APPLIED ELASTICITY

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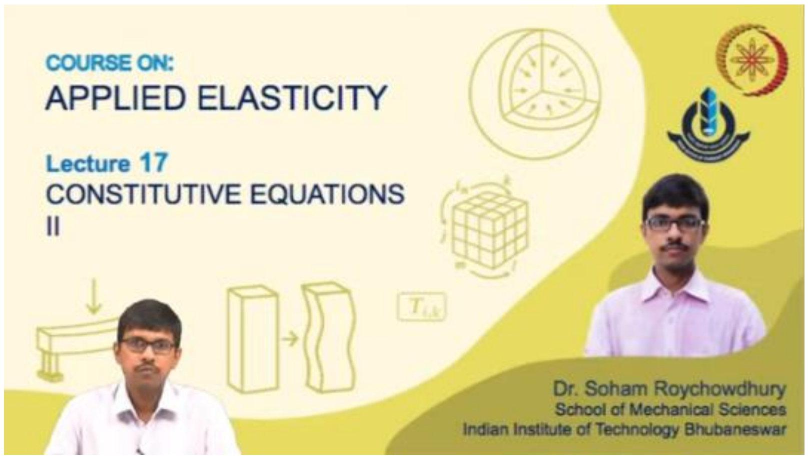
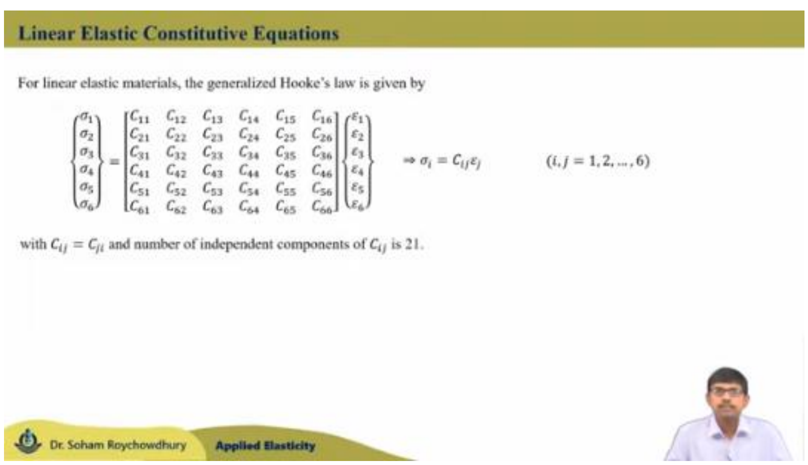
WEEK: 04
Lecture- 17

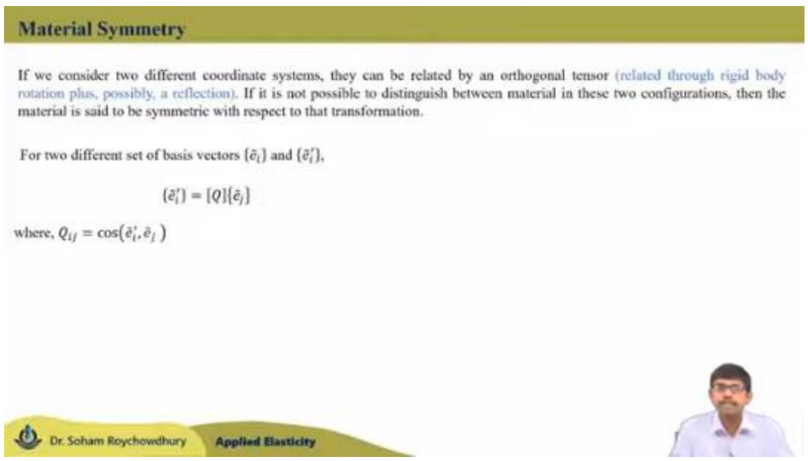


Welcome back to the course on applied elasticity. In the previous lecture, we had started our discussion on the constitutive equation. In today's lecture, we will continue with the discussion on constitutive equations, which which are nothing but the relation between the stress components and the strain components.



So first to have a quick recap what we had discussed in the last class the linear elastic constitutive materials which are having the assumption of small strain or small deformation and the linear behavior between the stress and strain, linear behavior between the applied force and resulting deformation, the generalized Hooke's law or the constitutive relation was defined like this. So, left hand side is the sigma stress tensor which is equals to C tensor acting over epsilon where epsilon is the strain tensor and C is called called the elastic stiffness tensor.

So, we can write in the engineering or void Kelvin notation σ_i is equals to $C_{ij}\varepsilon_j$ where i and j are varying from 1 to 6. this particular C matrix or C tensor is having 21 independent elastic constants and it is also a symmetric tensor. So, C_{ij} is equals to C_{ji}. Now, based on this we are going to discuss about the concept of material symmetry.

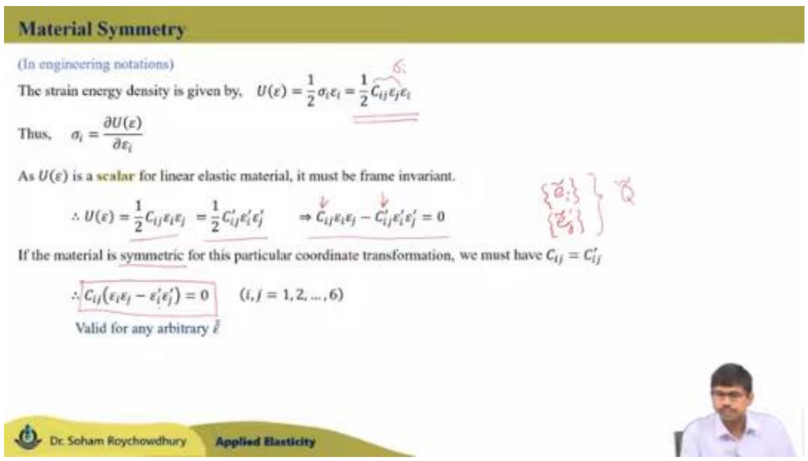


So, how is it defined? if we are considering different frames of reference or different coordinate systems which are related through an orthogonal tensor, orthogonal transformation tensor. So, earlier also we had discussed about this transformation of vectors, transformation of tensors, transformation of stress and strains. So, in the similar fashion if we are having 2 set of reference frame, 2 set of base vectors connected through one orthogonal tensor which physically means a rotation or a reflection or a combination of both rotation followed by a reflection that is the physical significance of any orthogonal tensor. So, if it is not possible to distinguish the material or the material properties in these two configurations. then the material is said to be symmetric with respect to that transformation.

So, this is the definition of the material symmetry. We are considering two coordinate systems,two different basis frames and with that we are writing the expressions for the material constitutive behaviour. If it is not possible to distinguish between the material properties in these two configurations described by two different coordinate systems, then the material is said to be symmetric with respect to this particular transformation through which these two coordinate systems are related.

Now, let us consider two sets of base vector systems, $\tilde{e}_i$ and $\tilde{e}_i'$ or $\tilde{e}_i'$. And these are related through one orthogonal transformation tensor Q. So, you can write the $\tilde{e}_i'$ vector as equal to Q acting over the $\tilde{e}_j$ vector, where $\tilde{e}_i'$ and $\tilde{e}_j$ are the base vectors of two different configurations in the same vector space. Now, what is Q?

Q is an orthogonal transformation tensor whose components are nothing but the cosine of the angle between $\tilde{e}_i'$ and $\tilde{e}_j$. So, if you are taking the cosine of the angle between $\tilde{e}_i'$ and $\tilde{e}_j$—basically, the dot product of $\tilde{e}_i'$ and $\tilde{e}_j$—that will give $i\ j$ component of the orthogonal transformation tensor Q.



Now, if you are writing the strain energy density the strain energy density, capital U, which is the strain energy stored within the body per unit volume, was defined as half of $\sigma_i\varepsilon_i$, and we are writing this in the engineering notation. Otherwise, it would be half of $\sigma_{ij}\varepsilon_{ij}$. For the engineering notation, we are associating one subscript with σ and one subscript with ε, where two subscripts are associated with C. Thus, in the engineering or Voigt-Kelvin notation, the strain energy density U would be half of $\sigma_i\varepsilon_i$.

And then, substituting the constitutive equation on the right-hand side, writing σ_i as $C_{ij}\varepsilon_j$. So, this is the constitutive equation. So, this is the constitutive equation. σ_i is equal to $C_{ij}\varepsilon_j$. So, $U(\varepsilon)$, the function of epsilon, would be half of $C_{ij}\varepsilon_j\varepsilon_i$. And σ_i is defined with respect to U as $\frac{\partial U}{\partial \varepsilon_i}$. Now, as this U, the strain energy density, is a scalar quantity for linear elastic materials, which is the inner product of sigma and epsilon divided by 2, it must be frame-invariant. Only the vector and tensor quantities—their components—may depend on the choice of the coordinate system or the choice of the frame. However, if you have a scalar quantity, as that is only having a magnitude, that must be frame-invariant. Now, u being a scalar strain energy density, a scalar quantity, should be invariant to the frame.

So, whatever frame—$\tilde{e}_i'$ frame or $\tilde{e}_i$ frame—whatever base vectors we choose, the value of this scalar U must be the same in both coordinate systems.

Now, U can be written as half $C_{ij}\varepsilon_i\varepsilon_j$ from this. Similarly, in the $\tilde{e}_i'$ frame, we can write u as half of $C_{ij}'\varepsilon_i'\varepsilon_j'$, and as we know that U is a frame-invariant quantity, both of them must be the same. ε_i and ε_i' may not be the same; C_{ij} and C_{ij}' may not be the same because those are vector or tensor quantities.

However, U being a scalar, U in the $\tilde{e}_i$ frame—that is, half $C_{ij}\varepsilon_i\varepsilon_j$—must be the same as U in the $\tilde{e}_i'$ frame—that is, half $C_{ij}'\varepsilon_i'\varepsilon_j'$. Now, taking the right-hand side term, this term on the left-hand side, and cancelling half, we can write that $C_{ij}\varepsilon_i\varepsilon_j$ minus $C_{ij}'\varepsilon_i'\varepsilon_j'$ equals 0. Now, coming to the definition of material symmetry: a material is said to be symmetric with respect to this transformation from $\tilde{e}_i$ vector frame to $\tilde{e}_j'$ vector frame only if the corresponding material properties are the same.

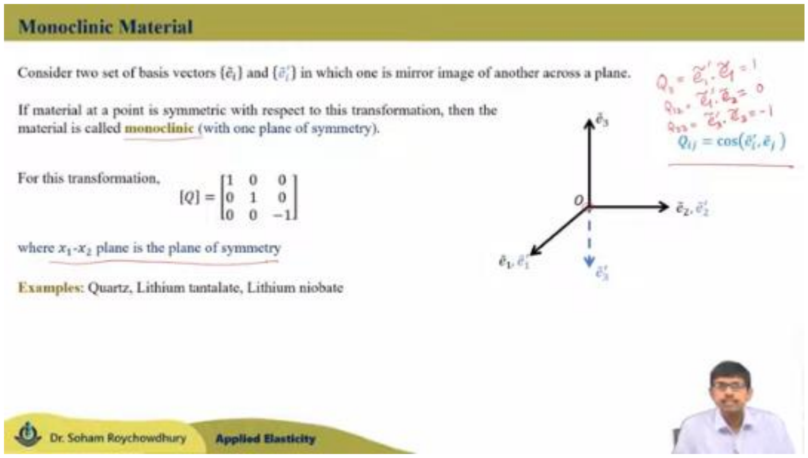
Now, in this equation, these C_{ij} or C_{ij}' terms are the terms corresponding to the material property. So, for the material to be symmetric, properties should be identical in both frames. Thus, we must have C_{ij} equal to C_{ij}' in both reference frames, and thus Taking C_{ij} to be common, the material symmetry equation becomes C_{ij} times $(\varepsilon_i\varepsilon_j - \varepsilon_i'\varepsilon_j')$ equals 0, where i and j can take all values from 1, 2, 3 up to 6.

And this is valid for any arbitrary epsilon. For any strain, if the material is symmetric with respect to a specific transformation, this particular expression must be satisfied. And you need to find out this ε_i', ε_j' with the help of the transformation matrix Q, which relates $\tilde{e}_i$ vector and $\tilde{e}_j'$ vector. Now, we will be taking up different materials that exhibit this symmetry behavior.

So, first we will start with monoclinic material, then proceed to orthotropic material, then transversely isotropic material, and finally, isotropic material. These are the four types of materials in which different material symmetries are observed. So, first, starting with the monoclinic material. Before having any kind of symmetry, before imposing material symmetry, we had 21 independent elastic constants in the elastic stiffness tensor C.

After accounting for the major and minor symmetries of C, 21 independent elastic constants remained. Now, for monoclinic, orthotropic, transversely isotropic, and isotropic materials, we shall impose further symmetries, and due to that, the number of independent elastic constants will reduce from 21.

Finally, for isotropic materials, we will show that this number is just 2. We need only 2 material constants to write the constitutive equation for isotropic materials.



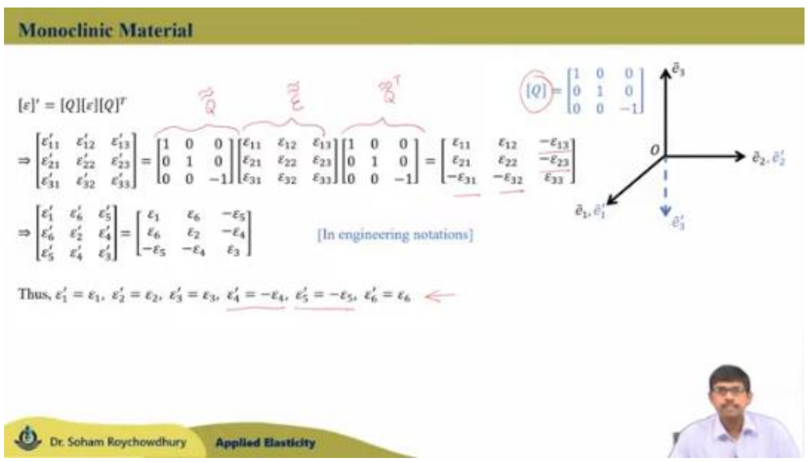
Now, first, starting with the monoclinic material. Let us consider two basis vectors for the same vector space. One is the $\tilde{e}_i$ vector, another is the $\tilde{e}'_i$ vector, where one is the mirror image of the other with respect to a plane. So, let us consider this first frame, the $\tilde{e}_i$ vector frame, where $\tilde{e}_1$, $\tilde{e}_2$, and $\tilde{e}_3$ are the three unit vectors with O being the origin. Now, the second frame $\tilde{e}'_i$ is considered like this, where $\tilde{e}'_1$ is the same as the $\tilde{e}_1$ vector, $\tilde{e}'_2$ vector is the same as the $\tilde{e}_2$ vector, but the $\tilde{e}'_3$ prime vector is just in the opposite direction of the $\tilde{e}_3$ vector. So, if you consider this $\tilde{e}_1$, $\tilde{e}_2$ plane, the horizontal plane, with respect to that, these two sets of base vectors are mirror images to each other as $\tilde{e}_3$ is equal to minus $\tilde{e}'_3$. Now, the material is defined to be monoclinic. At a point, it is symmetric with respect to this particular type of transformation. So, if you are having a transformation with one plane of symmetry and with respect to that transformation, the material behavior is symmetric, material properties are symmetric, then we call that material to be monoclinic.

So, if you consider point O, where the horizontal plane is the plane of symmetry and the $\tilde{e}_3$ vector and $\tilde{e}'_3$ vector are just opposite to each other. With respect to this transformation at point O, if the material is showing symmetric behavior at the plane of symmetry, then we call the material to be monoclinic. Now Now, our objective is to reduce the number of independent constants and find out the constitutive equation for this type of monoclinic material.

For that, this is the transformation matrix Q. Now, how to obtain Q? Q is defined as cos of $\tilde{e}'_i$ and $\tilde{e}_j$. So, Q_{ij}, let us say Q_{11}. Q_{11} would be $\tilde{e}_1$ dot $\tilde{e}'_1$ and both $\tilde{e}_1$ and $\tilde{e}'_1$ being in the same direction, this is equal to 1. Now, Q_{12} would be $\tilde{e}_1$ dot, no sorry, this would be E first one is prime as per our definition; you can just take the opposite one.

With that, Q can also be obtained. So, Q_{12} would be $\tilde{e}_1'$ dot $\tilde{e}_2$, which would be 0 because the angle between $\tilde{e}_1'$ and $\tilde{e}_2$ is 90 degrees. Similarly, if you go for Q_{33}, which is $\tilde{e}_3'$ dot $\tilde{e}_3$, this would be minus 1 because the angle between them is 180 degrees, and thus the Q transformation orthogonal transformation tensor would be 1 0 0 0 1 0 0 0 minus 1, which is defined through this particular expression. Now, with respect to this transformation, the material is x_1-x_2 is the plane of symmetry, and the plane containing $\tilde{e}_1 - \tilde{e}_2$ is the plane of symmetry.

Now, a few examples of this type of monoclinic material are quartz, lithium tantalate, and lithium niobate. These kinds of materials exhibit monoclinic symmetry or are called monoclinic materials, which have one plane of symmetry.

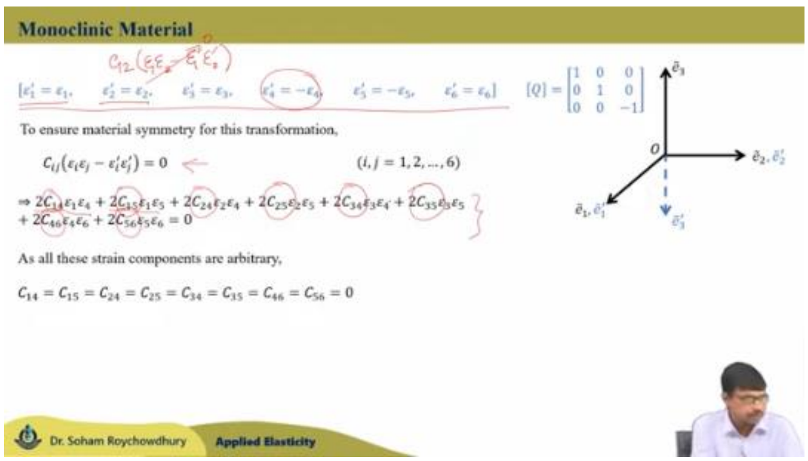


Now, moving further with this form of Q for this particular transformation with one plane of symmetry, we would like to relate the strain components in the $\tilde{e}_i$ frame and $\tilde{e}_i'$ frame, following the transformation of a second-order tensor. Epsilon, or strain, being a second-order tensor, if you are going for the transformation of a second-order tensor with respect to Q, the orthogonal transformation tensor $[\varepsilon]'$ can be written as $[Q][\varepsilon][Q]^T$. Now, I am writing all the terms of this strain tensor $[\varepsilon]'$. Prime means in the transformed frame of reference. So, ε_{11}', ε_{12}', ε_{13}', and so on.

This is equal to Q, the first one is Q, then this is epsilon, which is the strain before transformation, and the last one is $[Q]^T$. Q and $[Q]^T$ are the same for this particular Q. Now, if you multiply all these three, you would get ε_{11}, ε_{12}, $-\varepsilon_{13}$, and so on. You can check there are four locations where negative signs are present. If I now convert this into engineering notation for simplicity by writing ε_{11} as ε_1,

ε_{12} as ε_6, and so on, following the engineering notation, which is the standard one. And then, compare both sides; we would end up with this set of relations of strain components in two frames. So, here, ε_1' equals ε_1, ε_2' equals ε_1, ε_3' equals ε_3, and ε_6' equals ε_6. These four components of strain are identical in both frames. ε_4' is the negative of ε_4, and ε_5' is the negative of ε_5.

These are the two components where a negative sign is associated while relating the strains with respect to this transformation, having one plane of symmetry, which is the x_1-x_2 plane.

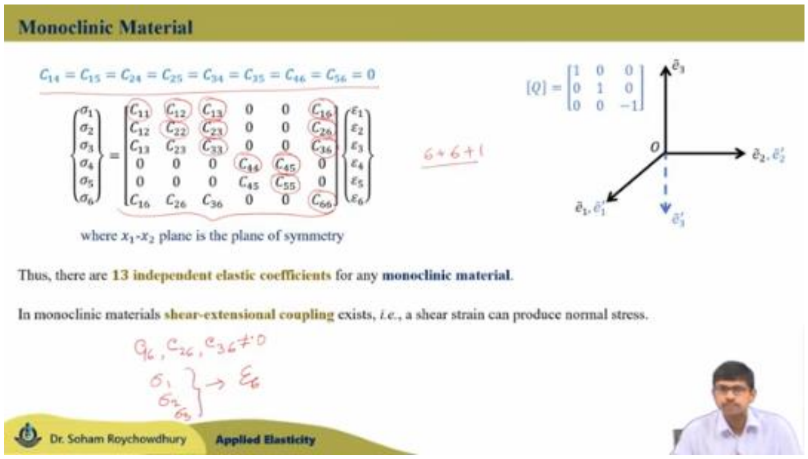


Now, using this epsilon relation between two frames, if you invoke the equation of material symmetry, what was that? $C_{ij}\left(\varepsilon_i\varepsilon_j - \varepsilon_i'\varepsilon_j'\right)$ equals 0 for ij varying from 1 to 6. Now, if you expand this equation for all values of i and j, This would be like this: $2C_{14}\varepsilon_1\varepsilon_4 + 2C_{15}\varepsilon_1\varepsilon_5 + 2C_{24}\varepsilon_2\varepsilon_4 + 2C_{25}\varepsilon_2\varepsilon_5 + 2C_{34}\varepsilon_3\varepsilon_4 + 2C_{35}\varepsilon_3\varepsilon_5 + 2C_{46}\varepsilon_4\varepsilon_6 + 2C_{56}\varepsilon_5\varepsilon_6 = 0$.

Here, if you carefully notice, you can see only eight constants are present, not all. Why? Because of this kind of relation between the epsilon. Let us consider the constant C_{12}. If you consider C_{12}, what would be C_{12} times ε_1, ε_2 minus ε_1', ε_2'?

This is the term corresponding to 1, 2. i is 1, j is 2. Now, using ε_1' being ε_1 and ε_2' being ε_2, this quantity will go to 0. In the same fashion, all the terms corresponding where both epsilon components are the same, they would go to 0. The non-zero terms are only those where one of the epsilons is related with a negative sign and another one is related with a positive sign.

So, if you combine ε_1 and 4, that term would be there as a non-zero term. Combining 1 and 5, a non-zero term. Combining 2 and 4, 2 and 5, 3 and 4, 3 and 5, 4 and 6, and 5 and 6. These are the combinations which would result in a non-zero term if you are expanding this equation.



So, it must associate either 4 or 5 and anyone else. If you combine 4 and 5, both having a negative relation, that would also go to 0. So, these are the 8 terms which would be the non-zero terms if you expand this equation and the right-hand side equals 0. Now, as this equation is valid for any arbitrary epsilon, any arbitrary stress component, or strain component, to have that, we must have all these constants: C_{14}, C_{46}, C_{15}, C_{56}, 24, 25, 34, and 35. All these constants must be 0.

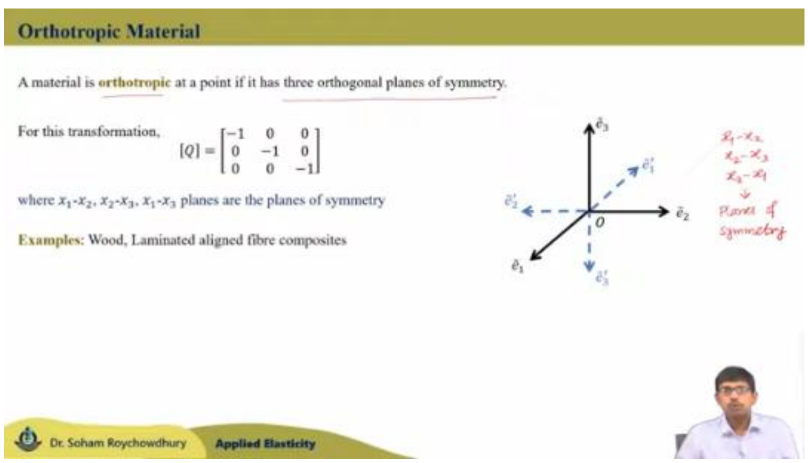
For any arbitrary epsilon, the above equation can be 0 only if all these 8 constants go to 0. Thus, for a monoclinic material with the x_1-x_2 plane being the plane of symmetry, we have 8 material constants equal to 0. And with these 8 material constants equal to 0, if we explicitly write the constitutive equation sigma equals C times epsilon and impose all these material constants to be 0 here, the C matrix would look like this. In the first row, C_{11}, C_{12}, C_{13}, then two 0s, then C_{16}, and it will continue as shown here.

Now, if you start counting, Here, you can see only 13 independent elastic coefficients are present for the monoclinic material with x1, x2 being the plane of symmetry. You can see all the diagonal terms: C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}. These 6 diagonal terms are independent. Along with that, we have C_{16}, C_{26}, C_{36}.

C_{11}, C_{12}, and C_{13}. So, 6 more and 1 here: C_{45}. So, in total, 13 independent elastic constants are there; all the rest are 0. This is the constitutive equation for a monoclinic material where, from 21, the number of independent elastic constants is reduced to 13. Now, if you carefully notice, the shear and extensional coupling exists.

Because of non-zero values of C_{16}, C_{26}, C_{36}. These terms are relating σ_1, σ_2 and σ_3. set of stresses are related with ε_6. Now, what is ε_6? ε_6 is one shear strain whereas, σ_1, σ_2, σ_3 are normal stresses along 1, 2 and 3 direction.

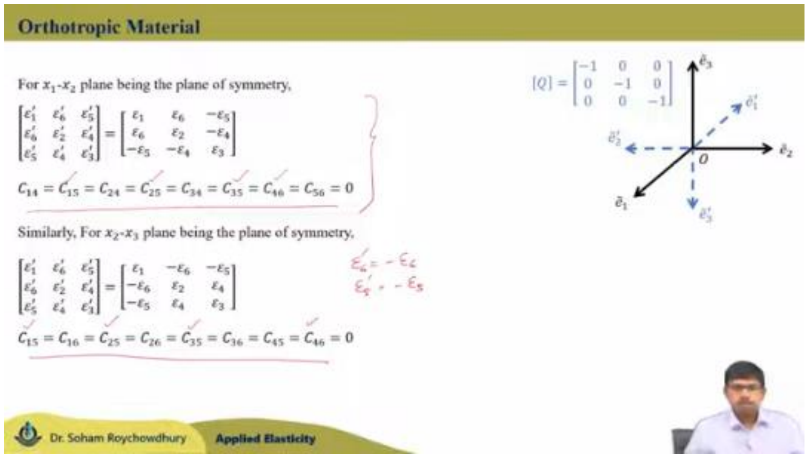
So, as these C_{16}, C_{26}, C_{36} are non-zero. Due to presence of these non-zero elements, there is a coupling between the extensional and shear strain and normal stress or shear stress and normal strain. So, thus if you are applying the normal stress to the normal load to the system that will result shear strain if you are applying shear strain to the system that would result generation of normal stress. So, shear strain can produce normal stress for the case of monoclinic materials.



Now, we will move further for the orthotropic material. Orthotropic material is defined to be a material which is having three mutually orthogonal planes of symmetry. So, we are considering one frame of reference $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ and another frame of reference $\tilde{e}'_1$, $\tilde{e}'_2$, $\tilde{e}'_3$ where $\tilde{e}'_1$ is in the opposite direction of $\tilde{e}'_1$, $\tilde{e}'_2$ is in the opposite direction of $\tilde{e}'_2$, $\tilde{e}'_3$ is in the opposite direction of $\tilde{e}_3$. So, thus for this transformation all these x_1-x_2, then x_2-x_3 and x_3-x_1, all these are planes of symmetry for this particular transformation. And for such cases, the transformation tensor Q can be written as minus 1, 0, 0, 0, minus 1, 0, 0, 0, minus 1. So, all the diagonal terms would be minus 1 because $\tilde{e}_1$ and $\tilde{e}'_1$ are in opposite directions, having an angle pi, and so on. Same for $\tilde{e}_2$, $\tilde{e}'_2$, $\tilde{e}_3$, and $\tilde{e}'_3$. So, all three planes are planes of symmetry, and all these three planes are mutually orthogonal planes.

So, any orthotropic material has three mutually orthogonal planes of symmetry, whereas a monoclinic material has only one plane of symmetry. Now, what are the examples?

Wood, laminated, fibre-reinforced composites—these are the materials which behave as orthotropic materials. Now, coming to the constitutive behavior with this form of Q, where minus 1 is present in all diagonal elements and non-diagonal elements are 0, we will impose the planes of symmetry one after another.

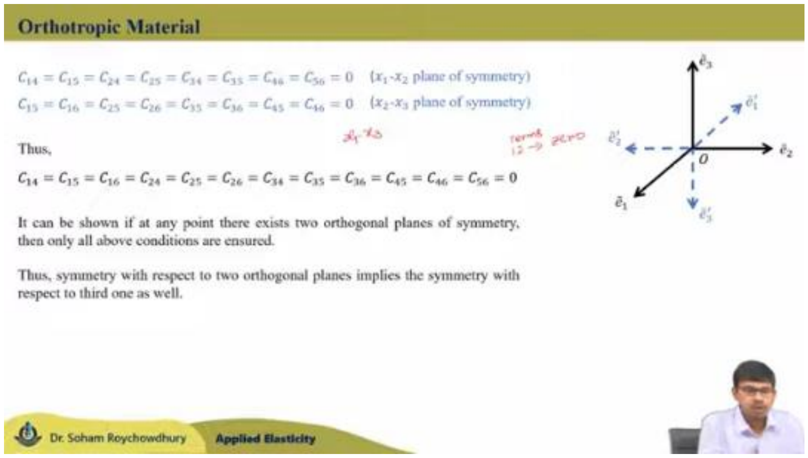


There are three orthogonal planes of symmetry for the orthotropic material: x_1-x_2; x_2-x_3; and x_1-x_3. First, let us start with the x_1-x_2 plane of symmetry, which is the same as the monoclinic material just discussed before this. So, with x_1-x_2 being the plane of symmetry, the relation between the strain components in two frames—the $\tilde{e}_i'$ frame and the $\tilde{e}_j$ frame—is like this. And with that, we had already shown for the monoclinic material that these 8 components—C_{14}, 15, 24, 25, 34, 35, 46, 56—are 0. Now, so I am not going to discuss this in detail.

This was already discussed for the monoclinic material with one plane of symmetry, where that plane of symmetry is the x_1-x_2 plane. Now, if you invoke another plane of symmetry—that is, the x_2-x_3 plane of symmetry—and follow a similar procedure as before, we can show that the strain components are related like this, where ε_6'... is equal to minus of ε_6, ε_5' prime is equal to minus of ε_5, and all the rest of the components are the same: ε_1 is ε_1', ε_2 is ε_2', ε_3 is ε_3', epsilon 4 is ε_4'.

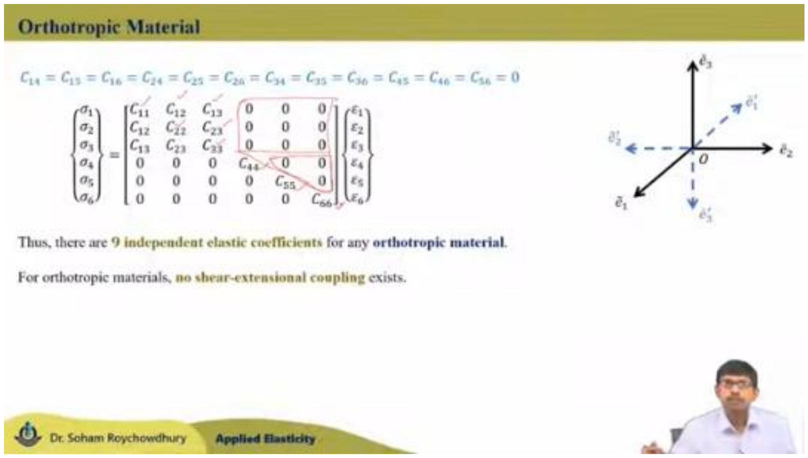
Now, using the material symmetry for this particular case and setting that to 0, you can show that these 8 constants—$C_{15}, C_{16}, C_{25}, C_{26}, C_{35}, C_{36}, C_{45}$,, and C_{46}—would come out to be 0. Now, comparing this, with this, So, the first case is x_1-x_2 being the plane of symmetry, and we are getting 8 constants to be 0.

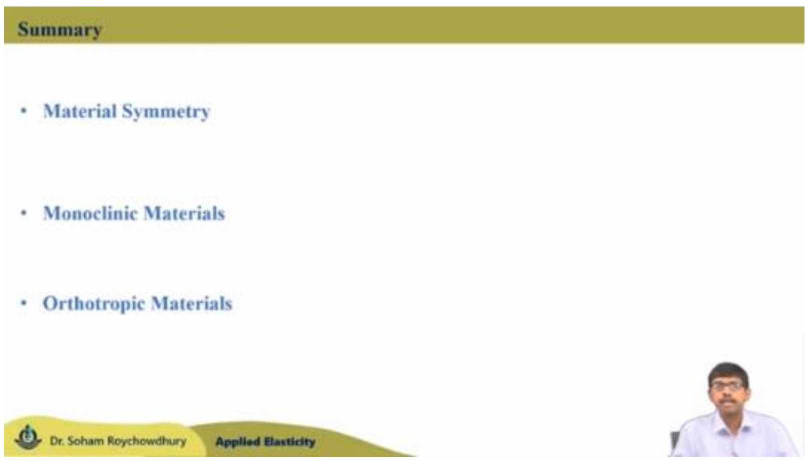
Now, invoking x_2-x_3 to be the plane of symmetry, once again, 8 constants are 0, but all these 8 constants for both cases are not independent; many of them are the same. If you carefully notice, C1 prime is present for both, C_{25} is present for both, C_{35} is present for both, I think C_{46}—that is also present for both. So, 4 are common terms, and 4 exclusive terms are there in the first one; 4 exclusive terms



are there in the second one as well. So, with x_1-x_2 and x_2-x_3 being planes of symmetry, if you combine them in total, you would be getting 12 independent terms going to 0. So, 12 C components are going to 0 if you have 2 mutually orthogonal planes of symmetry. So, this is monoclinic about x_1-x_2; monoclinic about x_2-x_3. But how was orthotropic defined? Three mutually orthogonal planes of symmetry. Now, we have to look for the third plane of symmetry, but as we have already invoked two planes of symmetry, with that, all 12 terms came out to be 0. There is no point in invoking the third plane of symmetry. Even if you do it, if you invoke the x1x3 plane of symmetry, with that, once again, you will get eight particular terms—eight components of c to be zero—which are already available in this equation. So, we have already proved those to be zero.

Thus, at any point, if there exist two orthogonal planes of symmetry, all the above conditions are ensured, and the third plane of symmetry already exists. So, symmetry with respect to two orthogonal planes directly implies symmetry with respect to the third plane as well. So, no need to invoke or write those equations separately. So, all these 12 constants would be 0 for the case of orthotropic material. Now, with these 12 constants being 0, if you write the constitutive equation for orthotropic material, it would look like this.





So, $\tilde{\tilde{\sigma}}$ is equals to $\tilde{\tilde{\tilde{\tilde{C}}}}$ $\tilde{\tilde{\varepsilon}}$, where you can see all this set of terms are 0. So, this top 9 terms, these are responsible for shear extensional coupling, whereas these 3 terms, these are responsible for the shear shear coupling and here all such couplings are 0. So, only 9 independent elastic coefficients non-zero coefficients are present in the orthotropic material C_{11}, C_{22}, C_{33}, C_{13}, 2 1 2 and 2 3 then C_{44}, C_{55}, C_{66}. These are the 9 non-zero independent elastic constants which are present in the orthotropic material and thus no shear and extensional coupling is existing for the orthotropic material. So, we started with 21 elastic constants for a general linear elastic solid with major minus symmetry. Then, for the monoclinic material having one plane of symmetry, the number of independent elastic constant has been reduced to 13 from 21. Now, for orthotropic material, the number of independent elastic constants is further reduced to 9 from 13, which is having three mutually orthogonal planes of symmetry. So, in this lecture we had discussed or introduced the concept of material symmetry and then we had discussed in details for two types of material monoclinic and orthotropic materials. Thank you.