APPLIED ELASTICITY

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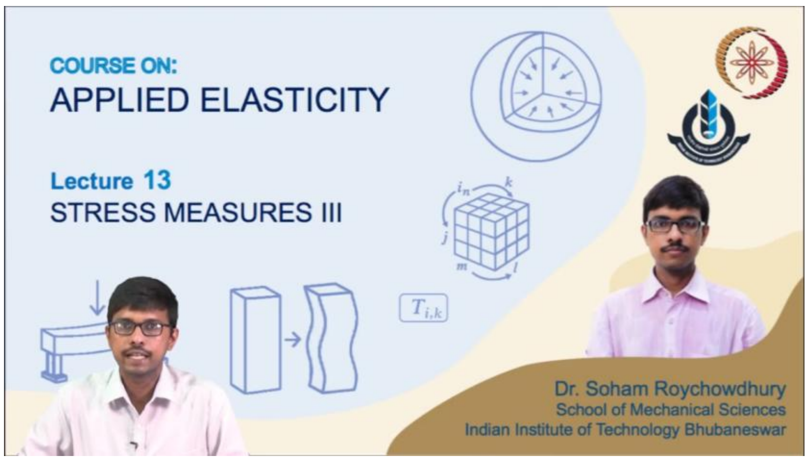
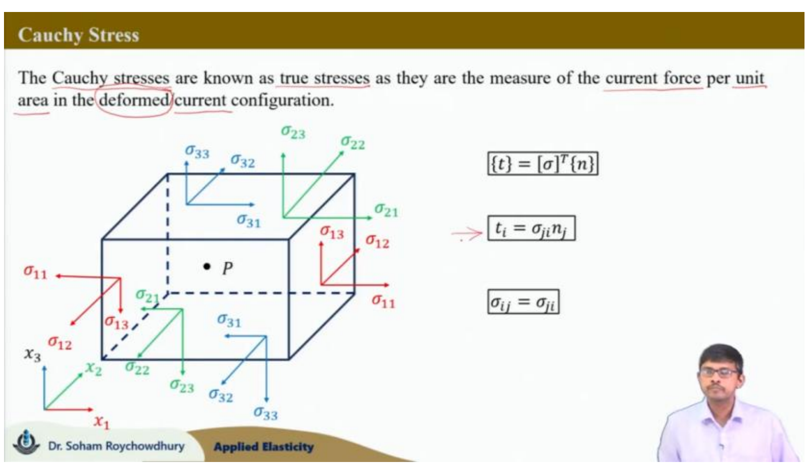
WEEK: 03
Lecture- 13



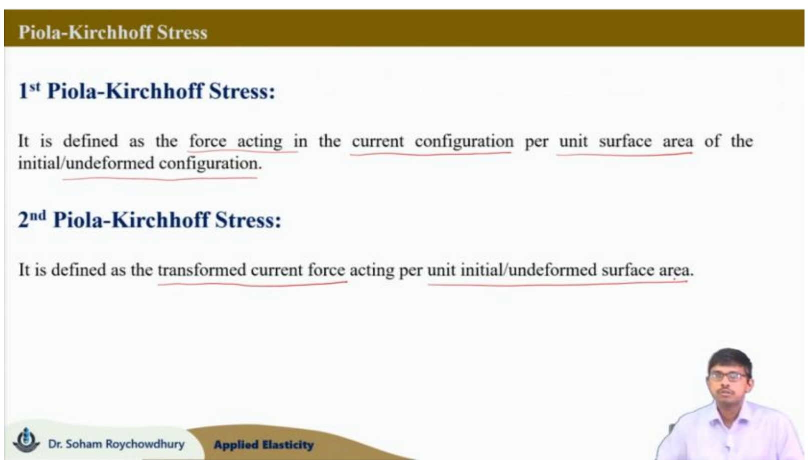
Welcome back to the course on applied elasticity. In the previous lecture, we were talking about stress measures, and in this lecture as well, we will continue with the topic of stress measures. Earlier, we were talking about the Cauchy stress tensor, which was a stress measure in the deformed coordinate system. Now, in today's lecture, we are going to talk about stress measures with respect to the undeformed coordinate system.



So, to have a quick recap, if we are considering a small infinitesimal volume or cube element centered at point P with sides aligned along x_1, x_2, x_3, the three orthogonal axes, the Cauchy stress tensors were defined as the true stress components, which are nothing but the measure of the current force. Force acting on the current configuration per unit current or deformed area. So, here, note these two points: the force we are considering is the current force, acting on the current configuration and defined per unit current area.

So, we are considering the force acting on the deformed configuration per deformed surface area, and that is defined as the Cauchy stress. This measure is also called true stress because we are considering the deformed surface area, not the original or undeformed surface area. So, in undergraduate solid mechanics, you must have heard about the difference between engineering stress and true stress. Engineering stress takes care

Of the undeformed surface area, whereas true stress takes into account the deformed surface area. As the Cauchy stress is defined as the current force per unit deformed surface area, this is a measure of true stress. We have shown that $\{t\}$ equals $[\sigma]^T\{n\}$ relation between the Cauchy stress tensor and the surface traction vector t, which in the initial notation can be written as t_i equals $\sigma_{ji}n_j$. We have also shown that the Cauchy stress tensor is symmetric; thus, σ_{ij} equals σ_{ji}.



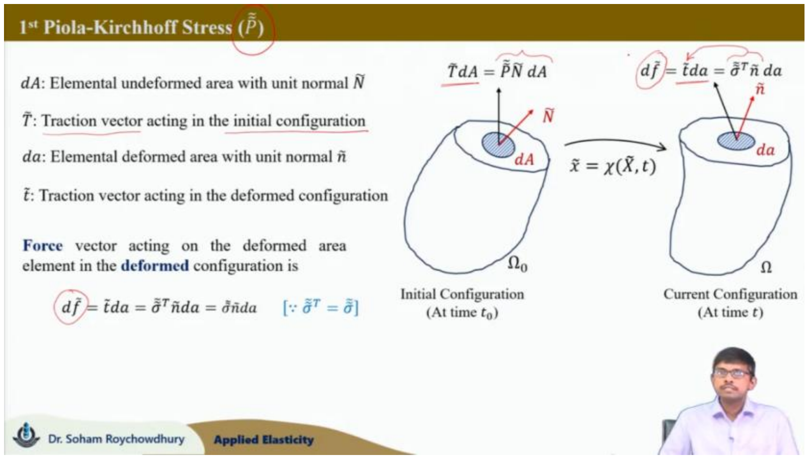
Now, moving forward to the definition of the Piola-Kirchhoff stress tensor. So, we have two different Piola-Kirchhoff stress measures. One is called the first Piola-Kirchhoff stress tensor, and the second one is called the second Piola-Kirchhoff stress tensor. These are the two stress measures that help us measure the stresses in the undeformed configuration with respect to the undeformed area.

These are equivalent to the engineering stress measures. So, how are they defined? The first Piola-Kirchhoff stress is defined as the force acting in the current configuration per unit surface area of the undeformed configuration. So, here the force is taken from the current configuration—the force acting in the current configuration divided by the area in the undeformed configuration.

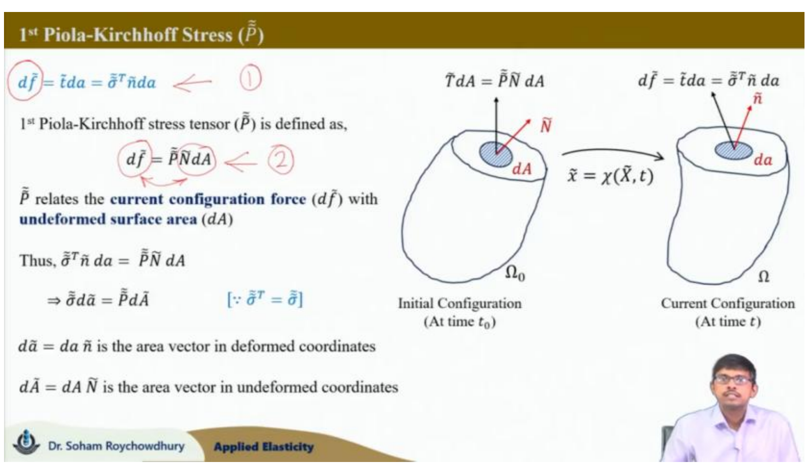
So, though we are taking the force from the deformed configuration, the area is taken from the undeformed configuration. This is the definition of the first Piola-Kirchhoff stress tensor. Whereas, for the second Piola-Kirchhoff stress tensor, the definition is like this: it is the transformed current force acting per unit undeformed or initial surface area. So, here instead of the current force, we are taking the transformed current force. So, some kind of transformation is required by which we are taking the current force back or mapping it to the initial configuration.

So, we are doing some force transformation here, and that divided by the unit undeformed surface area defines the second Piola-Kirchhoff stress. So, we will look into the mathematical description for both of these two stresses. So, starting with the first Piola-Kirchhoff stress tensor. So, this P tensor is used—you can see this capital P. This is used to denote the first Piola-Kirchhoff stress tensor.

Sigma is used to denote the Cauchy stress tensor, P is used to denote the first Piola-Kirchhoff stress tensor, and capital $\tilde{\tilde{S}}$ is used to denote the second Piola-Kirchhoff stress tensor. Now, let us consider this initial configuration of the body defined by capital Ω_0 at initial time t_0. We are considering a small area element dA with Unit normal capital $\widetilde{N}$ on the surface of this particular undeformed configuration body. Now, $\tilde{T}$, $\tilde{T}$ is the traction vector acting on the small area dA. The total vector traction we can write as capital T times dA. This is the force acting on this small area dA, dA in the undeformed configuration. And this can be written as $\tilde{\tilde{P}}\widetilde{N}\ dA$. How it comes, we will discuss afterwards. So, the force acting on the small area dA is capital $\tilde{T}dA$, where capital $\tilde{T}$ is the traction vector acting in the initial configuration.



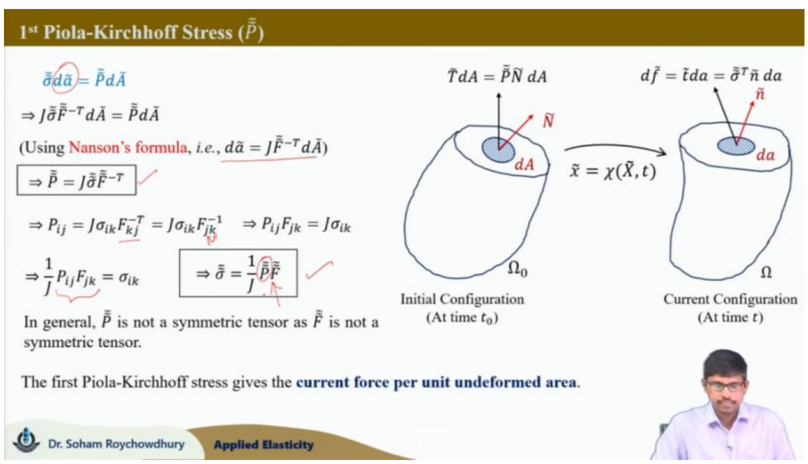
Now, moving forward, if you are going for the transformation which is defined through deformation gradient tensor $\tilde{\tilde{F}}$, in the current configuration, the body is shown here with capital Ω at any time t. The area dA is now deformed to da with unit normal $\tilde{n}$ and $\tilde{t}$ being the surface traction acting in the deformed configuration. So, the force acting on the da, the deformed surface area, is small $\tilde{t}$, the deformed surface traction, times da, the deformed surface area. Using the relation between the surface traction $\tilde{t}$ in the deformed coordinate and deformed coordinate stress measure, that is Cauchy stress measure, which is $\tilde{t}$ equals to $\tilde{\tilde{\sigma}}^T\tilde{n}$. This quantity is equal to $\tilde{t}$. $\tilde{\tilde{\sigma}}^T\tilde{n}$ is equal to $\tilde{t}$, as we had proved. So, the force acting on the current configuration on the deformed area da is equal to $\tilde{t}da$, and that is $\tilde{\tilde{\sigma}}^T\tilde{n}$ times da. So, the force vector acting on the deformed area element in the deformed configuration is denoted by $d\tilde{f}$, which is shown here in the figure, and that is equal to $\tilde{t}$, $\tilde{t}da$, equals to $\tilde{\tilde{\sigma}}^T\tilde{n}da$, and using and using the symmetry of the Cauchy stress tensor $d\tilde{f}$ vector is equals to sigma n d small a.



Now, moving forward $d\tilde{f}$ equals to $\tilde{\tilde{\sigma}}^T \tilde{n} da$ which we had just proved. Now, we are coming to the formal definition of first Piola Kirchhoff stress tensor which is defined as $d\tilde{f}$ is equals to $\tilde{\tilde{P}}\tilde{N}dA$.

So, this relates the current configuration force $d\tilde{f}$, $d\tilde{f}$ is the force in the current configuration and dA is the area in the undeformed state the initial configuration. So, area in the undeformed state and force in the current configuration $d\tilde{f}$ vector and dA. These two are related with the help of a stress measure

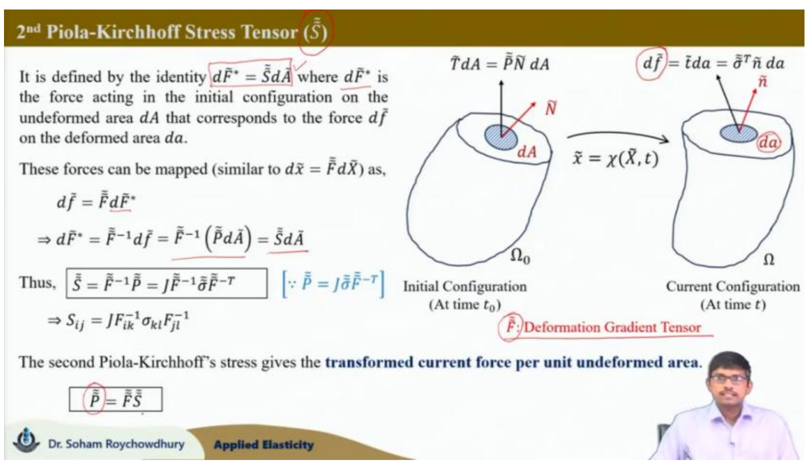
that is known as capital $\tilde{\tilde{P}}$ or first Piola-Kirchhoff stress tensor and through this particular transformation law $\tilde{\tilde{P}}$ is defined $d\tilde{f}$ is equals to $\tilde{\tilde{P}}\tilde{N}dA$. Now, using both of this equation this 1 and as left hand side of both the equations are same $d\tilde{f}$ we can equate the right hand side and right $\tilde{\tilde{\sigma}}^T \tilde{n}\ da$ equals to $\tilde{\tilde{P}}\tilde{N}\ dA$ and using the symmetry removing the transpose of $\tilde{\tilde{\sigma}} d\tilde{a}$ equals to $\tilde{\tilde{P}} d\tilde{A}$ where $d\tilde{a}$ is the deformed area vector $d\tilde{A}$ is the undeformed area vector.



Now, deformed area vector $d\tilde{a}$ and undeformed area vector $d\tilde{A}$ can be related by the commonly known Nansen's formula which we had discussed in the previous lectures one of the previous lectures. So, $d\tilde{a}$ can be transformed or related to $d\tilde{A}$ with the help of this Nansen's formula as $d\tilde{a}$ vector is equals to $J\tilde{\tilde{F}}^{-T}d\tilde{A}$ substituting that here in the left hand side. we are getting this equation $J\tilde{\tilde{F}}^{-T}d\tilde{A}$ equals to $\tilde{\tilde{P}}d\tilde{A}$. So, thus the first Piola Kirchhoff stress tensor $\tilde{\tilde{P}}$ is $J\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$ this is the relation between $\tilde{\tilde{P}}$ and $\tilde{\tilde{\sigma}}$ they are related through J that is Jacobian and $\tilde{\tilde{F}}^{-T}$ that is transpose of inverse of deformation gradient tensor. In the indicial notation, we can write the components of $\tilde{\tilde{P}}$ as P_{ij} equals to $J\sigma_{ik}F^{-T}_{kj}$. Now, removing the transpose sign from the last term and flipping the indices, this would be $J\sigma_{ik}F^{-1}_{jk}$. I have removed the transpose sign and flip these two indices j and k, these two are flipped.

Now, taking this F_{jk}^{-1} from right hand side to left hand side as F_{jk}. we are having $P_{ij}F_{jk}$ equals to J times capital J time σ_{ik} or σ_{ik} can be written as $\frac{1}{J}P_{ij}F_{jk}$. So, $P_{ij}F_{jk}$ is basically P times F_{ij} component and thus the Cauchy stress tensor $\tilde{\tilde{\sigma}}$ becomes $\frac{1}{J}\tilde{\tilde{P}}\tilde{\tilde{F}}$. So, Cauchy stress tensor $\tilde{\tilde{P}}$ and the first Piola-Kirchhoff stress tensor $\tilde{\tilde{P}}$ are related through this form either using this or using this. If anyone is known and we know the mapping deformation gradient tensor then we can find out the another stress measure.

Now in general $\tilde{\tilde{F}}$ is not symmetric the deformation gradient tensor is not symmetric in general and thus $\tilde{\tilde{P}}$ the first Piola-Kirchhoff stress tensor this stress measure is also not symmetric in general. So, that is one of the problem of this first Piola-Kirchhoff stress tensor. We are normally having stress tensors to be symmetric the cross shears are equal the, but here even though sigma is symmetric, but $\tilde{\tilde{F}}$ being not being symmetric $\tilde{\tilde{P}}$ in general is not symmetric. So, this first Piola of stress tensor or $\tilde{\tilde{P}}$ gives the current force, force acting in the deformed configuration per unit undeformed area. Now, inverse of this first Piola-Kirchhoff stress tensor is defined as a nominal stress tensor another stress measure, but this is not much used in practice. So, $\tilde{\tilde{N}}$ nominal stress tensor is defined as transpose of the $\tilde{\tilde{P}}$ first Piola-Kirchhoff stress tensor. Now, we are moving to the definition of second Piola-Kirchhoff stress tensor which is denoted by capital $\tilde{\tilde{S}}$.



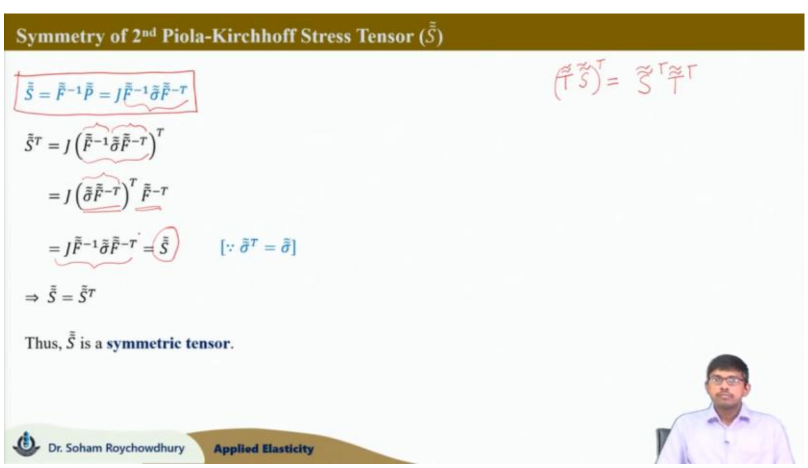
So, we are considering the same figure here in the initial and current configurations where the definition of second Piola of Kirchhoff stress tensor is given as the identity $d\tilde{F}^*$ equals to $\tilde{\tilde{S}}d\tilde{A}$. So, through this particular expression the $\tilde{\tilde{S}}$ second Piola of Kirchhoff stress tensor is defined. What is $d\tilde{F}^*$? $d\tilde{F}^*$ is the force acting in the initial or undeformed configuration on the undeformed area that corresponds to the force $d\tilde{f}$ which is acting on the deformed configuration on deformed area. So, as we had discussed while deriving the first Piola Kirchhoff stress tensor $d\tilde{f}$ is the force acting on the current configuration on deformed area da. Now, we are defining a force $d\tilde{F}^*$ through a mapping which maps this $d\tilde{f}$ force into $d\tilde{F}^*$. So, similar to the deformation mapping, mapping of $\tilde{x}$ we have seen through the deformation gradient tensor $\tilde{\tilde{F}}$, we are we are mapping $d\tilde{X}$ to $d\tilde{x}$. Using a similar mapping as $d\tilde{x}$ equals to $\tilde{\tilde{F}}$ acting over $d\tilde{X}$,

we are trying to define a force mapping using the same deformation gradient tensor. So, $d\tilde{f}$ the actual deformed configuration force acting on the deformed area $d\tilde{a}$ equals to deformation gradient tensor capital $\tilde{\tilde{F}}$ acting over $d\tilde{F}^*$ where $d\tilde{F}^*$ is the transformed force acting on per unit undeformed area. and from this $d\tilde{F}^*$ can be obtained as $\tilde{\tilde{F}}^{-1}d\tilde{f}$ and now $d\tilde{f}$ is nothing but $\tilde{\tilde{P}}d\tilde{A}$. $d\tilde{A}$ vector is undeformed area vector.

So, putting $d\tilde{f}$ as $\tilde{\tilde{P}}d\tilde{A}$ and comparing it with the definition $d\tilde{F}^*$ was defined as $\tilde{\tilde{S}}d\tilde{A}$. So, if you compare this with $\tilde{\tilde{S}}d\tilde{A}$, we simply get $\tilde{\tilde{S}}$ the second Piola-Kirchhoff stress tensor as $\tilde{\tilde{F}}^{-1}\tilde{\tilde{P}}$ and as we know $\tilde{\tilde{P}}$ was already shown to be $J\tilde{\tilde{F}}^{-1}\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$ substituting that here $\tilde{\tilde{S}}$ equals to $\tilde{\tilde{F}}^{-1}\tilde{\tilde{P}}$ becomes $J\tilde{\tilde{F}}^{-1}\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$.

And in the initial notation in terms of components, the components of second Piola-Kirchhoff stress tensor S_{ij} can be written as $JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$. So, second Piola-Kirchhoff stress gives the transformed current force defined per unit undeformed area and this transformation mapping of the force from the current configuration to undeformed configuration is done with the help of deformation gradient tensor $\tilde{\tilde{F}}$ which is used to transform the small elemental length $d\tilde{X}$ to $d\tilde{x}$.

Now with this definition of second Piola of Kirchhoff stress tensor, we are now going to check whether it is symmetric or not. We had already checked that the first Piola of Kirchhoff stress tensor $\tilde{\tilde{P}}$ was not symmetric and $\tilde{\tilde{P}}$ and $\tilde{\tilde{S}}$ are related through this relation $\tilde{\tilde{P}}$ equals to $\tilde{\tilde{P}} = \tilde{\tilde{F}}\tilde{\tilde{S}}$.

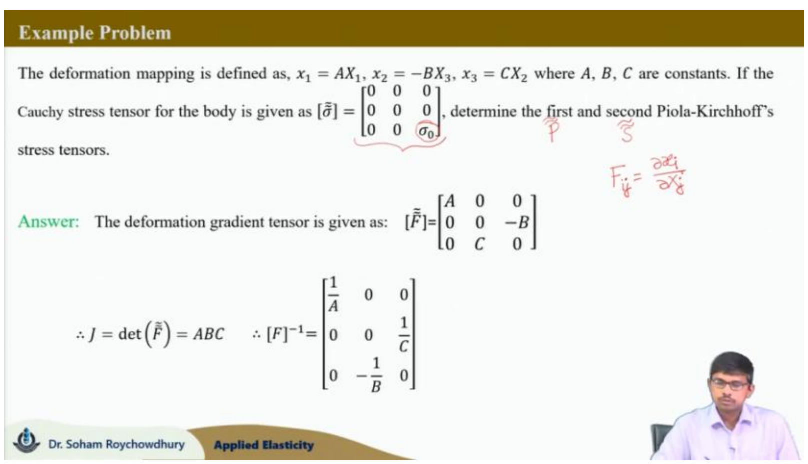


Now let us check whether $\tilde{\tilde{S}}$ is symmetric or not. So the relation between $\tilde{\tilde{S}}$, $\tilde{\tilde{P}}$ and $\tilde{\tilde{\sigma}}$ that is given as $\tilde{\tilde{S}}$ equals to $\tilde{\tilde{F}}^{-1}\tilde{\tilde{P}}$ equals to $J\tilde{\tilde{F}}^{-1}\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$.

Now, taking the transpose of $\tilde{\tilde{S}}^T$ taking transpose of both the sides of this equation $\tilde{\tilde{S}}^T$ transpose would be J which is a constant transpose cannot act over constant. So, J times transpose of this entire quantity $\tilde{\tilde{F}}^{-1}\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$. Now, we had discussed that the transpose of $\tilde{\tilde{T}}\tilde{\tilde{S}}$ by the transpose rule this is $\tilde{\tilde{S}}^T\tilde{\tilde{T}}^T$. So, using the same here.

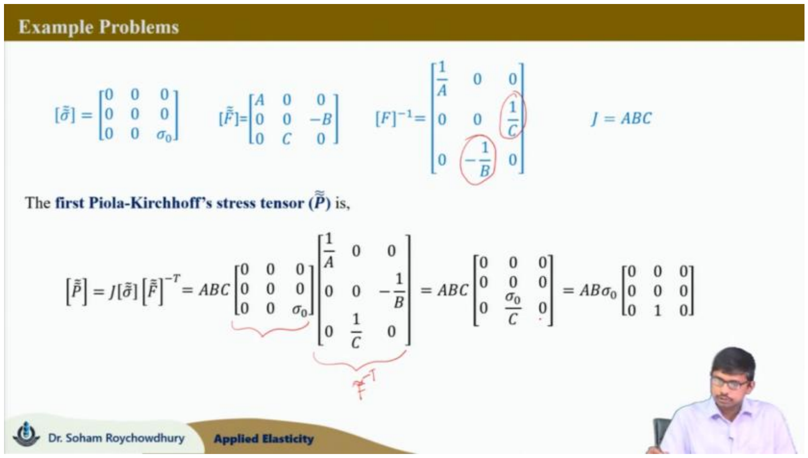
So, considering $\tilde{\tilde{F}}^{-1}$ to be 1 tensor $\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$ to be another tensor. $\tilde{\tilde{S}}^T$ can be written as J transpose of the second term first that is transpose of $\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$ and then transpose of the first term that is $\tilde{\tilde{F}}^{-T}$. Now using the same formula once again for simplifying this first term. this would be $J\tilde{\tilde{F}}^{-1}\tilde{\tilde{\sigma}}\tilde{\tilde{F}}^{-T}$ and I have already used the symmetry of Cauchy stress tensor sigma transpose equals to sigma here and thus you can see this $\tilde{\tilde{S}}^T$ is coming out to be $\tilde{\tilde{S}}$.

So, the second Piola of Kirchhoff stress tensor $\tilde{\tilde{S}}$ is a symmetric tensor and thus this is preferred over the over the first Piola-Kirchhoff stress tensor for measuring the measuring the stress in the undeformed or material configuration in the initial configuration because of its symmetry.



Now, we are going to solve one example problem. So let us consider this given deformation mapping, x_1 is equals to AX_1, x_2 is $-BX_3$, x_3 equals to CX_2, where A, B, C are some arbitrary constants which are known.

If the Cauchy stress tensor of the body is given as this, all terms are 0 apart from the 3 3 term. So, body is in uniaxial tension along the 3 axis along the x_3 axis then only the Cauchy stress tensor can look like this only σ_{33} term is non-zero. You are asked to find out the first and second Piola-Kirchhoff stress tensor. So, we need to find out $\tilde{\tilde{P}}$ which is the first Piola-Kirchhoff stress tensor and $\tilde{\tilde{S}}$ which is the second Piola-Kirchhoff stress tensor.

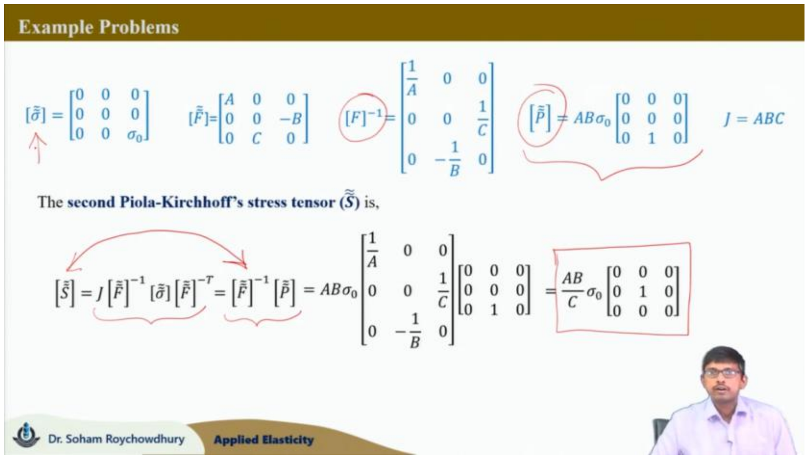


Now, for the given deformation mapping, we can easily find out the deformation gradient tensor by its definition. So, how is $\tilde{\tilde{F}}$ defined? The F_{ij} component of the deformation gradient tensor is del small x_i del capital X_j. So, small x_1 being AX_1, the first row of $\tilde{\tilde{F}}$ would be just a 0 0. Similarly, using x_2 and x_3 expressions, we can obtain the complete deformation gradient tensor as A 0 0 0 0 $-B$ and 0 C 0.

Now, J, which is required to be used for relating $\tilde{\tilde{\sigma}}$ $\tilde{\tilde{P}}$ and $\tilde{\tilde{S}}$. So, J is the determinant of $\tilde{\tilde{F}}$ or Jacobian, that is nothing but $A\ B\ C$ for this present deformation gradient tensor $\tilde{\tilde{F}}$, and using J, we can find out the $[F]^{-1}$. If you take the inverse of this deformation gradient tensor, $[F]^{-1}$ would be $\frac{1}{A}$ 0 0 0 0 $\frac{1}{C}$ 0 $-\frac{1}{B}$ and 0. So, with this matrix manipulation, now we are proceeding to find $\tilde{\tilde{P}}$ and $\tilde{\tilde{S}}$. First and second Piola-Kirchhoff stress tensors.

So, I have simply copied all the expressions which we had derived: $\tilde{\tilde{\sigma}}$ is given, $\tilde{\tilde{F}}$, $[F]^{-1}$, and J we had obtained. Now, the definition of the first Piola-Kirchhoff stress tensor $\tilde{\tilde{P}}$ is given through this transformation: $\tilde{\tilde{P}}$ equals to Jacobian J times $[\tilde{\tilde{\sigma}}]\left[\tilde{\tilde{F}}\right]^{-T}$. We already know $[\tilde{\tilde{\sigma}}]$, and we already know $\left[\tilde{\tilde{F}}\right]^{-T}$. So, $\left[\tilde{\tilde{F}}\right]^{-T}$ can be written like this.

So, J is ABC sigma is this where only one nonzero term is there at 3 3 location and this quantity is $\left[\tilde{\tilde{F}}\right]^{-T}$. So, I had flipped the $\frac{1}{C}$ and $-\frac{1}{C}$ these two particularentries in the $\left[\tilde{\tilde{F}}\right]^{-T}$. Now, if you multiply the 2 matrices, this would be resulting 0 0 0 0 0 0 0 σ_0 by C and 0 and taking σ_0 by C out, the $\tilde{\tilde{P}}$ first Piola-Kirchhoff stress tensor becomes $AB\sigma_0$ times all 0s in first and second row and third row is 0 1 0. So, this is the first Piola-Kirchhoff stress tensor corresponding to the given Cauchy stress tensor.



Now, moving forward to the calculation of second Piola-Kirchhoff stress tensor. So, here the obtained $\tilde{\tilde{P}}$ is also added which we had just now obtained along with the other quantity $[\tilde{\tilde{\sigma}}]$, $\left[\tilde{\tilde{F}}\right]^{-T}$ and J. Now, second Piola-Kirchhoff stress tensor $\tilde{\tilde{S}}$ is defined through this transformation. This is equals to $J\left[\tilde{\tilde{F}}\right]^{-1}[\tilde{\tilde{\sigma}}]\left[\tilde{\tilde{F}}\right]^{-T}$.

This is the relation between second Piola-Kirchhoff stress tensor $\tilde{\tilde{S}}$ and Cauchy stress tensor $\tilde{\tilde{\sigma}}$. Now, instead of this it is convenient to relate $\tilde{\tilde{S}}$ with $\tilde{\tilde{P}}$ then the expression is slightly shorter. We would be needing to multiply only 2 matrices if you are writing $\tilde{\tilde{S}}$ as $\left[\tilde{\tilde{F}}\right]^{-1}\left[\tilde{\tilde{P}}\right]$. you can also obtain $\tilde{\tilde{S}}$ using this particular rule, but here 3 big matrices you need to multiply. So, that is why it is convenient to use the relation between $\tilde{\tilde{S}}$ with $\tilde{\tilde{P}}$ the first and second Piola of stress tensor as $\tilde{\tilde{S}}$ equals to $\left[\tilde{\tilde{F}}\right]^{-1}\left[\tilde{\tilde{P}}\right]$.

So, $\left[\tilde{\tilde{F}}\right]^{-1}$ is already known. $\tilde{\tilde{P}}$ we had already calculated. So, Writing those two here, $\left[\tilde{\tilde{F}}\right]^{-1}$ times $\tilde{\tilde{P}}$ would be $AB\sigma_0$ multiplied with $\frac{1}{A}$ 0 0 0 0 $\frac{1}{C}$ 0 $-\frac{1}{B}$ 0. And $\tilde{\tilde{P}}$ is all zeros except the location of 3 2, which is equal to unity. Now, multiplying these two, we get the second Piola-Kirchhoff stress tensor $\tilde{\tilde{S}}$ to be $AB\sigma_0$ by C times 0 0 0 0 1 0 0 0 0.

So, this completes the present problem. This is the second Piola-Kirchhoff stress tensor corresponding to the given Cauchy stress tensor sigma. For any problem, if any one of these three stress tensors— $\tilde{\tilde{\sigma}}$, $\tilde{\tilde{P}}$, and $\tilde{\tilde{S}}$—are given, and you know the deformation mapping, that is, the deformation gradient tensor $\tilde{\tilde{F}}$, then it is possible to convert the given stress tensor into the remaining two, any two other stress tensors.

So, in total, in this particular lecture, first we introduced the concept of the stress measure in the undeformed or the initial configuration, involving both first and second Piola-Kirchhoff stress tensors, and then solved a few example problems. Thank you.