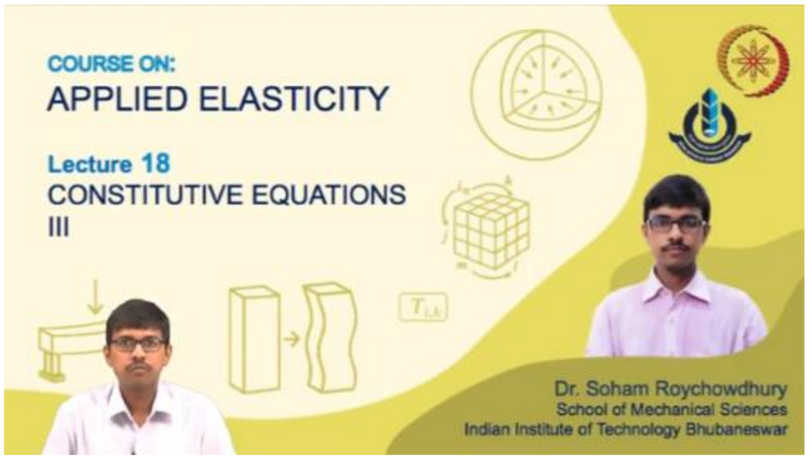
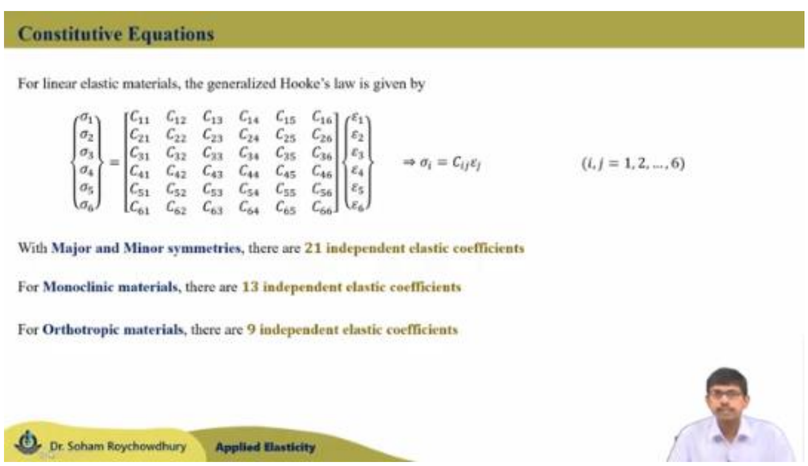
WEEK: 04
Lecture- 18

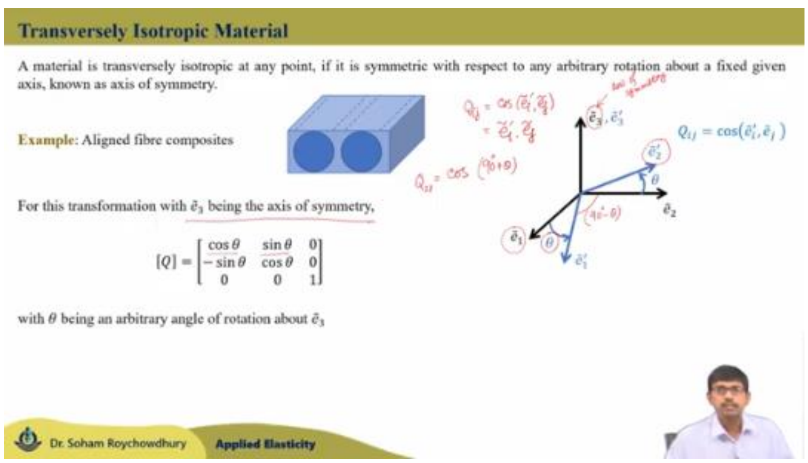


Welcome back to the course on applied elasticity. In the previous lecture, we discussed the constitutive equation, and in today's lecture, we will continue our discussion on the same topic.



So, to have a quick recap, for linear elastic solids, this was the generalized Hooke's law as discussed, where sigma is the stress component, epsilon is the strain component, which are related through C, the elastic stiffness tensor. So, we can write $\sigma_i = C_{ij}\varepsilon_j$, where C_{ij} are the components of the elastic stiffness tensor, and this equation is written in engineering notation. i and j can take the values of one, two, three, four, five, and six. C is also a symmetric tensor; thus, $C_{ij} = C_{ji}$. Imposing the major and minor symmetry of C, we had shown that in the elastic stiffness matrix, there exist 21 non-zero independent elastic constants. Then, after imposing different types of material symmetry, we can define various types of materials, such as monoclinic or orthotropic.

So, for the monoclinic material, the number of independent elastic constants in the C matrix is reduced to 13, where there is a single plane of symmetry for the case of monoclinic material. Now, moving forward, for the orthotropic material, which has three mutually orthogonal planes of symmetry, the number of independent elastic constants in the C matrix is further reduced to 9.



Now, in today's lecture, we will move forward to the next type of material, which is known as transversely isotropic material.

So, how is it defined? The transversely isotropic material is a material which at any particular point shows material symmetry with respect to any arbitrary rotation about a fixed given axis, which is also termed as the axis of symmetry. So, if you are considering a material and choosing one axis of symmetry, with respect to that axis of symmetry, if you give any arbitrary rotation. So, the transformation matrix is defined by this arbitrary rotation with respect to the fixed axis of symmetry, and with respect to this transformation, this fixed-axis rotation, if the material is symmetric, then we call that material a transversely isotropic material having one axis of symmetry.

Now, let us consider this frame of reference, the $\tilde{e}_i$ frame, where $\tilde{e}_1$ vector, $\tilde{e}_2$ vector, and $\tilde{e}_3$ vectors are the three mutually orthogonal base vectors. Now, we are choosing one of the axes, that is, the $\tilde{e}_3$ axis, to be the axis of symmetry. So here, this is chosen as the axis of symmetry. Now, by the definition of the transversely isotropic material, we can rotate the body, or rotate the frame, with respect to this axis of symmetry, which is $\tilde{e}_3$ here, by any arbitrary angle.

Let us say. So, we are rotating the system, the orthogonal triad—the right-hand triad—with respect to the $\tilde{e}_3$ axis through an arbitrary angle theta. With this rotation, $\tilde{e}_1$ moves to $\tilde{e}_1'$, and $\tilde{e}_2$ moves to $\tilde{e}_2'$. However, there is no change in $\tilde{e}_3$. $\tilde{e}_3$ and $\tilde{e}_3'$ are the same.

They are coinciding because that is the axis of symmetry. Now, aligned fiber composites, fiber-reinforced composites with fibers aligned along a particular direction, are a typical example of transversely isotropic material, where the fiber direction acts as the axis of symmetry. Now, for this particular transformation, with $\tilde{e}_3$ being the axis of symmetry, we can

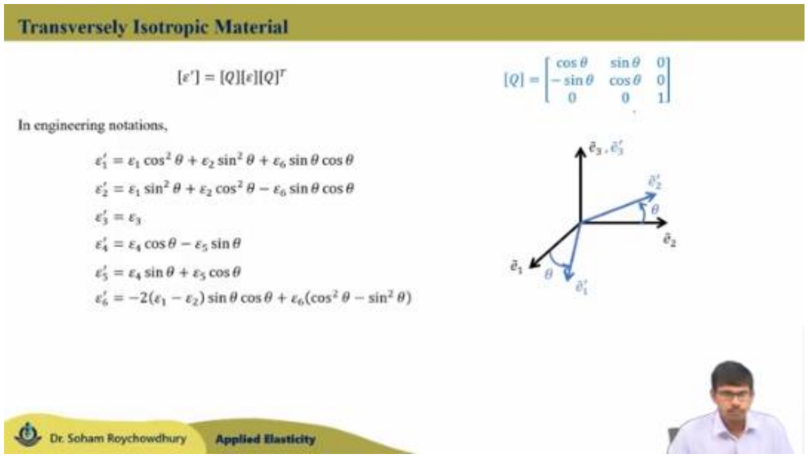
define the orthogonal transformation tensor Q as this. So, how was Q defined? Q_{ij} equals the cosine of the angle between $\left(\tilde{e}_i', \tilde{e}_j\right)$, or you can simply take the dot product of $\left(\tilde{e}_i', \tilde{e}_j\right)$ to get Q_{ij}. So, if you are taking $\tilde{e}_1$ and $\tilde{e}_1'$.

So, for finding Q_{11}, the cosine of the angle between $\tilde{e}_1$ and $\tilde{e}_1'$ is required, which is nothing but the cosine of this angle theta. Thus, the first term Q_{11} is cos of theta. Now, coming to the second term Q_{12}, that is the cosine of the angle between $\tilde{e}_1'$ and $\tilde{e}_2$. So, basically this angle. Now, how much is this angle?

$\tilde{e}_1$ and $\tilde{e}_2$ have an angle of 90 degrees, and $\tilde{e}_1$, $\tilde{e}_1'$ have an angle theta. So, the angle between $\tilde{e}_1'$ and $\tilde{e}_2$, which are all on the same plane, is basically 90 degrees minus theta. So, the cosine of this angle is nothing but sin theta. So, Q_{12} is sin theta. $\tilde{e}_1'$ and $\tilde{e}_3$ being perpendicular, the Q_{13} term is 0.

So, coming to the second row, the first term Q_{21} is the angle between $\tilde{e}_2'$ (this one) and $\tilde{e}_1$. So, the angle between $\tilde{e}_2'$ and $\tilde{e}_1$ is 90 degrees plus theta. So, the cosine of this is Q_{21}, which is minus sine theta, and so on. In the same fashion, you can obtain all the components of Q,

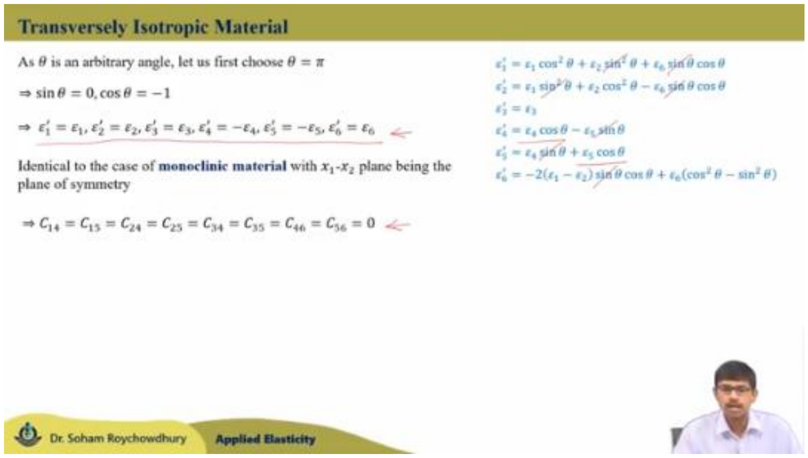
and Q would look like $\cos\theta \; \cos\theta \; 0 - \cos\theta \; \cos\theta \; 0 \; 0 \; 0 \; 1$ where theta is any arbitrary angle of rotation with respect to the axis of symmetry and here We have chosen $\tilde{e}_3$ to be our axis of symmetry. So, if the material has one axis of symmetry with respect to that axis for any arbitrary rotation if the material behavior is symmetric then we call the material a transversely isotropic material. Now, for this material which is transversely isotropic with respect to $\tilde{e}_3$ being the axis of symmetry, we are going to derive the constitutive equation and see how many independent elastic constants we will have. Now, this is the figure of E and $\tilde{e}_i$ and $\tilde{e}_i'$ vectors showing the transformation that is rotation by angle theta with respect to the $\tilde{e}_3$ axis and this is Q. the orthogonal transformation tensor.



Now, relating the strain components in two frames, the $\tilde{e}_i$ frame and the $\tilde{e}_i'$ frame. Through the transformation law of a second-order tensor, the strain tensor ε being a second-order tensor, the transformation is related or can be done with the help of this expression: $[\varepsilon'] = [Q][\varepsilon][Q]^T$, where Q we have already obtained. So, if you substitute the ε components and substitute this Q and $[Q]^T$ and expand it, you can relate the strain components in both the $\tilde{e}_i$ frame and the $\tilde{e}_i'$ frame as this. And I am writing the strain in the engineering notation with only one subscript, the subscript varying from 1 to 6. So, the first strain component in the deformed or transformed state ε_1' would be $\varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta + \varepsilon_6 \sin\theta\cos\theta$. ε_2' would be $\varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta - \varepsilon_6 \sin\theta\cos\theta$.

ε_3' is the same as ε_3. This is the normal strain along the axis of symmetry, and there is no change between $\tilde{e}_3$ and $\tilde{e}_3'$. Thus, the normal strain along that direction would be the same. That is, ε_3' is the same as ε_3. Now, ε_4' prime would come out to be $\varepsilon_4 \cos\theta - \varepsilon_5 \sin\theta$.

ε_5' would be $\varepsilon_4 \sin\theta + \varepsilon_5 \cos\theta$. ε_6' would be $-2(\varepsilon_1 - \varepsilon_2)\sin\theta\cos\theta + \varepsilon_6(\cos^2\theta - \sin^2\theta)$. Now, all these strain transformations—that is, the relations between the strain components in two frames—are written here, which we had obtained for the case of the transversely isotropic material with $\tilde{e}_3$ being the axis of symmetry. Now, theta is any arbitrary angle.

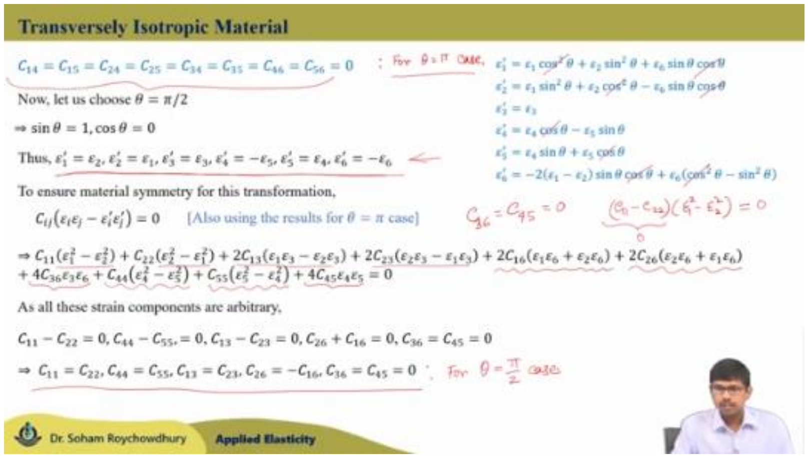


So, we can choose any value of θ, and with respect to any chosen value of θ, the material must show symmetric behavior. So, first, we are choosing theta to be π. So, for θ being π, $\sin\theta = 0$, and $\cos\theta = -1$. Now, if you look at this epsilon transformation, sine theta being 0, all these terms will cancel.

And thus, substituting $\cos\theta = -1$, we can write $\varepsilon_1' = \varepsilon_1$, $\varepsilon_2' = \varepsilon_2$, $\varepsilon_3' = \varepsilon_3$, $\varepsilon_4' = -\varepsilon_4$, $\varepsilon_5' = -\varepsilon_5$, $\varepsilon_6' = \varepsilon_6$. Because $\cos\theta$ is minus 1, ε_5' would be $-\varepsilon_5$, and ε_6' would be ε_6. So, for this particular choice of θ equals π, the relation between the strain components in two base vector frames would be like this. Now, if you compare this, this is the identical case for the monoclinic material with one plane of symmetry being the x_1-x_2 plane.

Now, for monoclinic material with x_1-x_2 being plane of symmetry, these constants were 0. These 8 constants, 8 components of C matrix were 0. So, here also strain relation being identical to the monoclinic material case with x_1-x_2 plane of symmetry, we must have these 8 constants to be 0. which is basically $\theta = \pi$. So, $\theta = \pi$ ensures this transversely isotropic material to be behaving as monoclinic material. theta is arbitrary.

It is not restricted only to a specific value of 180 degrees or π. It should be symmetric. The material should be symmetric for other values of θ as well. So, apart from this symmetry, we are supposed to have some more symmetries, and some other constants would also go to 0. Now, let us go for our second choice of π.



So, $\theta = \pi/2$. So, these 8 constants are already 0 from the $\theta = \pi$ case. And if constants are 0 for a specific θ case, those are 0 for all values of θ because constants are independent of θ. These are material properties. If we need to ensure one constant to be 0 to have material symmetry for a specific theta, those constants will be 0 for all values of theta throughout.

Now, choosing the second case with $\theta = \pi/2$, $sin\,\theta = 0$. 1 $\cos\theta = 0$, and $\cos\theta = 0$, these terms would go to 0. Thus, for this case, $\varepsilon_1' = \varepsilon_2$, $\varepsilon_2' = \varepsilon_1$, $\varepsilon_3' = \varepsilon_3$, $\varepsilon_4' = -\varepsilon_5$, $\varepsilon_5' = \varepsilon_4$, $\varepsilon_6' = -\varepsilon_6$. So, we will be getting this particular set of relations between the strain components in two base vector frames. Now, writing the expression of material symmetry for theta equals to π by 2 case with this set of relations between the ε and ε' components.

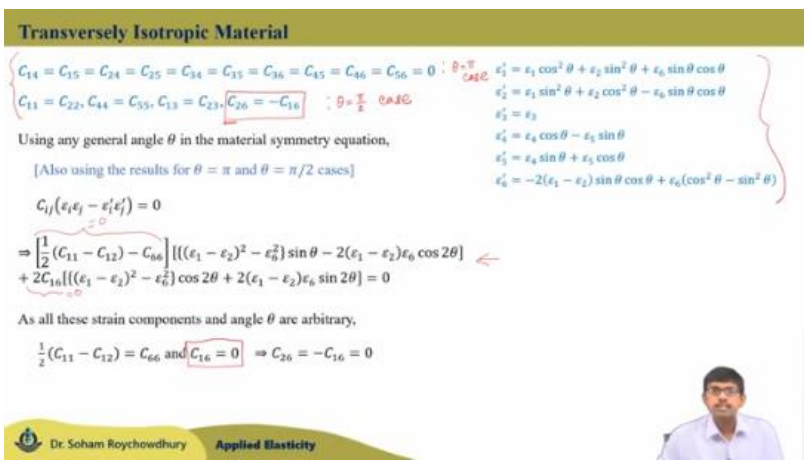
$C_{ij}(\varepsilon_i\varepsilon_j - \varepsilon_i'\varepsilon_j') = 0$. This is the general expression for any type of material symmetry. And we are using this strain component relation. We are also using the results of $\theta = \pi$ case, meaning these 8 constants are already set to 0. With that, if you expand this term, the material symmetry expression, it would look like this. So, you can see, I think we have 10 terms, 10 terms on the left-hand side, and the rest of the constants are neglected because many of the constants $C_{14}, C_{15}, C_{24}, C_{25}$, all these are 0. So, no need to write those terms. So, terms associated with the non-zero C values, whatever were non-zero after $\theta = \pi$, those were only taken, and also we have imposed these strain conditions.

So, with that, this would be the expression if you expand the left-hand side for i and j varying from 1 to 6. Now, the right-hand side being 0, for any arbitrary epsilon, we must impose some more conditions on the C, on some of the C components by looking at the left-hand side expression. So, if you look at the last term, 4 $C_{45}\varepsilon_4\varepsilon_5$, for any arbitrary epsilon, this term would go to 0 only if C_{45} equals 0.

Similarly, if you look at this term, $4C_{36}\varepsilon_3\varepsilon_6$, this would be 0 only if C_{36} is equal to 0. Now, look at the first term, $C_{11}(\varepsilon_1^2 - \varepsilon_2^2)$. From this, we cannot say C_{11} is 0. Why? Because

Compare the first term and second term. The second term is $C_{22}(\varepsilon_2^2 - \varepsilon_1^2)$. So, this expression of epsilon is the same in both the first term and second term, just with a negative sign. So, if you write $C_{11} - C_{22}$, this times $\varepsilon_1^2 - \varepsilon_2^2$. This is the combined term which is coming by combining the first term and second term of the left-hand side. No other term has this coefficient $\varepsilon_1^2 - \varepsilon_2^2$. So, similarly, these two terms, C_{13} term and C_{23} term, can be combined; C_{16} term and C_{26} terms can be combined. This term of C_{44} and C_{55} can also be combined. And after doing that and setting the coefficient of this combined term to 0, so for example, in this case, this would be 0 only if $C_{11} - C_{22}$ is 0.

So, for any arbitrary ε values this would be true, this equation would be true only if we are having this set of conditions to be satisfied. So, $C_{11} - C_{22}$ is 0, 4 4 minus 5 5 is also 0, $C_{13} - C_{23}$, $C_{26} - C_{16}$ and C_{36} and C_{45} are individually 0. Now, this first 3 expression can further be written as $C_{11} = C_{22}$, $C_{44} = C_{55}$, $C_{13} = C_{23}$, $C_{26} = -C_{16}$, and last 2 are 0, C_{36} and C_{45} are 0. So, these are the conditions we are getting for the θ equals to $\pi/2$ case. Now, apart from π and $\pi/2$, θ can take any other value.



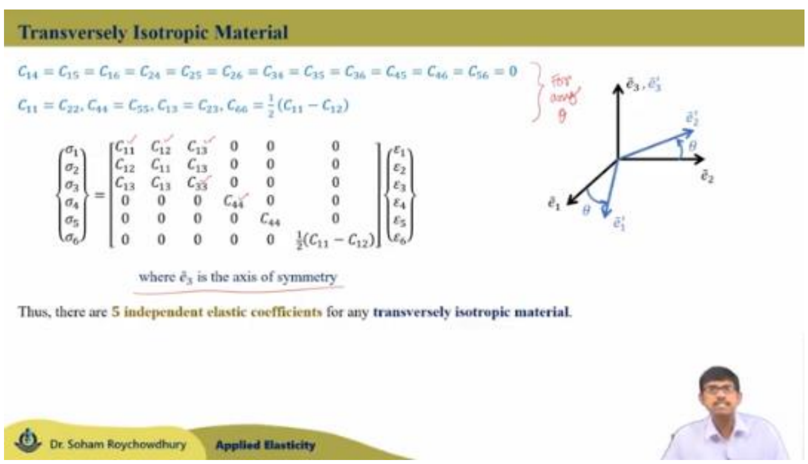
So, here the $\theta = \pi$ case and $\theta = \pi/2$ case results are noted at the top. Now, we are writing the material symmetry expression for any general angle θ. So, this set of expressions are coming for $\theta = \pi$ case These are, therefore, the $\theta = \pi/2$ case. And thus, these should be valid for any angle θ as well, because material properties are independent of θ.

Now, writing the material symmetry equation, $C_{ij}(\varepsilon_i\varepsilon_j - \varepsilon_i'\varepsilon_j') = 0$ for any arbitrary angle θ. Using all these ε transformation equations and also using the conditions which we had already obtained for $\theta = \pi$ and $\theta = \pi/2$ case, these would be the resulting equations. Many of the terms are already zero.

Some of the terms are related, like $C_{11} = C_{22}$, $C_{44} = C_{55}$, $C_{13} = C_{23}$, and $C_{26} = -C_{16}$. After incorporating all those, the expression would be like this. $\frac{1}{2}(C_{11} - C_{12}) - C_{66}$. This entire term multiplied by a function of ε and θ, plus the second term C_{16}, times another function of ε and θ, equals 0.

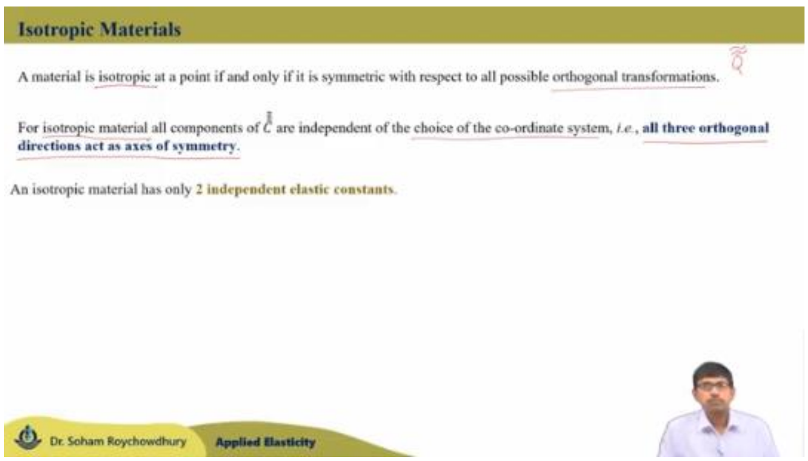
Now, for ε (the strain) as well as the angle of rotation θ, both being quite arbitrary, this equation can go to 0 only if the coefficients of these ε and θ functions in both terms go to 0 separately. So, thus, for arbitrary ε and θ, we must have $\frac{1}{2}(C_{11} - C_{12}) = C_{66}$ and $C_{16} = 0$. So, these are the two extra conditions coming from any arbitrary value of θ to impose material symmetry.

Now, if you look at this particular expression, C_{26} was obtained as $-C_{16}$. So, now, here we are getting C_{16} to be 0. Thus, C_{26} must also be 0. So, we are having many of the terms equal to 0.



Some other terms are directly related. So, we have written all those cases, and these are for any arbitrary value of theta. Now, if you are writing the constitutive equation explicitly in the matrix form, for the transversely isotropic material, it would look like this: $\sigma = C\varepsilon$, where If you count the number of independent material constants, that would come out to be only 5.

So, for transversely isotropic material, we have only 5 independent elastic constants or non-zero values, non-zero entries in the C matrix. Now, this particular equation is written for transversely isotropic material with $\tilde{e}_3$ being the axis of symmetry. You can derive a similar form of expression for transversely isotropic materials with either $\tilde{e}_1$ or $\tilde{e}_2$ as the axis of symmetry. We can only have a single axis of symmetry for transversely isotropic material. So, if you count, C_{11} is one, C_{33} is another, C_{44} is another, C_{12} is another, and C_{13} is another. These are the five constants, five non-zero independent elastic constants for transversely isotropic material.



Now, moving forward to the case of isotropic material. A material is said to be isotropic at a point if and only if it is symmetric with respect to all possible orthogonal transformations that can be defined at that point. So, orthogonal transformations are defined through this Q. Orthogonal transformation tensor Q. Now, the definition of isotropic material states that Q can be anything.

We can choose any possible orthogonal transformation, and for that, the material must be symmetric. If that is the case, then we call the material isotropic. For the isotropic material, all the components of C, the elastic stiffness tensor, are independent of the choice of the coordinate system because, for any expression of Q_{ij}, we want our material symmetry equation to be satisfied for the isotropic material, and this is possible only if all the components of $\tilde{\tilde{\tilde{\tilde{C}}}}$, the elastic stiffness tensor, are independent of the choice of the coordinate system. And this can be ensured only if all three orthogonal axes are axes of symmetry. So, we can choose our $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ along any direction. Any three orthogonal triads can be chosen for describing the material, and all such possible orthogonal triads.

Any choice of $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ would be a set of axes of symmetry. So, we are going to have three orthogonal directions or three orthogonal axes of symmetry for the case of isotropic materials. So, for the isotropic material, we are going to have only two independent elastic constants, which we will try to show. Now, starting with one axis of symmetry.

So, first let us say we are having the $\tilde{e}_3$ axis to be the axis of symmetry, and thus the non-zero components of the elastic stiffness tensor would be like this. That is the form for the transversely isotropic material with one axis of symmetry, $\tilde{e}_3$ being the axis of symmetry. $C_{22} = C_{11}$, $C_{55} = C_{44}$, $C_{23} = C_{13}$, $C_{66} = {}^1/_2\,(C_{11} - C_{12})$. So, this is for a material with one axis of symmetry, and that is the $\tilde{e}_3$ axis, meaning transversely isotropic about the $\tilde{e}_3$ axis. Now, if you consider $\tilde{e}_2$ to be the axis of symmetry instead of $\tilde{e}_3$.

So, this is for the transversely isotropic material with $\tilde{e}_2$ being the axis of symmetry; the non-zero components of the elastic stiffness tensor can be written like this. $C_{33} = C_{11}$, $C_{66} = C_{44}$, $C_{23} = C_{12}$, $C_{55} = {}^1/_2\,(C_{11} - C_{13})$. Now, we have imposed two axes of symmetry which are orthogonal to each other, and with that, these sets of conditions are coming, and all the rest of the constants are 0. Now, if you compare, $C_{11} = C_{22}$ from the first case, which is also equal to C_{33} from the next case. Similarly, , $C_{44}C_{55}$, and C_{66} are the same.

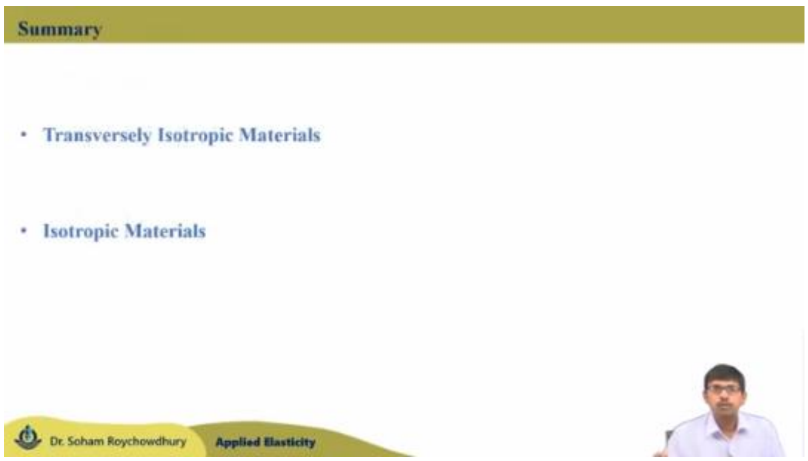
C_{23}, C_{13}, and C_{12} are also the same, and using the last equation, $C_{66} = {}^1/_2\,(C_{11} - C_{12})$, $C_{55} = {}^1/_2\,(C_{11} - C_{12})$, and also C_{55} and C_{66} being the same, C_{12} and C_{13} being the same. Finally, all these can be combined to only two non-zero independent elastic constants. Imposing the axis of symmetry for $\tilde{e}_2$ and $\tilde{e}_3$ automatically imposes $\tilde{e}_1$ being another axis of symmetry by default. So, if a material is transversely isotropic about two orthogonal axes of symmetry, then the third mutually orthogonal axis is already another axis of symmetry. This discussion is exactly similar to the case of orthotropic material.

If the material is monoclinic, r having a plane of symmetry about two orthogonal planes, then the third orthogonal plane is already another plane of symmetry, which ensures the material to be orthotropic. The same is true for this particular case. If a material has two axes of symmetry that are orthogonal to each other, then

the third orthogonal axis is also another axis of symmetry by default. Writing all these conditions with all three axes—all three orthogonal axes being axes of symmetry—$C_{11} = C_{22} = C_{33}$, $C_{12} = C_{23} = C_{13}$, $C_{44} = C_{55} = C_{66}$, and that equals ${}^1/_2\,(C_{11} - C_{12})$. The total constitutive equation for this linear elastic isotropic solid can be written like this, and you can see C_{11} is one independent elastic constant, C_{12} is another independent elastic constant, and all the rest.

are written in terms of that; even C_{44}, C_{55}, and C_{66} are also written in terms of C_{11} and C_{12}. Thus, only two independent elastic coefficients exist for the case of isotropic materials. Those are C_{11} and C_{12}. So, for the general case, we started with 81, then, by imposing different symmetries—major and minor symmetry—we reduced it to 21, followed

Then followed by 13, then 9, 5, and 2, as you go through different levels of material symmetry such as monoclinic, orthotropic, transversely isotropic, and isotropic materials, respectively. This homogeneous, isotropic, linear elastic solid is the one we use for most of our applications, most of the modeling problems,



which has only two independent elastic constants. So, in this lecture, we discussed two types of materials: transversely isotropic material and isotropic material. Thank you.