APPLIED ELASTICITY

Dr. Soham Roychowdhury

School of Mechanical Sciences

IIT Bhubaneswar

Week 3

Lecture 14: Balance Laws I
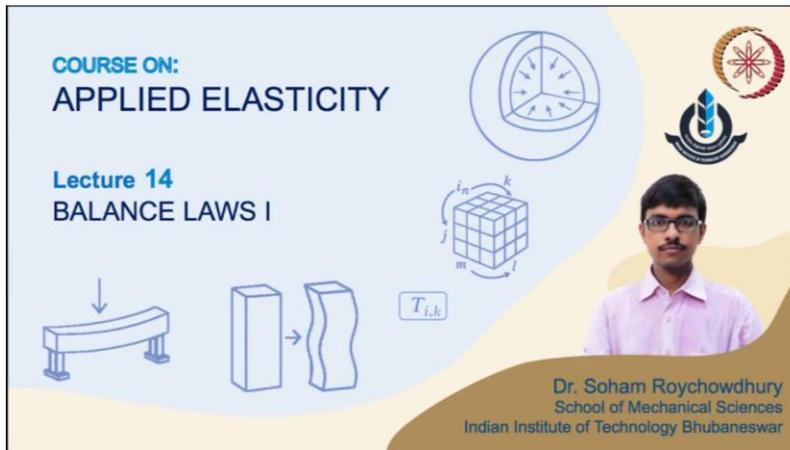
Welcome back to the course on Applied Elasticity. In this lecture, we are going to discuss about the balance laws. In last three lectures, we were talking about the different stress measures, and now, from the stress measure, we are going for the different balance laws. Now, these balance laws include mass balance, linear momentum balance, angular momentum balance, and energy balance.

Transport Theorem

For $f(\tilde{x}, t)$ being a continuous and differentiable scalar-valued function,

$$\frac{D}{Dt}\int_v f(\tilde{x}, t)\, dv = \int_v \frac{\partial f}{\partial t}\, dv + \int_\Gamma f\tilde{V}.\tilde{n}\, d\Gamma$$

Volume Integral   Surface Integral

$\tilde{V}$: Velocity vector

$\tilde{n}$: Unit normal vector on surface Γ

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Now, the first thing with which we are going to start is the transport theorem, and using this, all these different balance laws can be derived. So, transport theorem states that $f(\tilde{x}, t)$ being a continuous and differentiable scalar valued function for deformed position vector $\tilde{x}$ and time t. This is the statement of the transport theorem, where $\tilde{V}$ is the velocity vector and $\tilde{n}$ is the unit normal vector on the boundary surface Γ. So, $\frac{D(\ )}{Dt}$, total time derivative of this $f(\tilde{x}, t)$ integrated over the total volume of the body. So, we are considering a body defined by this v and the boundary surface Γ.

Now, $f(\tilde{x}, t)$ is any property of the body which is a scalar valued function defined by the deformed position vector $\tilde{x}$ and time. So, total time derivative of integral of $f(\tilde{x}, t)$ over the total deformed volume dv is equal to summation of these two terms. First term is local time derivative $\int_v \frac{\partial f}{\partial t} dv$ plus $\int_\Gamma f\tilde{V}.\tilde{n}d\Gamma$, the surface integral of f times $\tilde{V}.\tilde{n}$, where $\tilde{V}$ is the velocity vector, and $\tilde{n}$ is the unit normal vector to the surface Γ. So, one term on the right hand side, the first term, is volume integral, and second term is surface integral.

So, first term takes care of the local changes of the property f within the volume or within the body, whereas, the second term, convective term, takes care of the changes due to property f due to change of the boundary surface, and in total, summation of these two results the total time derivative of this property $f(\tilde{x}, t)$ within the body.

Now, moving forward, first we are going to talk about the mass balance. Let us define the density as a scalar field: $\rho(\tilde{x}, t)$, which is defined as $\lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v}$, where Δv is the volume of a small element whose mass is given by Δm. Now, total mass of the body can be obtained by integrating this ρ over the entire volume. Now, if this volume integral is with respect to the deformed volume dv, then we define that mass with respect to current configuration, and if this volume integral is with respect to the undeformed volume dV, then we define the mass as integral of $\int_v \rho_0(\tilde{X}, t)\, dV$, *i.e.*, integration over the undeformed volume in the initial configuration.

So, this expression, if ρ is function of $\tilde{x}$ and integral is over dv, then it is current configuration definition of mass. If ρ is function of $\tilde{X}$ and integral is over dV, undeformed volume, then this is the definition of mass in the initial configuration. Now, for the principle of conservation of mass, this implies the total mass of the system remains constant at each and every configuration.

Thus, rate of change of this mass, total time derivative of this mass of the system, must be equal to 0. $\dot{m}$ - here, this dot refers to the total time derivative $\frac{D(\ )}{Dt}$, where D is all capital. $\frac{Dm}{Dt}$, or $\dot{m}$, rate of change of mass, equals to 0, and we are going to write the expressions of all these balance laws in the current or the deformed configuration. So, we are taking the definition of mass in the current configuration and thus, $\frac{D}{Dt} \int_v \rho(\tilde{x}, t)\, dv = 0$ is the mass balance in the current or Eulerian description.

$$\dot m = \frac{D}{Dt}\int_v \rho(\tilde x, t)dv = 0$$

Using the Transport Theorem in Eulerian description, [Choosing $f(\tilde x, t)$ as $\rho(\tilde x, t)$]

$$\int_v \frac{D}{Dt}[\rho(\tilde x, t)]dv = \int_v \frac{\partial \rho}{\partial t}dv + \int_\Gamma \left(\rho(\tilde V.\tilde n)d\Gamma\right)$$

Unit normal

Velocity

Using divergence theorem,

$$\dot m = \int_v \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\tilde V)\right]dv = 0$$

For dv being an arbitrary volume element, $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\tilde V) = 0$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + (\rho V_i)_{,i} = 0}$$ → Mass balance in Eulerian description

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Now, we are simplifying the left hand side of this equation - $\frac{D}{Dt}\int_v \rho(\tilde x, t)dv$ - using the transport theorem, where $f(\tilde x, t)$, that scalar valued function, is chosen as $\rho(\tilde x, t)$. So, if you look back at the transport theorem, and replace f with ρ, where f can be any scalar valued function, and for mass balance we are choosing that function to be the density ρ as a function of $\tilde x$ and time. So, substituting that in the transport theorem, we will be getting this $\frac{D}{Dt}\int_v \rho(\tilde x, t)dv = \int_v \frac{\partial \rho}{\partial t}dv + \int_\Gamma \rho(\tilde V.\tilde n)d\Gamma$. Now, using the divergence theorem for the last term.

If we use the divergence theorem, we can convert this last term from surface integral to volume integral, and that would be, this term $\int_\Gamma \rho(\tilde V.\tilde n)d\Gamma$, would be $\int_v \operatorname{div}(\rho\tilde V)$. So, whatever is the coefficient of $\tilde n$, divergence of that would be added and $d\Gamma$ term would be changed to dv term. Surface integral will be changed to volume integral if you are invoking divergence theorem. Thus, the left hand side $\dot m$ becomes $\int_v \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\tilde V)\right]dv$, where $\tilde V$ is the velocity.

Do not get confused with the undeformed volume. We are not involving undeformed volume in this particular calculation because all our description is in the Eulerian description. The volume is dv whereas, $\tilde V$ refers to the velocity field. Now, for dv being an arbitrary quantity and this integral to be 0, we must have $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\tilde V) = 0$. So, in the indicial notation, we can write $\operatorname{div}(\rho\tilde V)$ as $(\rho V_i)_{,i}$. So, $\frac{\partial \rho}{\partial t} + (\rho V_i)_{,i} = 0$ is the mass balance expression in the Eulerian description.

$$\frac{\partial \rho}{\partial t} + (\rho V_i)_{,i} = 0$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t} + \rho_{,i} V_i\right) + \rho V_{i,i} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \, \text{div}(V) = 0 \qquad \left[\because \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \rho_{,i} V_i\right]$$

For **incompressible materials** with $\frac{D\rho}{Dt} = 0$, the mass balance equation becomes

$$\text{div}(V) = 0$$

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Now, we can further simplify this. This $', i'$ quantity can be expanded into two terms: once taking the derivative of ρ, and another one taking the derivative of V_i. So, two terms would be resulted and it would be $\frac{\partial \rho}{\partial t} + \rho_{,i} V_i + \rho V_{i,i}$. Now, combining these two terms and using the definition of the material time derivative of ρ. In the Eulerian description, $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \rho_{,i} V_i$. So, first two terms in total results this $\frac{D\rho}{Dt}$, total time derivative of ρ, or density of the material, and next term is $\rho V_{i,i}$, where $V_{i,i} = \text{div}(\tilde{V})$.

So, second term is $\rho \, \text{div}(\tilde{V})$. So, mass balance can also be written in this alternate form as $\frac{D\rho}{Dt} + \rho \, \text{div}(\tilde{V}) = 0$. And if the material is incompressible, ρ is constant; $\frac{D\rho}{Dt}$ is 0 for such cases. The mass balance equation becomes $\text{div}(\tilde{V}) = 0$. So, for incompressible material, this is the mass balance equation.

dv: Elemental volume with boundary $d\Gamma$ in the current configuration

$\tilde{n}$: Unit outward surface normal vector

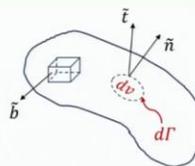$\tilde{t}$: Surface traction acting on the boundary Γ

$\tilde{b}$: Distributed body force per unit volume

$\tilde{V}$: Velocity vector

The linear momentum of the body is $\int \tilde{V} dm = \int_v \rho \tilde{V} dv$

The **principle of conservation of linear momentum** implies that rate of change of linear momentum is equal to the net resultant force acting on the body, *i.e.*,

$$\frac{D}{Dt} \int_v \rho \tilde{V} dv = \int_\Gamma \tilde{t} \, d\Gamma + \int_v \tilde{b} \, dv$$

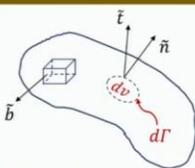Dr. Soham Roychowdhury   **Applied Elasticity**

Now, moving forward to the linear momentum balance. So, let us consider a body. Within the body, we are taking a small volume element dv bounded by small surface $d\Gamma$. Now, $\tilde{n}$ being the unit outward surface normal vector on this surface $d\Gamma$, $\tilde{t}$ being the surface traction force acting on this boundary Γ, $\tilde{b}$ being the uniformly distributed body force defined per unit volume, and $\tilde{V}$ being the velocity vector, we can write the linear momentum of the body as mass times velocity integrated over the entire mass. So, $\tilde{V}$, the velocity integrated over the entire mass defines the linear momentum of the body. Writing $dm = \rho dv$ in the Eulerian or the deformed configuration, the linear momentum would be $\int_v \rho \tilde{V} dv$.

Now, principle of conservation of linear momentum states that the rate of change of linear momentum, which is this particular quantity, is equal to the net resultant external force acting on the body. And it can be mathematically written as $\frac{D(\ )}{Dt}$, rate of change of linear momentum. So, $\frac{D}{Dt} \int_v \rho \tilde{V} dv$ is the rate of change of linear momentum, which is equal to net work done. So, there are two terms: one is $\int_\Gamma \tilde{t} d\Gamma$, surface integral, and another is $\int_v \tilde{b} dv$, volume integral.

So, this first term, $\int_\Gamma \tilde{t} d\Gamma$, that is the total work done by the surface traction forces acting on the boundary surface Γ, whereas, the second term, $\int_v \tilde{b} dv$, is the total work done by the body forces which is spanned over the entire material volume. So, these are the two components of work done. Two external forces are acting on the system: $\tilde{t}$ is acting on

the surface, and $\tilde{b}$, body force is acting within the body. Work done by both the forces are included here; total surface traction and total body force are added.

So, the rate of change of linear momentum, this left hand side, is defined to be the net external force acting on the body, that is the surface traction integral, $\int_\Gamma \tilde{t} d\Gamma$, and volume integral, $\int_v \tilde{b} dv$. Now, using the Reynolds transport theorem with $f(\tilde{x}, t) = \rho\tilde{V}$. So, here the scalar valued function f is chosen as $\rho\tilde{V}$ and substituting that in the transport theorem, left hand side becomes $\int_v \frac{\partial}{\partial t}(\rho\tilde{V}) dv + \int_\Gamma \rho\tilde{V}(\tilde{V}.\tilde{n}) d\Gamma$.

So, these are the two terms coming by using the transport theorem on the left hand side of the equation with $f(\tilde{x}, t)$ replaced with $\rho\tilde{V}$, and right hand side remains the same. Now, I have written $\tilde{t}$ as $\tilde{\tilde{\sigma}}\tilde{n}$ by using the definition of the Cauchy stress tensor integrated over $d\Gamma$, and last term on the right hand side is $\int_v \tilde{b} dv$. Now, considering the material to be incompressible and also using the divergence theorem.

So, there are two surface integrals: the second term of left hand side and first term of right hand side, both of them are surface integrals. These are required to be converted into volume integral with the help of divergence theorem, and considering the material to be incompressible, this ρ can be taken out. $\frac{\partial}{\partial t}(\rho\tilde{V})$ can be written as $\rho\frac{\partial\tilde{V}}{\partial t}$, as ρ is not changing with time. So, by assuming material incompressibility and using divergence theorem, this equation would be reduced to this particular form.

So, if you use the divergence theorem for this particular term, this would be $\int_v \rho\tilde{V}\text{div}(\tilde{V}) dv$, whereas, this particular term would be $\int_v \text{div}(\tilde{\tilde{\sigma}}) dv$. So, with this, if you are substituting both of them in the final equation, and then use the definition of this material total time derivative $\frac{D\tilde{V}}{Dt} = \frac{\partial\tilde{V}}{\partial t} + \tilde{V}\text{div}(\tilde{V})$. So, $\tilde{V}\text{div}(\tilde{V})$ term comes from this. and this $\frac{\partial\tilde{V}}{\partial t}$ term is already there in the first term, combining both of them, the left hand side is having a single term: $\rho\frac{D\tilde{V}}{Dt}$. Thus, $\int_v \rho\frac{D\tilde{V}}{Dt} dv = \int_v[\text{div}(\tilde{\tilde{\sigma}}) + \tilde{b}] dv$ integrated over the total deformed volume. So, this is the linear momentum conservation expression.

Linear Momentum Balance

$$\int_v \rho\left(\frac{D\tilde{V}}{Dt}\right)dv - \int_v \left(\text{div}\left(\tilde{\tilde{\sigma}}\right) + \tilde{b}\right)dv = 0$$

$$\Rightarrow \int_v \left(\rho\tilde{a} - \text{div}\left(\tilde{\tilde{\sigma}}\right) - \tilde{b}\right)dv = 0 \qquad \text{where } \tilde{a} = \frac{D\tilde{V}}{Dt} = \text{Acceleration vector}$$

As dv is an arbitrary volume element,

$$\boxed{\rho\tilde{a} = \text{div}\left(\tilde{\tilde{\sigma}}\right) + \tilde{b} \Rightarrow \rho a_i = \sigma_{ij,j} + b_i} \longrightarrow \text{Linear momentum balance in Eulerian description}$$

For static problem with velocity field $(\tilde{V})$ being zero,

$$\boxed{\text{div } \tilde{\tilde{\sigma}} + \tilde{b} = 0} \longrightarrow \boxed{\text{Static equilibrium equations}}$$

$$\sigma_{ij,j} + b_i = 0 \qquad (i = 1, 2, 3)$$

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Now, simplifying that further, we can define $\tilde{a}$ to be the acceleration vector, which is total time derivative of $\tilde{V}$, i.e., velocity vector. So, in the first term, this $\frac{D\tilde{V}}{Dt}$ is written as $\tilde{a}$. So, $\rho\tilde{a}$, i.e., ρ times acceleration, minus $\text{div}\left(\tilde{\tilde{\sigma}}\right)$ minus $\tilde{b}$ integrated over total volume is 0. Now, as this elementary volume dv is quite arbitrary, to have this integral to be always 0, we must have the total integrand: $\rho\tilde{a} - \text{div}\left(\tilde{\tilde{\sigma}}\right) - \tilde{b}$ to be 0. So, $\rho\tilde{a} - \text{div}\left(\tilde{\tilde{\sigma}}\right) - \tilde{b} = 0$ implies $\rho\tilde{a} = \text{div}\left(\tilde{\tilde{\sigma}}\right) + \tilde{b}$. In the indicial notation, $\rho a_i = \sigma_{ij,j} + b_i$.

This is the linear momentum balance in the Eulerian description and this is valid for both static as well as dynamic problem, because we are considering the acceleration component a_i in this particular linear momentum balance. Now, if we are just considering the static problem, $\tilde{V}$, the velocity field is 0, thus, acceleration field is 0. The left hand side of this equation would go to 0; thus, it would be $\text{div}\left(\tilde{\tilde{\sigma}}\right) + \tilde{b} = 0$. This is called the static equilibrium equation, one very important equation which will be used multiple times within this course.

So, for solid mechanics or elasticity problems, as we are not considering the dynamics of the system, $\text{div}\left(\tilde{\tilde{\sigma}}\right) + \tilde{b} = 0$ gives us the static equilibrium equation. In the indicial notation it would be $\sigma_{ij,j} + b_i = 0$, where i, the free index, varies from 1, 2, and 3. So, we can have three static equilibrium equations along x_1, x_2, and x_3 directions. So, linear momentum balance is basically giving us the static equilibrium equation for strength of material or elasticity problems.
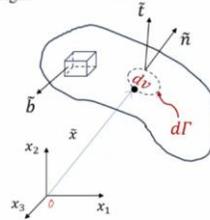
Angular Momentum Balance

The **principle of angular momentum balance** states that the rate of change of angular momentum about the origin is equal to the net moment of all the forces acting on the body about the origin.

$\tilde{x}$: Position vector in the deformed configuration

$$\frac{D}{Dt}\int_v (\tilde{x} \times \rho \tilde{V})\, dv = \int_\Gamma (\tilde{x} \times \tilde{t})\, d\Gamma + \int_v (\tilde{x} \times \tilde{b})\, dv$$

$$\Rightarrow \frac{D}{Dt}\int_v e_{ijk}\rho x_j V_k\, dv = \int_\Gamma e_{ijk}x_j t_k\, d\Gamma + \int_v e_{ijk}x_j b_k\, dv$$

Considering the material to be **incompressible**,

$$\Rightarrow \int_v \rho e_{ijk}\frac{D}{Dt}(x_j V_k)\, dv - \int_\Gamma e_{ijk}x_j \sigma_{lk}n_l\, d\Gamma - \int_v e_{ijk}x_j b_k\, dv = 0$$

$$[\because t_k = \sigma_{kl}^T n_l = \sigma_{lk}n_l]$$

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So, after the linear momentum balance, now we are going to discuss about the angular momentum balance law. So, let us consider the same body where we are considering origin here for this x_1, x_2, x_3 axes and body force $\tilde{b}$ per unit volume is considered acting over the small volume dv and the surface traction $\tilde{t}$ is acting over the surface $d\Gamma$ with $\tilde{n}$ being the unit normal on the surface.

So, the principle of angular momentum balance states that the rate of change of angular momentum of the continuum about a fixed point, that is origin O for this case, is equal to the net moment of all the forces acting on the body about that same fixed point origin. So if you are calculating the angular momentum of the body about point O, rate of change of that angular momentum is equal to the net moment created by all the forces acting on the body about that same origin. Now, two types of forces are acting on the body: one is body force, another is surface traction force.

So, the moment created by the body forces and moment created by the surface traction forces, their summation is equal to the rate of change of angular momentum of the body. Considering $\tilde{x}$ to be the position vector of any point within the body in the deformed configuration, the mathematical statement for this angular momentum balance principle can be written like this. So, left hand side of this equation is the rate of change of angular momentum.

So, this $\tilde{V}$ is the velocity and this $\int_v \rho \tilde{V} dv$ is the linear momentum. Now, as we are taking the angular momentum, here that is written as the moment of linear momentum. So, if

you are taking the moment of linear momentum about point O, that would be $\int_v (\tilde{x} \times \rho\tilde{V}) \, dv$. So, this total thing is actually angular momentum of the body and total time derivative of that, $\frac{D}{Dt}\int_v (\tilde{x} \times \rho\tilde{V}) \, dv$ is the rate of change of angular momentum.

Now, this should be equal to the moment of all the forces. So, first term on the right hand side is moment of surface traction, $\int_\Gamma (\tilde{x} \times \tilde{t}) \, d\Gamma$, plus the second term is the moment caused by the body force $\tilde{b}$, which is $\int_v (\tilde{x} \times \tilde{b}) \, dv$ integrated over the entire volume of the material.

So, rate of change of angular momentum, $\frac{D}{Dt}\int_v (\tilde{x} \times \rho\tilde{V}) \, dv = \int_\Gamma (\tilde{x} \times \tilde{t}) \, d\Gamma + \int_v (\tilde{x} \times \tilde{V}) \, dv$. Now, writing this equation in terms of the indicial notation, $\tilde{x} \times \rho\tilde{V}$, I am writing using the permutation symbol as $e_{ijk}\rho x_j V_k$, volume integral of that and $\frac{D}{Dt}$, rate of change of this quantity on the left-hand side. Similarly, this $\tilde{x} \times \tilde{t}$ can be written as $e_{ijk}x_j t_k$, and $\tilde{x} \times \tilde{b}$ can be written as $e_{ijk}x_j b_k$. All these cross products are written with the help of permutation symbol.

Now, considering the material to be incompressible, this ρ is a constant. So, we can take that ρ out of this time differentiation. e, the permutation symbol is also constant. So, the left hand side term can be written as $\int_v \rho e_{ijk} \frac{D}{Dt}(x_j V_k) \, dv$ minus second and third terms, which are the terms coming from right-hand side to left-hand side, and thus are having a change in sign.

So, this t_k, the surface traction component can be written in terms of Cauchy stress resultant as $t_k = \sigma_{kl}^T n_l$, and using the symmetry or removing this transpose, we can write $\sigma_{kl}^T n_l = \sigma_{lk} n_l$, where n_l are the components of the unit vector. So, hence, the second term becomes minus $\int_\Gamma e_{ijk}x_j \sigma_{lk} n_l \, d\Gamma$, and finally the last term remains as it is, that is, the moment due to body force is minus $\int_v e_{ijk}x_j b_k \, dv$.

$$\int_v \rho e_{ijk} \frac{D}{Dt}(x_j V_k)\, dv - \int_\Gamma e_{ijk} x_j \sigma_{lk} n_l\, d\Gamma - \int_v e_{ijk} x_j b_k\, dv = 0$$

$$\Rightarrow \int_v \rho e_{ijk}\left(x_j \frac{DV_k}{Dt} + V_j V_k\right) dv - \int_v e_{ijk}(x_j \sigma_{lk})_{,l}\, dv - \int_v e_{ijk} x_j b_k\, dv = 0 \quad \text{[Using divergence theorem]}$$

$$\Rightarrow \int_v e_{ijk}\left(x_j\left[\rho \frac{DV_k}{Dt} - \sigma_{lk,l} - b_k\right] - \sigma_{jk}\right) dv = 0$$

$$[\because e_{ijk} V_j V_k = \tilde{V} \times \tilde{V} = 0] \quad \frac{\partial x_j}{\partial x_l} = \delta_{jl}$$

$$[\because (x_j \sigma_{lk})_{,l} = x_j \sigma_{lk,l} + \delta_{jl}\sigma_{lk} = x_j \sigma_{lk,l} + \sigma_{jk}]$$

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Now moving further, the first term can be rewritten in this fashion. This time derivative, is first acting over x_j and then acting over V_k. If we treat x_j as constant, the first term is $x_j \frac{DV_k}{Dt}$, and then keeping V_k constant, this would be $V_k \frac{Dx_j}{Dt}$. Now, x_j being position vector, $\frac{Dx_j}{Dt} = V_j$, the velocity in j.

So, the second term would be $V_j V_k$. So, $\frac{D}{Dt}(x_j V_k)$ is having two terms $x_j \frac{DV_k}{Dt} + V_j V_k$ in the first integral term. Now, the second integral term, which was a surface integral, is now converted into volume integral using the Gauss divergence theorem like this. So, surface integral $e_{ijk} x_j \sigma_{lk} n_l$ is now written as $\left(e_{ijk} x_j \sigma_{lk}\right)_{,l}$ integrated over volume and last term remains as it is.

Now, these can be simplified further like this. So, consider this particular term, $e_{ijk} V_j V_k$. This is nothing but the cross product of the same vector $\tilde{V}$, $\tilde{V} \times \tilde{V}$. So, cross product of the same vector $\tilde{V}$ is simply 0 and thus, this term is not going to have any contribution. So, first term we are having as it is, $i.e.$, $e_{ijk} x_j \rho \frac{DV_k}{Dt}$. Then, coming to this term $\left(x_j \sigma_{lk}\right)_{,l}$, this can be expanded into two terms: once we are taking partial derivative with respect to x_l for the first term x_j, and another once for the second term σ_{lk} keeping another term constant. So, one term would be x_j, constant, then $\sigma_{lk,l}$. Another term is σ_{lk}, constant, and we have $\frac{\partial x_j}{\partial x_l}$.

Now, this term $\frac{\partial x_j}{\partial x_l}$ is nothing but δ_{jl}. So, this is the Kronecker's delta. Now, using the property of the Kronecker delta, we can write $\delta_{jl}\sigma_{lk} = \sigma_{jk}$. So, this particular term $(x_j\sigma_{lk})_{,l}$ is replaced with this. Hence, combining all the terms which have one x_j common, and clubbing them together, we have these three terms. So, $\int_v e_{ijk}\left[x_j\left(\rho\frac{DV_k}{Dt} - \sigma_{lk,l} - b_k\right) - \sigma_{jk}\right]dv = 0$.

Now, once again starting from this, and writing $\frac{DV_k}{Dt}$ as a_k, the acceleration component, the first term would become $\int_v e_{ijk}\left[x_j\left(\rho a_k - \sigma_{kl,l} - b_k\right) - \sigma_{jk}\right]dv = 0$. Now, if you recall the linear momentum balance, which was $\rho a_k = \sigma_{kl,l} + b_k$.

So that was nothing but $\rho\tilde{a} = \text{div}(\tilde{\sigma}) + \tilde{b}$, this was the linear momentum balance. In the indicial notation, if you express that, that would be resulting this particular term, and thus, this entire term goes to 0. Hence, we are left with only a single term for the angular momentum balance, that is $\int_v e_{ijk}\sigma_{jk}dv = 0$. Now, dv being a small arbitrary volume element, for this volume integral to be 0, we must have $e_{ijk}\sigma_{jk} = 0$.

Now, if we are having this element to be 0, we can write that this element is same as minus of that element. So, if you are able to write $A = -A$, that implies $A = 0$ because $2A = 0$ and then, A should be 0. We are trying to write the same thing here. So, $e_{ijk}\sigma_{jk} = -e_{ijk}\sigma_{jk}$.

Now, slightly manipulating this term by changing the indices, first I am flipping k and j, so j is replaced with k and k with j. Interchanging these two indices, this becomes $-e_{ikj}\sigma_{kj}$ and then, this $-e_{ikj}$ can be written as $+e_{ijk}$, which is equal to $-e_{ikj}$. This initial quantity $e_{ijk}\sigma_{jk}$ is shown to be equal to $e_{ijk}\sigma_{kj}$. Now e_{ijk} being common, we can cancel that, and thus, we will have $\sigma_{jk} = \sigma_{kj}$, which means $\tilde{\tilde{\sigma}} = \tilde{\tilde{\sigma}}^T$, i.e., the Cauchy stress tensor is symmetric. So, from the angular momentum balance equation, we are once again coming back to the symmetry of the Cauchy stress tensor.

So, angular momentum balance equation ensures the symmetry of the Cauchy stress tensor in the deformed coordinate. We are writing all these balance equations in the deformed coordinate and as Cauchy stress tensor is the deformed stress measure, we are getting the symmetry of the Cauchy stress tensor by the angular momentum balance equation in the deformed coordinate.

So, in this lecture, we had first introduced the transport theorem and with the help of that in the deformed coordinate, we had derived the expressions of mass balance, then linear momentum balance, which basically results the static equilibrium equation for the static problem, and then finally, angular momentum balance resulting the equality of the cross shear of the Cauchy stress components.

Thank you.