APPLIED ELASTICITY

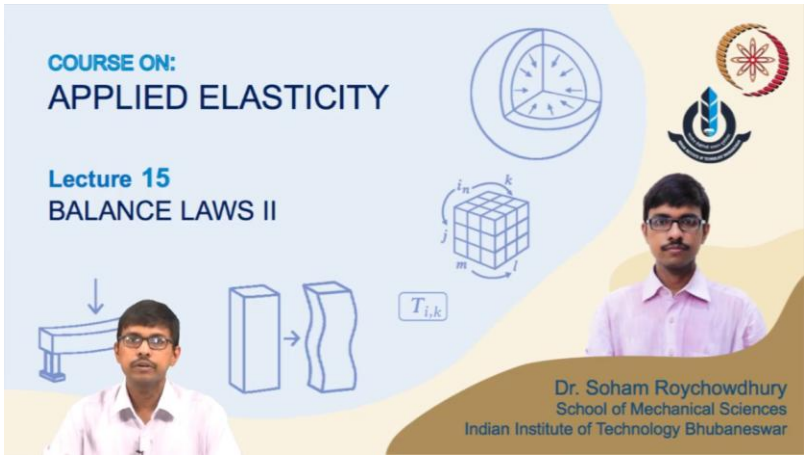
Dr. Soham Roychowdhury

School of Mechanical Sciences

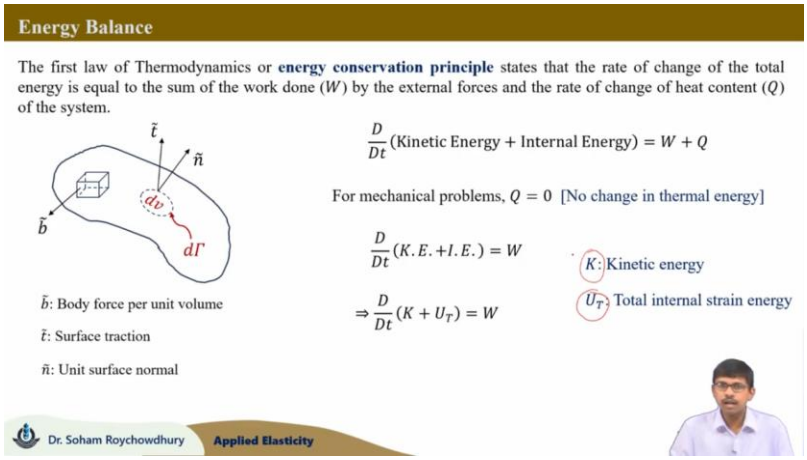
IIT Bhubaneswar

Week 3

Lecture 15: Balance Laws II



Welcome back to the course of Applied Elasticity. In today's lecture, we are going to continue our discussion on the balance laws which we had started in the previous lecture. In the last lecture, we had discussed about the transport theorem and using that, we had derived the expressions for mass balance, linear momentum balance and angular momentum balance.

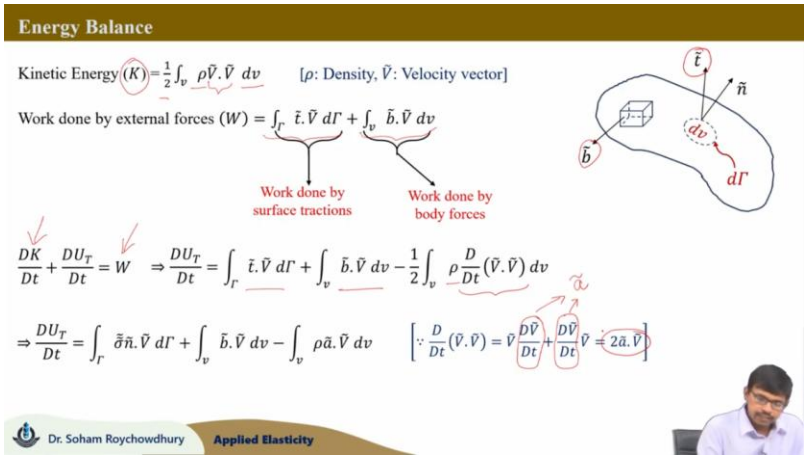


Now, today, we are going to have the formulation on energy balance. So, the energy balance principle, or energy conservation principle, which is also known as first law of thermodynamics states that the rate of change of total energy of the system is equal to the summation of the work done by all the external forces acting on the system and the rate of change of heat content of the system.

So, here let us consider this system which is subjected to surface traction $\tilde{t}$ on the boundary surface Γ. This is subjected to body force $\tilde{b}$ per unit volume, within the volume v. At any point let $\tilde{n}$ be the unit surface normal vector and considering W to be the total work done by all these external forces, which contains both surface traction and body force, and Q being the rate of change of heat content of the system, this energy balance principle can be written as $\frac{D(\ )}{Dt}$, that is time derivative of the total energy, which is having two parts. One is kinetic energy of the system, and another is the internal energy of the system. So, change in total energy is $\frac{D(\ )}{Dt}$ of kinetic energy plus internal energy and as per the principle of energy conservation, this is equal to $W + Q$.

Now, if we are having a purely mechanical problem, in that case $Q = 0$. No change in the thermal energy is associated for such mechanical problems. This particular equation would become $\frac{D(\ )}{Dt}$ of kinetic energy plus internal energy equal to W, where internal energy is nothing but the strain energy stored in the system. So, we are writing internal energy or strain energy stored in the system as U_T, whereas K is used to denote the kinetic energy.

Thus, the energy balance principle is $\frac{DK}{Dt} + \frac{DU_T}{Dt} = W$, the work done by all the external forces.



Now, we will try to write the expressions of all these terms explicitly and expand this equation. So, $K = \frac{1}{2}\int_v \rho \tilde{V}.\tilde{V} dv$, where ρ is the density of the material and $\tilde{V}$ is the velocity vector at any point within the material. Kinetic energy can be thought of as $\frac{1}{2}mV^2$.

Now, m is the total mass of the system, which is obtained by integrating ρ over the entire volume. So, $\int_v \rho dv$ is giving us mass, then half times that mass times V^2 (i.e., velocity squared), which is velocity vector dotted velocity vector. So, this is the kinetic energy of the system.

For a single particle we can write kinetic energy as $\frac{1}{2}mV^2$. But here as it is a continuum, a distributed body, we have to write kinetic energy with the help of volume integral, and that is written as $\frac{1}{2}\int_v \rho \tilde{V}.\tilde{V} dv$. Now coming to W, the work done by all the external forces. Here, two types of external forces are acting. One is $\tilde{t}$, the surface traction acting on surface Γ. Another is $\tilde{b}$, the body force acting over the entire volume.

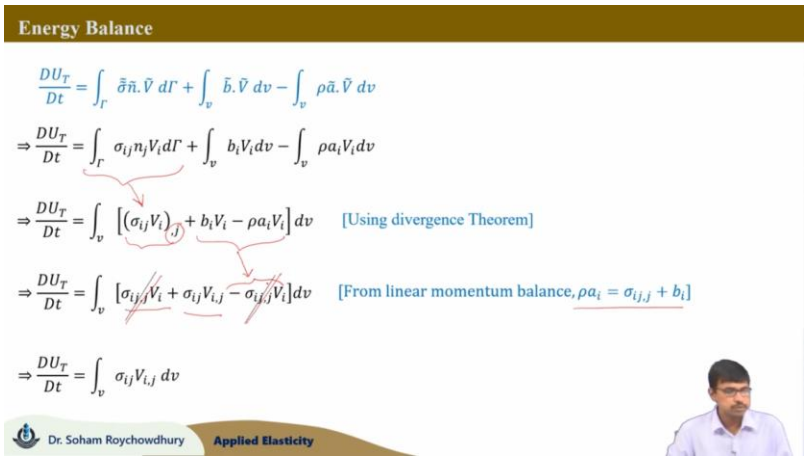
So, work done by external forces includes work done by the surface traction, which is the first term, and work done by body force, the second term. So, $\int_\Gamma \tilde{t}.\tilde{V} d\Gamma$ is the work done

by the surface traction, whereas the second term, $\int_v \tilde{b}.\tilde{V}dv$, is work done by all the body forces, which is spanning within the entire material.

Now, coming back to the energy conservation principle $\frac{DK}{Dt} + \frac{DU_T}{Dt} = W$, after neglecting Q, considering purely mechanical problems. Now, $\frac{DU_T}{Dt}$ is the rate of change of total strain energy of the system. Substituting the expressions for K and W in this, and keeping $\frac{DU_T}{Dt}$, which is kind of an unknown term at this stage, on the left hand side, we can write this as $\frac{DU_T}{Dt} = W - \frac{DK}{Dt}$. So, both the terms of W are written here and coming to the last term minus $\frac{DK}{Dt}$, we are taking the time derivative of this kinetic energy term K.

Considering a material to be incompressible, ρ is taken out of the time derivative, and thus, $-\frac{DK}{Dt}$ becomes $-\frac{1}{2}\rho\int_v \frac{D}{Dt}(\tilde{V}.\tilde{V})dv$. Now, moving forward this particular term, $\frac{D}{Dt}(\tilde{V}.\tilde{V})$ can be simplified as $\tilde{V}.\frac{D\tilde{V}}{Dt} + \frac{D\tilde{V}}{Dt}.\tilde{V}$. Now, both of these two terms: $\frac{D\tilde{V}}{Dt}$ and this $\frac{D\tilde{V}}{Dt}$, both can be written as acceleration vector $\tilde{a}$, the rate of change of velocity vector.

Thus, $\frac{D}{Dt}(\tilde{V}.\tilde{V})$ term becomes $2\tilde{a}.\tilde{V}$. So, substituting this here in the last term, $\frac{DU_T}{Dt}$ becomes $\int_\Gamma \tilde{\tilde{\sigma}}\tilde{n}.\tilde{V}d\Gamma + \int_v \tilde{b}.\tilde{V}dv - \int_v \rho\tilde{a}.\tilde{V}dv$. And here also you note, this $\tilde{t}$, surface traction, is written as $\tilde{\tilde{\sigma}}\tilde{n}$, by using the relation between the Cauchy stress component and the surface traction vector.

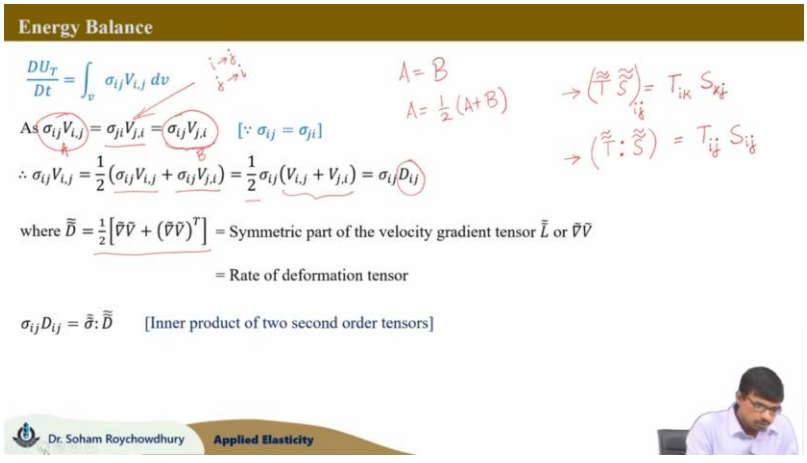


Now, moving forward, we are trying to simplify it by writing it in the indicial notations. So, $\frac{DU_T}{Dt} = \int_\Gamma \sigma_{ij} n_j V_i d\Gamma + \int_v b_i V_i dv - \int_v \rho a_i V_i dv$. Now, using the divergence theorem to convert this first surface integral term into volume integral, we can rewrite this as $\int_v (\sigma_{ij} V_i)_{,j} dv$.

So, this first term is written as the volume integral using the Gauss divergence theorem and next two terms remain as it is. Now, expanding this $(\sigma_{ij} V_i)_{,j}$, we can take this derivative with respect to x_j, as comma j refers to derivative with respect to x_j. So, we are taking derivative of σ_{ij} first and then taking derivative of V_i. So, while taking derivative of σ_{ij}, this becomes $\sigma_{ij,j}$, and V_i is kept constant, that is the first term, and then keeping σ_{ij} constant, the derivative is taken over V_i, so $\sigma_{ij} V_{i,j}$.

Then, these two terms: $b_i V_i - \rho a_i V_i$, is written as $-\sigma_{ij,j} V_i$ using the linear momentum balance equation, which was discussed in the last lecture. So, what was that? The linear momentum balance equation was $\rho a_i = \sigma_{ij,j} + b_i$. So, from here, we can write $b_i - \rho a_i = -\sigma_{ij,j}$ and that is substituted here.

So, this is equal to $-\sigma_{ij,j} V_j$. You can see the first term of this equation is getting cancelled with the last term of this equation and thus, $\frac{DU_T}{Dt}$ is having only one term, $\int_v \sigma_{ij} V_{i,j} dv$. So, this is rate of change of total strain energy stored within the system.



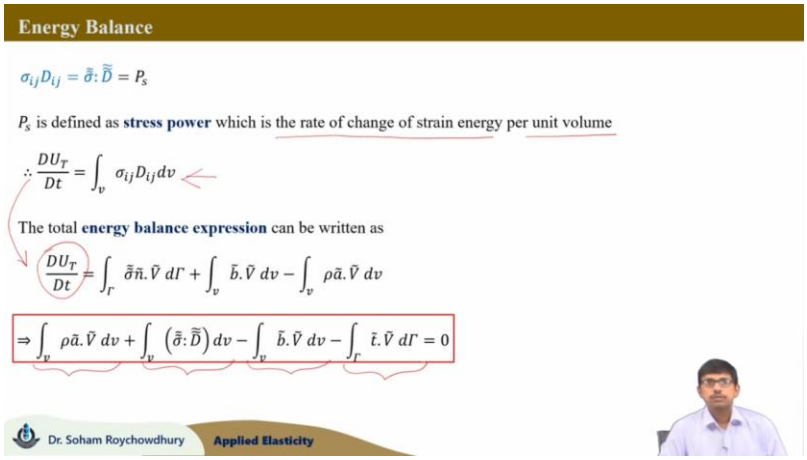
Now, we will try to rewrite this in a different fashion. So, first writing $\sigma_{ij}V_{i,j}$ as $\sigma_{ji}V_{j,i}$. So, here in this step, I have interchanged the two dummy indices. i is replaced with j and j is replaced or renamed with i. And with that we can write $\sigma_{ij}V_{i,j}$ as $\sigma_{ji}V_{j,i}$. Then, using the property of the symmetricity of σ, the Cauchy stress tensor, we can write σ_{ji} as σ_{ij}, and thus it would become $\sigma_{ij}V_{j,i}$. So, the first term $\sigma_{ij}V_{i,j}$ is equal to $\sigma_{ij}V_{j,i}$ and thus, as these two terms are equal, we can write $\sigma_{ij}V_{i,j}$ as half of summation of these two terms.

So, this is something like $A = B$, where this is A and this is B. So, now we can write A as $\frac{1}{2}(A + B)$. We are rewriting $\sigma_{ij}V_{i,j}$ in this particular form, as $\frac{1}{2}(A + B)$, where A is $\sigma_{ij}V_{i,j}$, and B is $\sigma_{ij}V_{j,i}$, and taking σ_{ij} out, this $\frac{1}{2}\left(V_{i,j} + V_{j,i}\right)$ is nothing but D_{ij}. So, what is that? $\tilde{\tilde{D}}$ is the symmetric part of the velocity gradient tensor, i.e., $\tilde{\nabla}\tilde{V}$. So, symmetric part of $\tilde{\nabla}\tilde{V}$ is $\frac{1}{2}\left[\tilde{\nabla}\tilde{V} + \left(\tilde{\nabla}\tilde{V}\right)^T\right]$ and that was written here in the indicial notation.

So, symmetric part of velocity gradient tensor is named as D_{ij}, and we had also discussed about this specifically; it was named as rate of deformation tensor. Thus, $\frac{DU_T}{Dt}$, which was equal to $\int_v \sigma_{ij}V_{i,j}dv$, that can be rewritten as $\int_v \sigma_{ij}D_{ij}dv$. Now, this, $\sigma_{ij}D_{ij}$, is nothing but the inner product of two second order tensors: $\tilde{\tilde{\sigma}}$ and $\tilde{\tilde{D}}$. So, what is inner product?

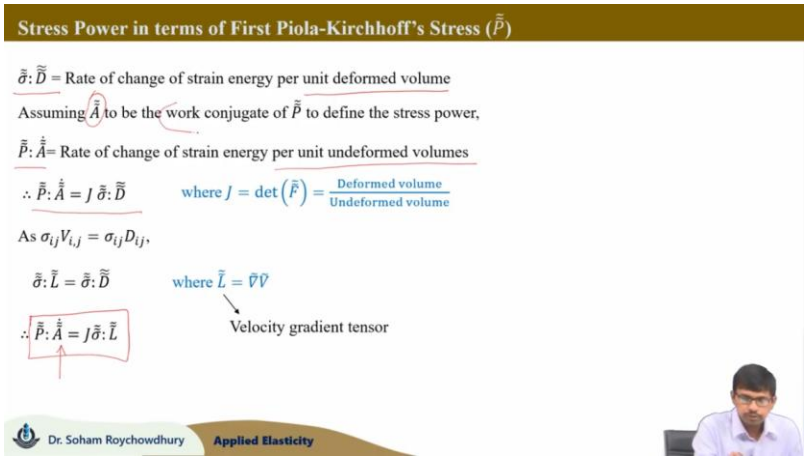
We had earlier discussed about the product of two tensors, $\tilde{\tilde{T}}$ and $\tilde{\tilde{S}}$. So, in indicial notation, this can be written as $T_{ik}S_{kj}$. The second index of first tensor is same as first index of second tensor, whereas, for the inner product, we use this colon (:) symbol in between these two. So, second one, this refers to inner product of two tensors.

The indices are same in both; this is $T_{ij}S_{ij}$, and this results in a scalar quantity, whereas, product of these two tensors results in a tensor quantity. So, as this is a scalar there is no point of writing this i and j. No indices are required; both i and j are dummy indices on the right hand side of the scalar, which is inner product of two tensors.



Thus, we are now defining a quantity called stress power, P_s, which is defined as inner product of $\tilde{\tilde{\sigma}}$ and $\tilde{\tilde{D}}$, Cauchy stress tensor and rate of deformation tensor, and which is nothing but the rate of change of strain energy per unit volume of the material. So, rate of change of strain energy $\frac{DU_T}{Dt} = \int_v P_s dv$. With the help of this, replacing $\frac{DU_T}{Dt}$ in this form, the total energy balance can be rewritten as $\frac{DU_T}{Dt} = \int_\Gamma \tilde{\tilde{\sigma}}\tilde{n}.\tilde{V}d\Gamma + \int_v \tilde{b}.\tilde{V}dv - \int_v \rho\tilde{a}.\tilde{V}dv$.

Replacing this $\frac{DU_T}{Dt}$ term with this, it would be $\int_v \rho\tilde{a}.\tilde{V}dv + \int_v \left(\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}\right)dv - \int_v \tilde{b}.\tilde{V}dv - \int_\Gamma \tilde{t}.\tilde{V}d\Gamma = 0$. So the first term on the left hand side is rate of change of kinetic energy. Second term on the left hand side is rate of change of internal or strain energy. Summation of these two is equal to total work done, having taken work done terms on the left hand side. So, first term is work done due to body force $\int_v \tilde{b}.\tilde{V}dv$ and then $\int_\Gamma \tilde{t}.\tilde{V}d\Gamma$, which is the work done due to all the surface tractions acting on the boundaries. So, this is the expression for energy balance for any mechanical system where the thermal term Q term is taken to be 0.



Now, moving forward to the concept of stress power and then extending this for Piola-Kirchhoff stress tensors. So we know that Cauchy stress tensor is the measure of stress in the deformed state, whereas first and second Piola-Kirchhoff stress tensors are used as the stress measure in the undeformed or initial state. So, we should be able to have some kind of stress power expression where instead of Cauchy stress $\tilde{\tilde{\sigma}}$, we can express it in terms of first or second Piola-Kirchoff stress tensors, either $\tilde{\tilde{P}}$ or $\tilde{\tilde{S}}$.

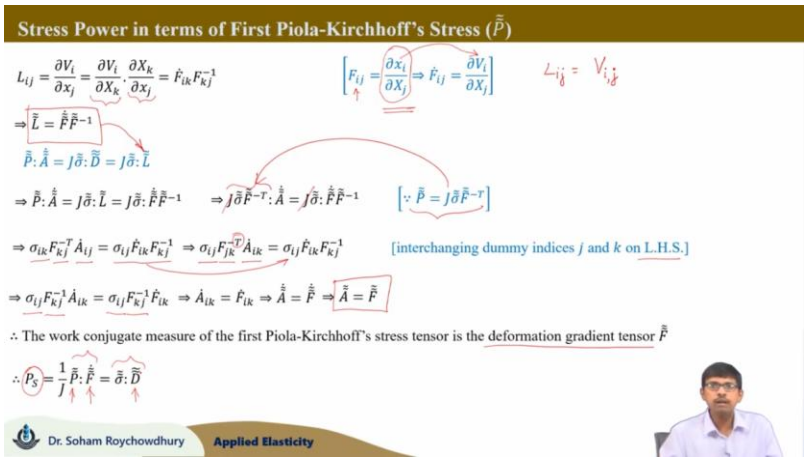
Now, we had defined the rate of change of strain energy per unit volume as $\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$, and as $\tilde{\tilde{\sigma}}$ is defined in the deformed configuration, this particular unit volume is per unit deformed volume. So, $\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$ is rate of change of strain energy per unit deformed volume.

Now, let us assume a second order tensor $\dot{\tilde{\tilde{A}}}$, which is unknown at this stage, which we are trying to find out as the work conjugate of $\tilde{\tilde{P}}$, so that we can define the stress power in terms of $\tilde{\tilde{P}}$ as $\tilde{\tilde{P}}:\dot{\tilde{\tilde{A}}}$, where $\dot{\tilde{\tilde{A}}}$ is the rate of change of $\tilde{\tilde{A}}$. So, we are taking the stress measure to be first Piola-Kirchoff stress, taking inner product of that with $\dot{\tilde{\tilde{A}}}$. $\tilde{\tilde{A}}$ is some unknown tensor at this stage and this gives us the rate of change of strain energy per unit undeformed volume.

Why undeformed? Because $\tilde{\tilde{P}}$ is a stress measure in the undeformed state or initial state. So, both of these: $\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$ defines the stress power but in the deformed configuration, $\tilde{\tilde{P}}:\dot{\tilde{\tilde{A}}}$ defines the stress power but in the undeformed configuration, i.e., rate of change of strain energy per unit undeformed volume.

Now, as the deformed and undeformed volumes are related through J, determinant of the deformation gradient tensor or the Jacobian of the system, as $J = \det\left(\tilde{\tilde{F}}\right)$ equals to ratio of deformed volume by undeformed volume, we can relate these two quantities, $\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$ and $\tilde{\tilde{P}}:\dot{\tilde{\tilde{A}}}$ as $\tilde{\tilde{P}}:\dot{\tilde{\tilde{A}}} = J\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$, with the help of Jacobian J. Now, in the indicial form $\sigma_{ij}V_{i,j} = \sigma_{ij}D_{ij}$, which we had just proved before this. We can also write this $V_{i,j}$ as L_{ij}, $\tilde{\tilde{L}}$ being the velocity gradient tensor.

So, $\tilde{\tilde{\sigma}}:\tilde{\tilde{L}} = \tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$. So, starting from this equation, rewriting $\tilde{\tilde{D}}$ as $\tilde{\tilde{L}}$, we can write $\tilde{\tilde{P}}:\dot{\tilde{\tilde{A}}} = J\tilde{\tilde{\sigma}}:\tilde{\tilde{L}}$. Now, what is our objective for this particular formulation? Our objective is to find out the form of $\tilde{\tilde{A}}$. We are choosing some $\tilde{\tilde{A}}$, a second order tensor, which is called work conjugate, and which we can associate with $\tilde{\tilde{P}}$ so that the stress power can be defined in this particular fashion.



So, moving forward, first we are writing L_{ij}, which is the velocity gradient tensor, in terms of its components as $L_{ij} = \frac{\partial V_i}{\partial x_j}$. So, L_{ij} is nothing, but $V_{i,j}$, that is written in terms of $\frac{\partial V_i}{\partial x_j}$. This can be rewritten as $\frac{\partial V_i}{\partial X_k}\frac{\partial X_k}{\partial x_j}$, where the component of $\tilde{\tilde{F}}$, deformation gradient tensor, F_{ij} can be written as $\frac{\partial x_i}{\partial X_j}$, and rate of change of $\tilde{\tilde{F}}$, $\dot{F}_{ij}$ can be written as $\frac{\partial V_i}{\partial X_j}$. So, I am taking the time derivative of F_{ij}, and thus, x_i is now changed to V_i. So, $\dot{F}_{ij}$ is $\frac{\partial V_i}{\partial X_j}$.

Now, the first term of L_{ij} is nothing but $\dot{F}_{ik}$, and second term of L_{ij} is nothing but inverse of this. So, that we can write as F_{kj}^{-1}.

Thus, L_{ij} can be alternately written as $\dot{F}_{ik}F_{kj}^{-1}$, or $\dot{\tilde{\tilde{L}}} = \dot{\tilde{\tilde{F}}}\tilde{\tilde{F}}^{-1}$. Now, as we had defined the stress power in this fashion: $\tilde{\tilde{P}}: \dot{\tilde{\tilde{A}}} = J\tilde{\tilde{\sigma}}: \tilde{\tilde{\tilde{D}}}$, which is equal to $J\tilde{\tilde{\sigma}}: \tilde{\tilde{L}}$, Replacing this definition of $\tilde{\tilde{L}}$ here, it would be $\tilde{\tilde{P}}: \dot{\tilde{\tilde{A}}} = J\tilde{\tilde{\sigma}}: \tilde{\tilde{L}} = J\tilde{\tilde{\sigma}}: \dot{\tilde{\tilde{F}}}\tilde{\tilde{F}}^{-1}$.

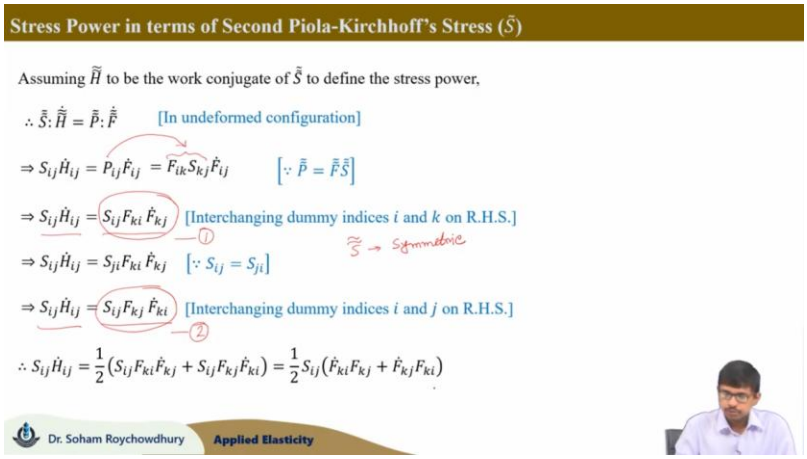
Writing this in the indicial notation, $J\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}: \dot{\tilde{\tilde{A}}}$ would be $J\tilde{\tilde{\sigma}}: \dot{\tilde{\tilde{F}}}\tilde{\tilde{F}}^{-1}$. So, what are we doing? The relation between the first Piola-Kirchoff stress tensor $\tilde{\tilde{P}}$, and the Cauchy stress tensor $\tilde{\tilde{\sigma}}$, $\tilde{\tilde{P}} = J\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$, is replaced here. So, this term was $\tilde{\tilde{P}}$, which is rewritten in terms of Cauchy stress tensor.

Now, expanding this inner product on both sides in terms of its components, or in terms of indicial notation, we can write it as $\sigma_{ik}F_{kj}^{-T}\dot{A}_{ij}$, and that is equal to $\sigma_{ij}\dot{F}_{ik}F_{kj}^{-1}$, where j has been cancelled, which is scalar from both the sides. Now, interchanging the dummy indices j and k on the left hand side. So, our objective is to simplify both the sides and get some of the common terms cancelled and for that we have to go for manipulation on the indices.

So, first on the left hand side, we change j to k and k to j. So, σ_{ik} would be σ_{ij}. F_{kj}^{-T} would be F_{jk}^{-T} and $\dot{A}_{ij}$ would be $\dot{A}_{ik}$. The right hand side remains same as it is. Now, you can see σ_{ij} is a common term on both the sides and removing this transpose and writing F_{jk}^{-T} as F_{kj}^{-1}. Then, both the sides are having two common terms σ_{ij} and F_{kj}^{-1} which you can easily cancel, and that would result $\dot{A}_{ik} = \dot{F}_{ik}$, which means $\dot{\tilde{\tilde{A}}} = \dot{\tilde{\tilde{F}}}$ or $\dot{\tilde{\tilde{A}}} = \dot{\tilde{\tilde{F}}}$.

Thus, this work conjugate $\tilde{\tilde{A}}$ for the first Piola-Kirchhoff stress tensor is nothing but the deformation gradient tensor. With respect to that, we can easily define the stress power as $\frac{1}{J}\tilde{\tilde{P}}: \dot{\tilde{\tilde{F}}}$, and that is equal to $\tilde{\tilde{\sigma}}: \tilde{\tilde{\tilde{D}}}$. So, if we are having Cauchy stress measure as our stress measure, then we can use $\tilde{\tilde{\tilde{D}}}$ for finding stress power or strain energy per unit volume.

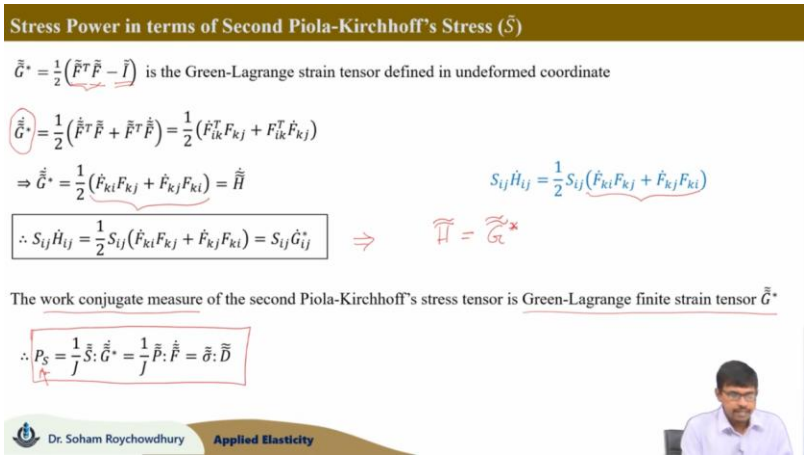
If you are using Piola-Kirchhoff stress tensor, then $\dot{\tilde{\tilde{F}}}$, the rate of change of deformation tensor, should be multiplied with it. $\tilde{\tilde{P}}:\dot{\tilde{\tilde{F}}}$ should be taken for finding strain energy per unit volume or P_s, the stress power. This $\frac{1}{J}$ is coming because this is for the undeformed state while this is for the deformed state. For relating the undeformed unit volume to deformed unit volume, $\frac{1}{J}$ term is introduced.



Now, in the similar fashion, we will proceed forward for finding the stress power for second Piola-Kirchoff stress tensor. Assuming $\tilde{\tilde{\tilde{H}}}$ to be the work conjugate of $\tilde{\tilde{\tilde{S}}}$, with which we can define the stress power as $\tilde{\tilde{\tilde{S}}}:\dot{\tilde{\tilde{\tilde{H}}}} = \tilde{\tilde{P}}:\dot{\tilde{\tilde{F}}}$. No J is associated here because both the Piola-Kirchoff stress tensors are used in the undeformed configuration. Thus, $S_{ij}\dot{H}_{ij} = P_{ij}\dot{F}_{ij}$ in the indicial notation. Now, using the relation between the first and second Piola-Kirchhoff stress tensors $\tilde{\tilde{P}}$ and $\tilde{\tilde{S}}$, we can replace this P_{ij} as $F_{ik}S_{kj}$ and $\dot{F}_{ij}$ sits at the end.

So, $S_{ij}\dot{H}_{ij} = S_{ij}F_{ki}\dot{F}_{kj}$. Here, I am interchanging the dummy indices i and k on the right hand side of the equation, and then rewriting S_{ij} as S_{ji} because we know that $\tilde{\tilde{S}}$, the second Piola-Kirchoff stress tensor, is a symmetric stress tensor and thus, $S_{ij} = S_{ji}$, which we cannot use for $\tilde{\tilde{P}}$, the first Piola-Kirchoff stress tensor, as it was not a symmetric stress tensor.

So, writing S_{ij} as S_{ji}, and then once again interchanging the dummy indices i and j on the right hand side, the right hand side would become $S_{ij}F_{kj}\dot{F}_{ki}$. So, $S_{ij}\dot{H}_{ij}$, this left hand side, is equal to $S_{ij}F_{kj}\dot{F}_{ki}$ here, at this stage. It was equal to $S_{ij}F_{ki}\dot{F}_{kj}$ at the previous stage somewhere here. So, let us consider this to be equation 1, and this to be equation 2 and for both of them the left hand side term is same. So, we can write the left hand side as half of summation of right hand sides of equation 1 and 2, similar to the approach used for finding stress power for the first Piola-Kirchoff stress tensor. So, $S_{ij}\dot{H}_{ij}$ is half of this plus this: $S_{ij}F_{ki}\dot{F}_{kj} + S_{ij}F_{kj}\dot{F}_{ki}$, and then, I have taken out S_{ij} and remaining term is kept within bracket. So, this manipulation of indices was done to get this equation 2 from equation 1 and rewrite $S_{ij}\dot{H}_{ij}$ in this particular form.



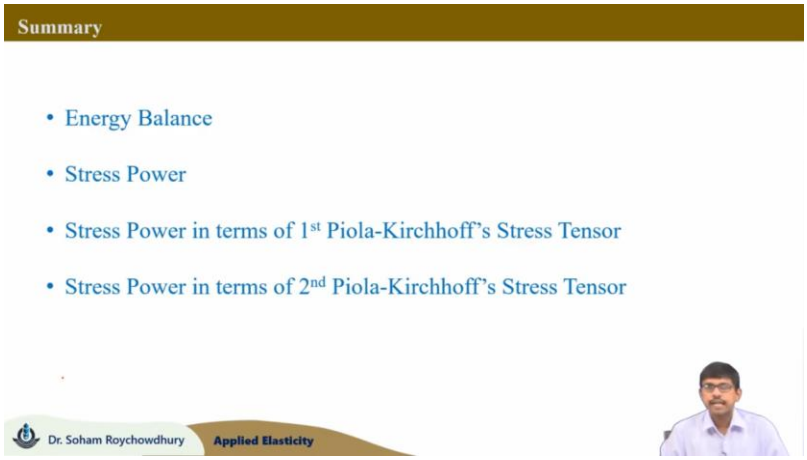
Now, to simplify it further in terms of Green-Lagrange strain tensor, we will look back into the definition of Green-Lagrange strain tensor, which was a strain measure used in the undeformed or initial state and was defined as $\frac{1}{2}\left(\tilde{\tilde{C}} - \tilde{\tilde{I}}\right)$ or $\frac{1}{2}\left(\tilde{\tilde{F}}^T\tilde{\tilde{F}} - \tilde{\tilde{I}}\right)$, with $\tilde{\tilde{F}}$ being the deformation gradient tensor. Now, if we consider the rate of change of this Green-Lagrange strain tensor, $\tilde{\tilde{G}}^*$, so $\frac{D}{Dt}\left(\tilde{\tilde{G}}^*\right)$ or $\dot{\tilde{\tilde{G}}}^*$ is the rate of change of Green-Lagrange strain tensor. So, we are taking time derivative of the right hand side, and $\tilde{\tilde{I}}$ being a constant identity tensor, its time derivative would be 0.

But time derivative of this first term would result two terms. So, $\frac{1}{2}\left[\dot{\tilde{\tilde{F}}}^T\tilde{\tilde{F}} + \tilde{\tilde{F}}^T\dot{\tilde{\tilde{F}}}\right]$. In the first term, we are taking the time derivative of $\tilde{\tilde{F}}^T$. In the second term, we are taking the time derivative of $\tilde{\tilde{F}}$. We can write this in the indicial form as $\frac{1}{2}\left[\dot{F}_{ik}^T F_{kj} + F_{ik}^T \dot{F}_{kj}\right]$.

So, $\dot{\tilde{\tilde{G}}}^*$, the rate of change of Green-Lagrange strain tensor, is equal to $\frac{1}{2}\left[\dot{F}_{ki}F_{kj} + \dot{F}_{kj}F_{ki}\right]$. Now, if we compare this expression and the expression of $S_{ij}\dot{H}_{ij}$, which we had derived in the previous slide, you can see that the quantity or term within bracket is same for both of them. Thus, $S_{ij}\dot{H}_{ij} = \frac{1}{2}S_{ij}\left(\dot{F}_{ki}F_{kj} + \dot{F}_{kj}F_{ki}\right)$, and that is nothing but $S_{ij}\dot{G}_{ij}^*$. So, the work conjugate measure of the second Piola-Kirchhoff stress tensor is the Green-Lagrange finite stress tensor $\tilde{\tilde{G}}^*$.

So, from here, basically $H_{ij} = G_{ij}^*$, and $\tilde{\tilde{H}}$, we had assumed to be the work conjugate for the second Piola-Kirchoff stress tensor, and that is coming out to be nothing but $\tilde{\tilde{G}}^*$, the Green-Lagrange finite strain tensor, which is used in the undeformed coordinate. So, stress power in terms of all three different stress measures can be written like this. In terms of second Piola-Kirchoff stress tensor, it is $\frac{1}{J}\tilde{\tilde{S}}:\dot{\tilde{\tilde{G}}}^*$. Then, in terms of first Piola-Kirchoff stress tensor, it is $\frac{1}{J}\tilde{\tilde{P}}:\dot{\tilde{\tilde{F}}}$. In terms of Cauchy stress tensor, it is $\tilde{\tilde{\sigma}}:\tilde{\tilde{D}}$, where $\tilde{\tilde{D}}$ is the symmetric part of velocity gradient tensor.

So, this is the relation of work conjugate in three different stress measures, and with respect to that, we can define the stress power. Stress power, P_s, is defined as the strain energy density, i.e., the total strain energy stored per unit deformed volume, the time derivative of that particular quantity. So, $\frac{DU_T}{Dt}$, the rate of change of strain energy per unit deformed volume. And as these two are in the undeformed state, we are dividing it with J, the Jacobian or $\det\left(\tilde{\tilde{F}}\right)$.



So, in this lecture, we started with discussion of energy balance principle, defined the concept of stress power, and wrote the stress power in terms of Cauchy stress, first Piola-Kirchoff stress and second Piola-Kirchoff stress tensors.

Thank you.