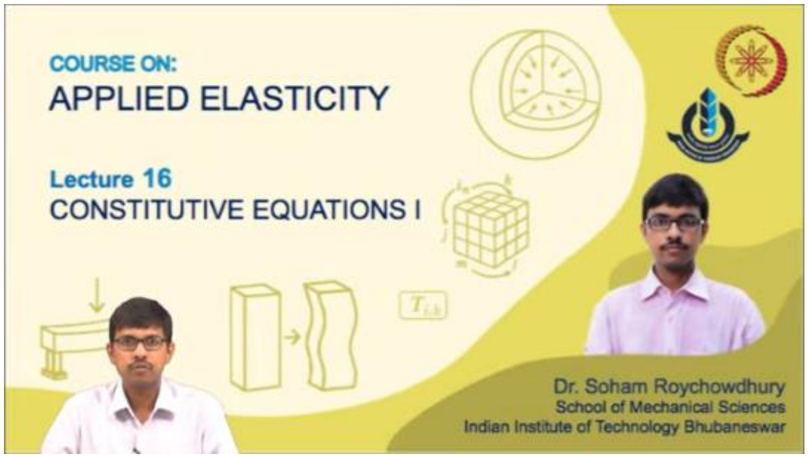
APPLIED ELASTICITY

Dr. SOHAM ROYCHOWDHURY

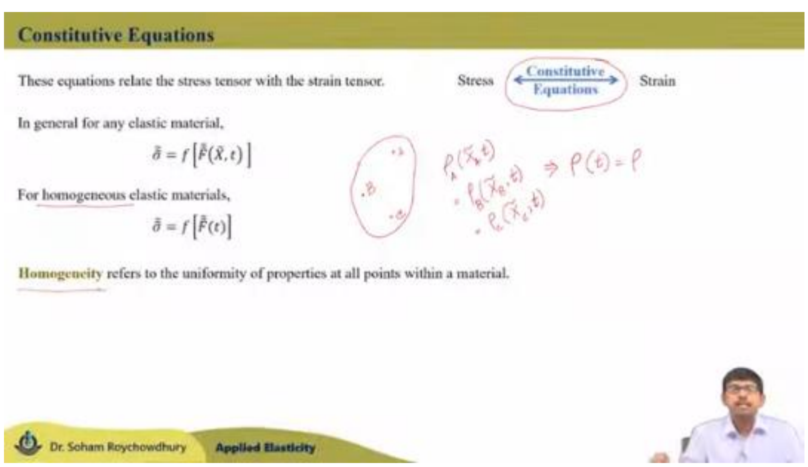
SCHOOL OF MECHANICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR

WEEK: 04
Lecture- 16



Welcome back to the course on applied elasticity. In this course, till now, we have discussed the different strain measures, which was followed by a discussion on different stress measures. We have also talked about the balance laws. Now, in the lectures of this particular week, we are going to talk about the constitutive equations, which is the relation between the stress measure and the strain measure.



So, the constitutive equations are the equations that relate the stress tensor with the strain tensor. So, we have various stress and strain measures. We have discussed three different stress measures: Cauchy stress, which is defined in the deformed frame, the current force acting on the current or deformed unit area. Then, we have also talked about the first and second Piola-Kirchhoff stress tensors.

The first Piola-Kirchhoff stress tensor was defined as the current force acting per unit initial or undeformed area. Whereas, the second Piola-Kirchhoff stress tensor was defined as the transformed current force acting per unit undeformed area. So, these are the three different stress measures we have discussed. Now, coming to the strain, strain measures can be defined in either the Eulerian frame or the Lagrangian frame. In the Lagrangian or the material description, the name of the strain measure which we have discussed was the Green-Lagrange strain tensor.

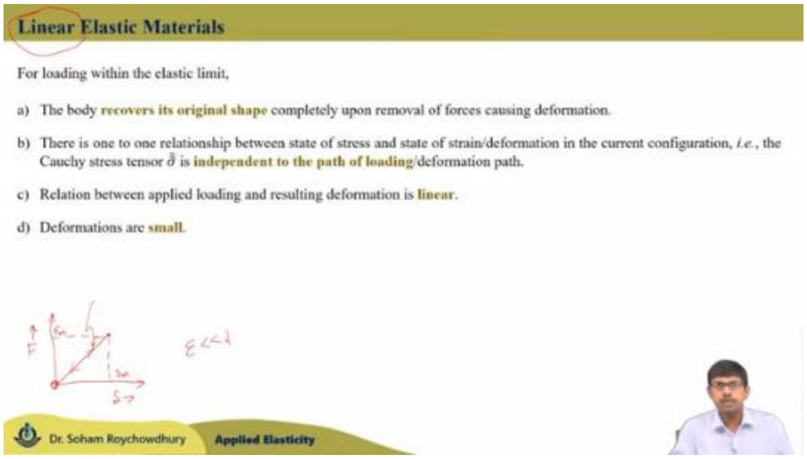
And in the Eulerian frame or the spatial frame current configuration, the strain measure was named the Euler-Almansi strain tensor. We had also shown that under the assumption of small deformation, the deformation being small, thus neglecting the nonlinear terms for both strain measures, the Green-Lagrange strain tensor G star as well as the Euler-Almansi strain tensor small e star, both reduce to the same form, which we had defined as the infinitesimal small linear strain tensor denoted by epsilon.

Now, any of these stress measures can be coupled with any of the strain measures with the help of constitutive equations. So, constitutive equations are nothing but the set of equations that relate the stress components of the stress tensor with the components of the strain tensor. Now, in general, the stress tensor is a function of the deformation gradient tensor for any elastic body undergoing deformation. So, $\tilde{\tilde{F}}$ is defined as the deformation gradient tensor, which is in general a function of both time and material coordinate capital $\tilde{X}$.

Capital $\tilde{X}$ is the coordinate of any point in the undeformed state in the material description. So, in general, $\tilde{\tilde{F}}$ is a function of both the position vector capital $\tilde{X}$ and time, and thus sigma is also dependent on both the position vector and time. Now, if we have a homogeneous elastic solid, for homogeneous elastic materials, the stress sigma must not be a function of the position vector $\tilde{X}$. Thus, we can define this property of homogeneity of any material as the uniformity of all properties at all different points within the material.

So, let us consider a material and different points A, B, C, and so on. Now, if you choose any property ρ, $\tilde{X}$, t. So, that property at point A should be the same as that property at point B, and that should be the same as that property at point C. So, for point A, the position vector is $\tilde{X}_A$; for point B, the position vector is $\tilde{X}_B$; and for point C, the position vector is $\tilde{X}_C$.

And if all these properties are the same, at all three different points, that means, ρ is a function of t only and not a function of $\tilde{X}$. It is independent of the location of the point, and if this is true, then we call that material a homogeneous material. Thus, homogeneity is nothing, but the uniformity of the material properties at all different material points.



Now, moving forward to the linear elastic material. So, first, we are going to discuss the constitutive equations for linear elastic materials and then move forward to the constitutive equations for non-linear elastic materials at the end of this week. So, for loading within the elastic limit, whatever force we apply on the material, if that is within the elastic limit, these observations or assumptions can be imposed for the linear elastic material.

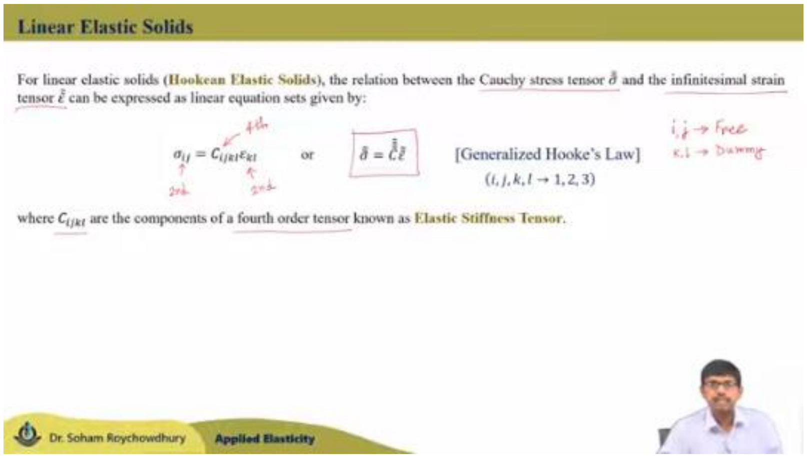
The first one is the body recovers completely to its initial or original shape upon removal of the force causing deformation. So, if you try to plot force versus deformation, F versus delta curve, say we are loading up to this particular force F max and the corresponding deformation is delta max. So, this is the path of loading. Now, once you are unloading it, it would be coming back to its initial position.

Upon complete removal of the force causing the deformation, body must recover to its original shape. Thus, there should not be any permanent set or permanent deformation present once the unloading process is completed. Now, coming to the next point, there is one-to-one relationship between the state of stress and state of strain or deformation in the current configuration. For one particular value of sigma, you can just have a single particular value of epsilon.

Or with respect to force displacement curve, any one particular force corresponds to a specific displacement only. That means the Cauchy stress tensor sigma is independent to the deformation path, independent to the path of loading. In whatever way, in whatever path we are applying the load, Irrespective of that, once a specific load is reached, that corresponds to a specific value of displacement. The Cauchy stress, that is stress measured in the deformed configuration is independent to the path of deformation or independent to the path of loading for the linear elastic materials.

Now, coming to the next point, the relation between the applied loading F and the resulting deformation sigma is linear. And that is why the name 'linear' is used with this kind of material. So, the F versus delta curve is a straight line. This particular line is a straight line. Similarly, the sigma versus epsilon curve would also be a straight line.

And deformations are assumed to be small. So, the value of delta—whatever deformation we are having or whatever strain we are having—is less than 1, which ensures the assumption of small deformation. So, under all these assumptions, we can define the constitutive equation for linear elastic materials. Now, coming to the constitutive equation of linear elastic solids, which is also known as Hookean elastic solid. We are trying to relate the stress measure sigma, which is the Cauchy stress tensor, with the infinitesimal or linear strain tensor epsilon.



As we are having the assumption of small strain—or with the assumption of small deformation—both material as well as spatial coordinate strain measures are coming down to the same form of the small linear strain tensor epsilon. So, there is no difference between the spatial and material description of strain under the assumption of small deformation or small strain, which is valid for linear elastic solids. So, now, our objective is to relate the Cauchy stress tensor sigma with the infinitesimal strain tensor epsilon. And as sigma is linearly related with epsilon, as stated in one of the assumptions,

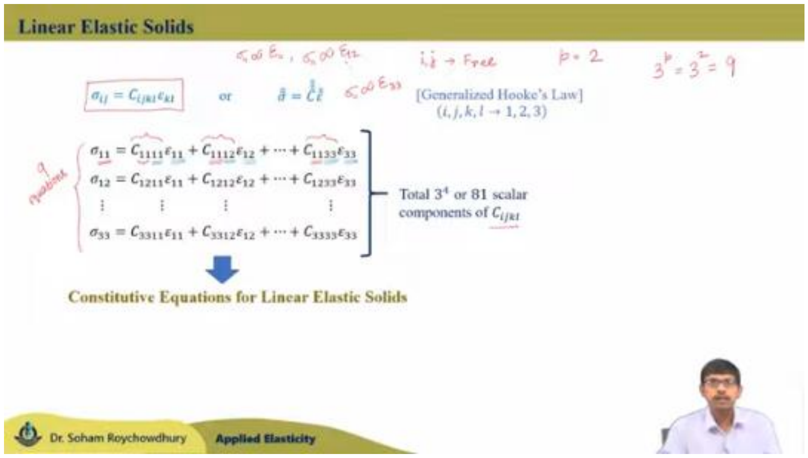
The constitutive equations can be written like this in this particular form. σ_{ijkl}, the stress tensor component, is $C_{ijkl}\varepsilon_{kl}$, or in the tensor form, the $\tilde{\tilde{\sigma}}$ tensor equals the $\tilde{\tilde{\tilde{\tilde{C}}}}$ tensor acting over $\tilde{\tilde{\varepsilon}}$, where i, j, k, l are the four indices which can take the values of 1, 2, 3. And in this particular equation, you can see i and j. These are appearing in each term on both the left-hand side and the right-hand side.

Thus, i and j are free indices, whereas k and l are not appearing on the left-hand side. They are appearing on the right-hand side twice: you can see k here once and here once. Similarly, l also appears here once and here once. So, k and l are dummy indices.

So, we have two dummy indices and two free indices here. This form of equation is called the generalized Hooke's law, where $\tilde{\tilde{\tilde{\tilde{C}}}}$ is known as the elastic stiffness tensor. The components of $\tilde{\tilde{\tilde{\tilde{C}}}}$ are denoted as C_{ijkl}. Since four subscripts are associated with this, it is a fourth-order tensor.

So, both $\tilde{\tilde{\sigma}}$ and $\tilde{\tilde{\varepsilon}}$ are second-order tensors. However, these two second-order tensors are related with the help of a fourth-order tensor. So, two second-order tensors are related through a transformation via a fourth-order tensor. Two vectors are related through a linear transformation of a second-order tensor. Similarly, two second-order tensors can be related through a transformation similar to this,

with the help of a fourth-order tensor, and that fourth-order tensor is related to nothing but the elastic stiffness tensor with components C_{ijkl}. This particular expression is called the generalized Hooke's law, which is the constitutive equation for linear elastic solids. Now, if we explicitly write all those expressions—if you expand this indicial notation—we are going to get these many equations. So, what will be the total number of equations if you expand σ_{ij} equals $C_{ijkl}\varepsilon_{kl}$?

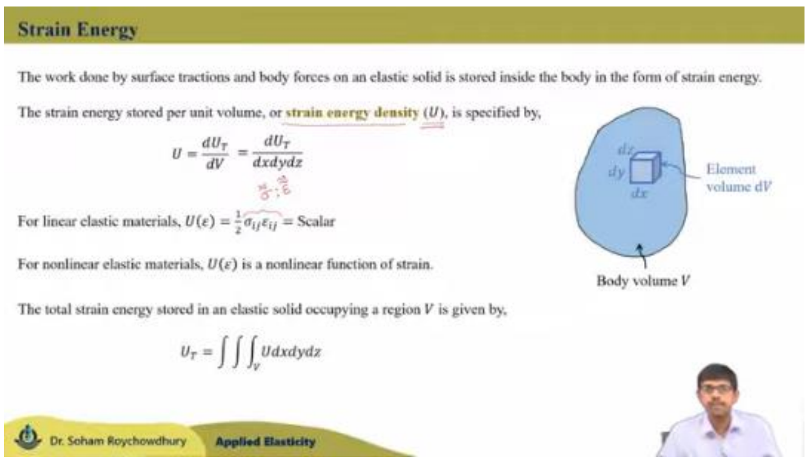


As I had already mentioned, i and j are the two free indices, and thus the number of free indices p equals 2. We had discussed that the total number of equations in the expanded form would be 3 to the power p, where p is the number of free indices. So here, it is 3 squared, or 9. We are supposed to have 9 equations. So here, we have 9 equations, and on the right-hand side

we are having all the epsilon terms. So, the left-hand side is any one stress. Let us say you are taking the first equation. So, σ_{11} equals $C_{1111}\varepsilon_{11}$, $C_{1112}\varepsilon_{12}$, and it continues up to the last term, $C_{1133}\varepsilon_{33}$. So, if you denote—if you carefully look at this—the first two subscripts, of C_{ijkl} are the same as the subscripts of σ, whereas the next two subscripts (these two) are the same as the subscripts of ε. That is valid for all the cases. For the second term, the last two subscripts are 1 2, same as the subscripts of ε; for the last term, 3 3, same as the subscripts of ε. As the left-hand side σ subscript is 1 1, thus all the C terms of the first row are having the first two subscripts as 1 1. The same method is followed for writing all the remaining equations.

So, each of the stress components σ_{ij} is proportional to each of the strain components separately, and thus you can think of σ_{11} being proportional to ε_{11}, where C_{1111} is this proportionality constant. Similarly, σ_{11} is also proportional to ε_{12}, with C_{1112} being the second proportionality constant. Like that, you continue until σ_{11} is proportional to ε_{33}, and C_{1133} is the last proportionality constant. So,

If you are using the linear superposition of all these, the net σ_{11} would be dependent on all 9 strain components, and thus we will get the first equation. Similarly, if you are taking all remaining 8 stress components on the left-hand side—σ_{12}, σ_{22}, σ_{13}, σ_{23}, σ_{33}, σ_{13}, all those are coming on the left-hand side, and we can expand the right-hand side in a similar fashion, and thus we In total, these 9 equations would involve 81 or 3 to the power of 4 scalar components of C_{ijkl} because we have 9 components in sigma and 9 components in epsilon, each of them directly proportional to each other, and thus, a total of 9 cross 9, 81 scalar components of Cijk are required to relate this stress component σ with the corresponding strain components ε. In total, this set of 9 linear equations is known as the constitutive equations for linear elastic solids, which are often called Hookean solids as well. Now, moving forward, let us define the strain energy quantity with the help of these constitutive equations. As we already know, whatever work the external forces do on the body, it gets stored within the body as internal energy, and we call that strain energy.

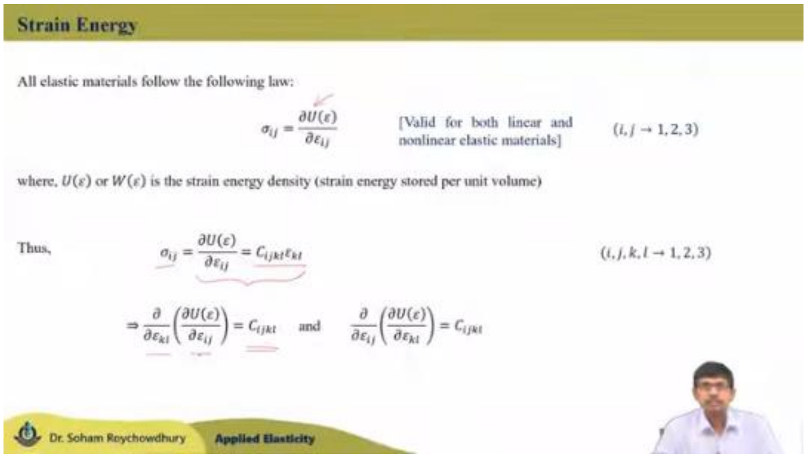


So, the work done by all external forces includes work done by surface tractions acting on the boundaries and work done by body forces present within the material volume. The total work done by surface tractions and body forces within an elastic solid is stored as the strain energy of the body. So, strain energy per unit volume of the body is defined as strain energy density, which I am writing as capital U.

So, let us consider a body as shown here with volume V. Now, taking a small elementary volume dV somewhere within the body, strain energy density can be defined as $\frac{dU_T}{dV}$. So, du is equal to strain energy defined per unit volume, and U_T is the total strain energy stored within the body. So, $\frac{dU_T}{dV}$ is equal to U, which is the strain energy stored per unit volume, and for the small cuboidal elementary volume, dV is equal to $dx\ dy\ dz$.

Thus, we can write U as dU_T by $dx\ dy\ dz$. For any linear elastic material where σ and ε are related linearly. We can write u as half of $\sigma_{ij}\varepsilon_{ij}$, and u or strain energy is always a scalar quantity. So, this is nothing but the inner product of σ and ε.

The σ tensor's inner product with the ε tensor, and half of that, is going to give us the strain energy for the linear elastic solid. Now, for the non-linear elastic solid, U would be the strain energy density function would be a non-linear function of strain or ε, and what will be the form of that, we will discuss later.

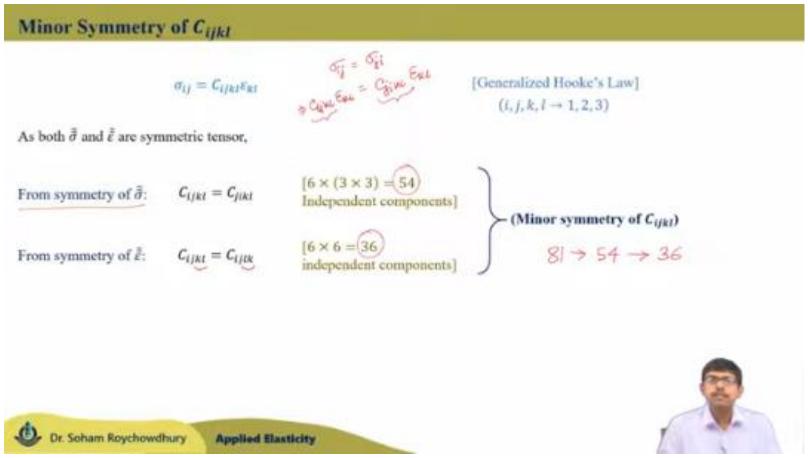


Now, the total strain energy stored within the body of volume V can be written as the integral of U. of U, strain energy density, over the total volume, which is dx, dy, dz. Now, for all the elastic materials, we can rewrite the constitutive equation in the form of strain energy as σ_{ij} is equal to $\frac{\partial U(\varepsilon)}{\partial \varepsilon_{ij}}$. So, if you take the derivative of the strain energy density with respect to ε_{ij}, we will get back the corresponding stress component σ_{ij}.

And this is valid for both linear as well as nonlinear elastic solids. So, whether it is linear or nonlinear, Constitutive behaviour being linear or non-linear with that, this equation is not going to get affected. So, only the form of U would change. For linear elastic, this is half $\sigma_{ij}\varepsilon_{ij}$.

For non-linear elastic solid, this would be non-linear function of epsilon. Now, U or sometimes it is written as W, which is the strain energy stored per unit volume. If you are taking the derivative of that with respect to strain component ε_{ij}, we would be getting the corresponding stress component ε_{ij}. And σ_{ij} we can write as $C_{ijkl}\varepsilon_{kl}$ with the help of the generalized Hooke's law. σ_{ij} is nothing but $C_{ijkl}\varepsilon_{kl}$.

So, $\partial\sigma \frac{\partial}{\partial \varepsilon_{kl}}$ if you are taking the partial derivative of this equation this part of the equation with respect to ε_{kl}. this left hand side would be del 2 u $\partial\varepsilon_{kl}\ \partial\varepsilon_{ij}$, whereas right hand side would be just C_{ijkl}. And if you flip the order of these two partial derivatives with respect to ε_{ij} and ε_{kl}, this we can rewrite as $\frac{\partial}{\partial \varepsilon_{ij}}$ of $\frac{\partial U(\varepsilon)}{\partial \varepsilon_{kl}}$ which is C_{ijkl}.

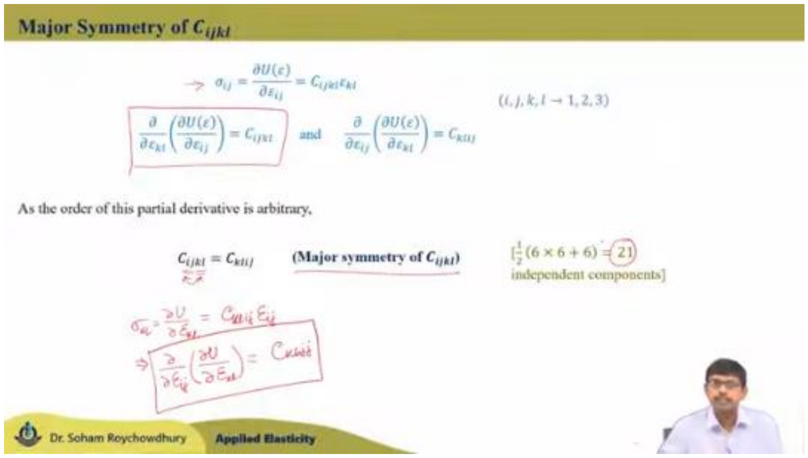


Now, coming to the symmetric properties of this elastic stiffness tensor C, σ_{ij} equals to $C_{ijkl}\varepsilon_{kl}$ $ijkl$ varying from 1 to 3, this is the generalized Hooke's law. Now, this C, C_{ijkl} is having two types of symmetry. One is called minor symmetry of C_{ijkl}, second one is called major symmetry of C_{ijkl}. For the minor symmetry, we are invoking the principle of symmetry of Cauchy stress tensor σ. We know that σ_{ij} is equals to σ_{ji}.

Now, σ_{ij} we can write as $C_{ijkl}\varepsilon_{kl}$ whereas, σ_{ij} we can write as $C_{ijkl}\varepsilon_{kl}$. And as σ_{ij} and σ_{ji} are same, we can say that C_{ijkl} is same as C_{jikl}. And due to this, from the 81, the number of independent constants in the C would come down to 54. If there is no symmetry of C matrix,

For a fourth-order tensor C, there would be no reduction in the number of independent constants, which would remain at 81. After imposing the symmetry of sigma, the number of independent elastic constants would decrease from 81 to 54. Now, further using the symmetry of epsilon in the same fashion, we can show that C_{ijkl} equals C_{ijlk}. So, the last two can be flipped, which are associated with epsilon, and thus from 54, the number of elastic constants in C would further reduce to 36.

So, by invoking the minor symmetry of C_{ijkl}, starting from 81, we reduce it to 54 due to the symmetry of σ, and then further reduce the number of independent components to 36. by invoking the symmetry of the strain tensor epsilon, and this is called the minor symmetry property of C_{ijkl}. Now, we will examine the major symmetry property of C. As we have already defined σ_{ij} as $\frac{\partial U(\varepsilon)}{\partial \varepsilon_{ij}}$, which equals $C_{ijkl}\varepsilon_{kl}$.



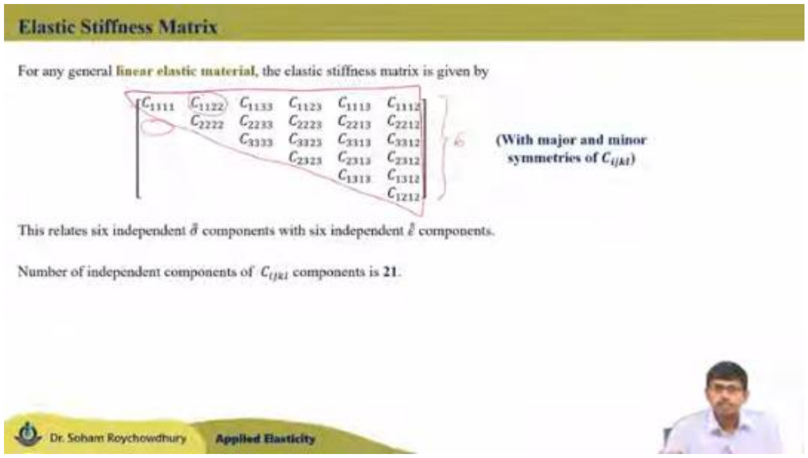
And once again, taking the partial derivative with respect to kl, we can show that $\frac{\partial}{\partial \varepsilon_{kl}}\left(\frac{\partial U(\varepsilon)}{\partial \varepsilon_{ij}}\right)$ equals C_{ijkl}. Now, since the order of these partial derivatives is arbitrary, we can always change the order of differentiation. It can be easily shown that C_{ijkl} equals C_{klij}. So, there is a first group of two subscripts, ij, and a second group of another two subscripts, kl.

For major symmetry, we are flipping these groups entirely. So, kl is coming first, ij is going at the end. For minor symmetry, only the first and second, i and j were flipped, and k and l were flipped separately. Here, a group of ij and a group of kl are being flipped for the major symmetry of C_{ijkl}. This can be shown in this fashion. Let us say, $\frac{\partial U(\varepsilon)}{\partial \varepsilon_{kl}}$.

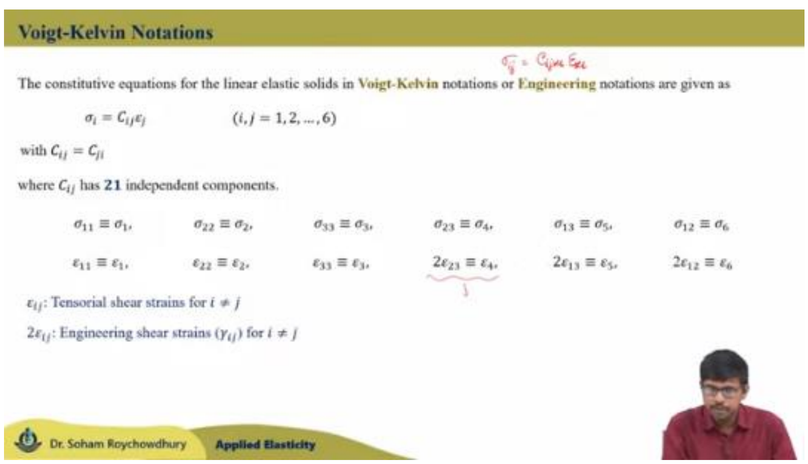
This we can write as σ_{kl}, following this $\sigma_{ij} = \frac{\partial U(\varepsilon)}{\partial \varepsilon_{ij}}$. Instead of ij, we can write σ_{kl} as $\frac{\partial U(\varepsilon)}{\partial \varepsilon_{kl}}$, and this is $C_{klij}\varepsilon_{ij}$ by using the constitutive equation σ_{kl} can be written as $C_{klij}\varepsilon_{ij}$. Now, taking the derivative with respect to ε_{ij} from both sides, the left-hand side would be $\frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial U(\varepsilon)}{\partial \varepsilon_{kl}}\right)$, and that would be C_{klij}. Now, compare this equation with this equation.

The left-hand side of both these equations is the same, just we are changing the order of partial derivatives, and thus the right-hand side must be the same: $C_{ijkl} = C_{klij}$, which is the major symmetry property of the C elastic stiffness tensor. With that, from 81, the total number of independent elastic constants is coming down to 21. With minor symmetry, we had reduced 81 to 36.

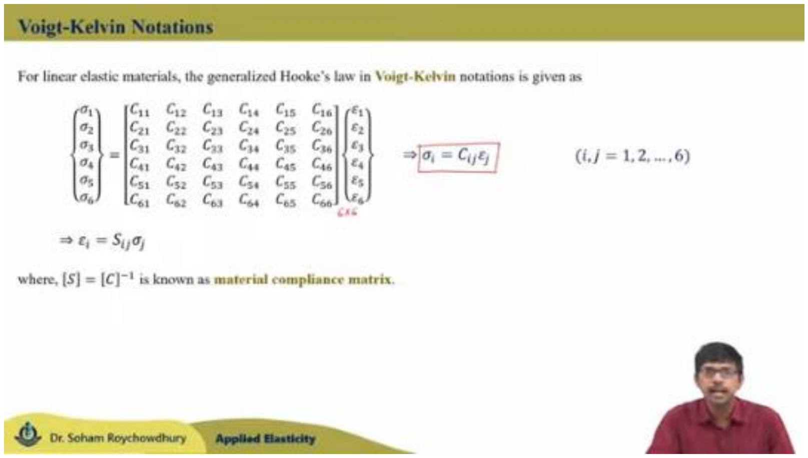
Now, further, we are reducing it from 36 to 21 with the help of major symmetry of C. And thus, After imposing this minor and major symmetry, we can write the elastic stiffness matrix as this. Only this set of independent terms are present. You can see a triangle is forming over the diagonal, and terms below this diagonal are the same as the cross terms or whatever was here, that same term would be sitting here and so on.



So, with the major and minor symmetry, the C matrix looks like this, which relates 6 independent sigma components with 6 independent epsilon components because out of 9 stress or 9 strain components, only 6 are independent after invoking the symmetry of sigma and epsilon. So, now, the number of equations or the number of rows and columns is reduced to 6 instead of 9 after invoking the symmetry of sigma and epsilon. Thus, the number of independent elastic constants here is 21 for the linear elastic solids in the elastic stiffness matrix.



Now, if I express ε in terms of σ by taking the inverse of this equation, ε_i can be written as $S_{ij}\sigma_j$, where S is nothing but the inverse of C. S is known as the material compliance matrix. C was the material stiffness matrix or material stiffness tensor. S is the inverse of that, which is defined as the material compliance matrix. So, in this lecture, we have discussed the constitutive equations for linear elastic solids.



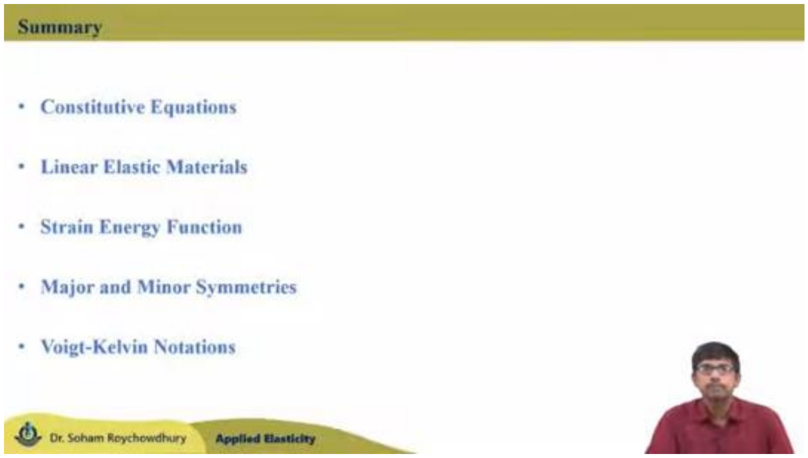
Now, coming to the Voigt-Kelvin notation for describing the elastic stiffness tensor or elastic stiffness matrix, which involves only two indices, two subscripts instead of four in the elastic stiffness tensor, so we call this notation as engineering notation or Voigt-Kelvin notation where sigma and epsilon are related like this: σ_i equals $C_{ij}\varepsilon_j$, with i and j varying between 1 to 6. So, earlier we had the relation as $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$.

Now, instead of 2 indices with sigma and epsilon and 4 indices with C, we can express this constitutive equation with 1 index in sigma and epsilon and 2 indices in C. So, this particular notation is called the engineering notation, where the symmetry of this C matrix is still valid. So, Cij is equal to Cji. Cij, this C matrix has 21 independent elastic components.

Now, σ_i with i varying from 1 to 6 refers to the 6 non-zero Cauchy stress components. σ_{11} is 1, σ_{22} is 2, σ_{33} is 3. These 3 are normal stresses. Now, coming to the shear stresses, σ_{23} is named as σ_4; σ_{13} is named as σ_5; σ_{12} is named as σ_6. So, i equals to 4, 5, 6 refers to the shear stress components.

Similarly, the strain components ε_{11}, ε_{22}, ε_{33}, normal strains are named as ε_1, ε_2, and ε_3 in engineering notation. And the shear strain, tensorial shear strain 2 ε_{23} is named as ε_4; 2 ε_{13} is named as ε_5; 2 ε_{12} is named as ε_6. These are the three engineering shear strains in the engineering notation. So, ε_{ij} are known as tensorial shear strains with i not equal to j.

And $2\varepsilon_{ij}$ are known as engineering shear strain for i not equal to j, which are also written as γ_{ij}. So, these terms are γ_{23}, γ_{13}, and γ_{12}, renamed as ε_4, ε_5, ε_6 in the engineering notations. Now, moving forward, for the linear elastic material, the generalized Hooke's law in the engineering notation or Kelvin-Voigt notation can be written like this. So, σ and ε are the column vectors with 6 components, and C_{ij} is a 6×6 matrix which is symmetric and has 21 independent elastic constants in the tensorial form. In the indicial notation, it can be written as $\sigma_i = C_{ij}\varepsilon_j$, with i and j varying between 1 to 6.



We have also discussed the concept of strain energy functions, followed by the concept of major and minor symmetry of the C elastic stiffness matrix, and finally described the elastic stiffness matrix in the Voigt-Kelvin notations. Thank you.