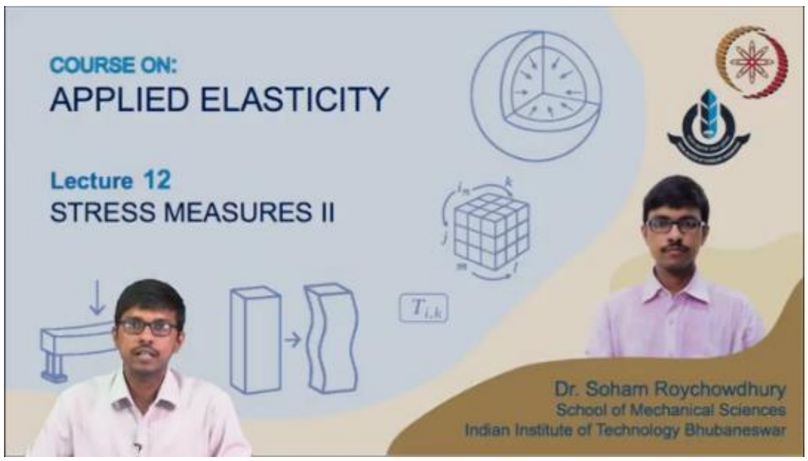
APPLIED ELASTICITY

Dr. SOHAM ROYCHOWDHURY

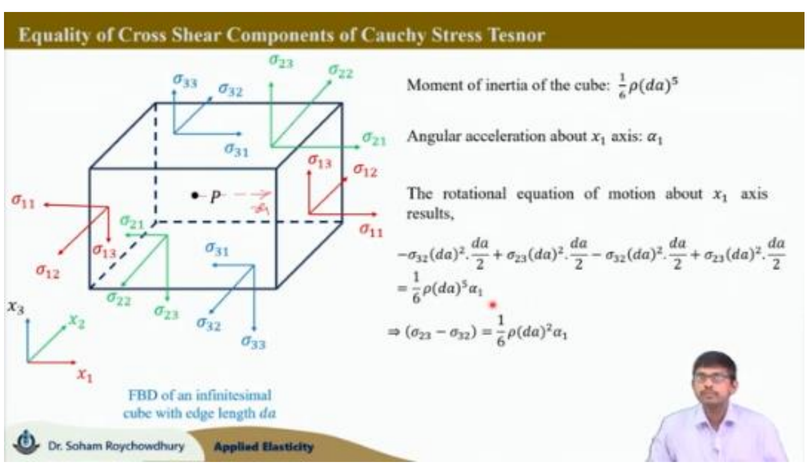
SCHOOL OF MECHANICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR

WEEK: 03
Lecture- 12



Welcome to the course on applied elasticity. In this lecture, we are going to continue our discussion on stress measures. In the previous lecture, we started our discussion on different stress measures, introduced the Cauchy stress components, and their relation with the traction vector.



Today, we are first going to show the equality of the cross-shear components of a Cauchy stress tensor. So, let us consider this infinitesimal cube element with edge length da and edges parallel to the x_1, x_2, and x_3 axes of the coordinate system, respectively.

Then, considering a point P at the center of this cube and drawing all the stress components—Cauchy stress components—on the x_1 plane, σ_{11}, σ_{12}, and σ_{13} are the Cauchy stress components on the x_1 plane. Then, on the x_2 plane—that is, on the front and back faces—the components are σ_{21}, σ_{22}, σ_{23}, and on the x_3 planes—the top and bottom faces of this cube— σ_{31}, σ_{32}, and σ_{33} are the Cauchy stress components, as we discussed in the last lecture. Now, our objective here is to show that the cross-shear components of this Cauchy stress tensor are equal.

So, σ_{ij} equals σ_{ij}—that is what we are trying to prove through this particular approach. Now, considering the moment of inertia of the cube about any one axis—x_1, x_2, or x_3—that would be (1/6) $\rho(da)^5$, where da is the length of each side of this cube and ρ is the density of the material. Considering α_1 to be the angular acceleration of the body with respect to the x_1 axis, if we write the rotational equation of motion of this cube about the x_1 axis, that would be

I times α, the moment of inertia of the cube, multiplied with angular acceleration about the x_1 axis. So, this 1 by 6 ρ da to the power 5 times α_1 equals the net moment due to all the external forces acting on all different faces about the x_1 axis. So, if you express the net moment about the x_1 axis due to all the Cauchy stress components acting on the 6 different faces of the cube. So, if you look at the first term, minus σ_{32} da square into da by 2.

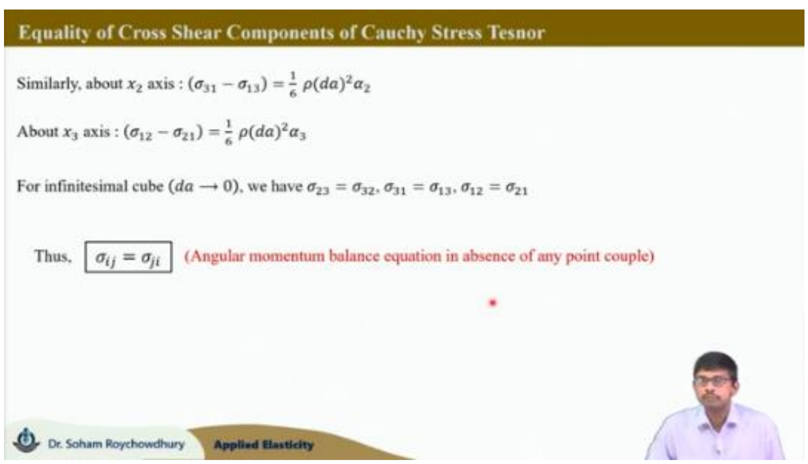
So, this is the moment coming due to this σ_{32} term. So, here just consider that the origin is at point P. So, the x_1 axis, x_2 axis, and x_3 axis are attached to point P. Now, we are taking the moment balance about the x_1 axis centered at point P. So, σ_{32} multiplied with the area on which σ_{32} is acting. So, the area of the bottom face is da square. So, σ_{32} multiplied with area da square into the distance of the bottom face from point P, that is da by 2, half of the height of the cube.

In total, this creates a moment about the negative x_1 axis because if you just look at the direction of the moment created by this σ_{32} term, that is about the negative x_1 axis, and if you look at the moment direction created by σ_{32} acting on the top face, that is also about the negative x_1 axis. Thus, the first term and third term, these are the contribution of σ_{32} times area da square into da by 2, whereas considering the two faces which are on the x_2

plane, that is the front face where σ_{32} is there and on the back face where σ_{23} upward is there, both of them are causing moment about the positive x_1 axis, so σ_{23} multiplied with corresponding area da square into

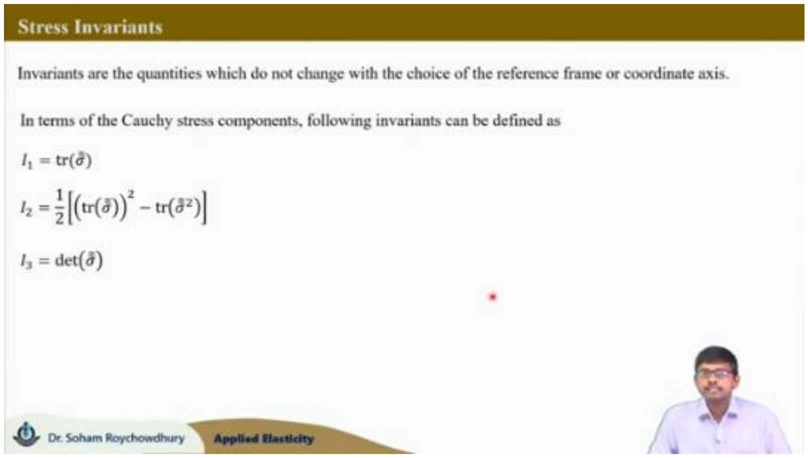
The horizontal distance from point P to the two side x_2 planes is da by 2 once again. So, thus the second term and the fourth term are the moments created by σ_{23} about the positive x_1 axis. The rest of the Cauchy stress components—σ_{11}, σ_{22}, σ_{33}, all of these, and σ_{12}—cannot create any moment about the x_1 axis because either they are parallel to the x_1 axis or they are passing through the x_1 axis. So, the moment of all the rest of the Cauchy stress components is 0 in the rotational equation of motion about the x_1 axis. So, these left-hand side four terms are the net moment about the x_1 axis, which equals I times the moment of inertia times the angular acceleration α_1 about the x_1 axis.

Now, if you simplify the left-hand side, it would simply become σ_{23} minus σ_{32}. Cancelling da cube from both sides, the right-hand side becomes 1 by 6 ρ da square times α_1. So, this is the moment balance or rotational equation of motion about the x_1 axis. If you repeat it for the x_2 and x_3 axes.



Similarly, you will get these two equations. So, considering the rotational equation about all three axes, for this small infinitesimal cube with da tending to 0, we must have σ_{23} equal to σ_{32}, σ_{31} equal to σ_{13}, and , σ_{12} equal to σ_{21}. All the Cauchy cross-shear components—the non-diagonal components—should be equal to the corresponding non-diagonal components. σ_{ij} equals σ_{ji}. This proves that the Cauchy stress tensor must be symmetric. Now, note that this proof is valid as long as

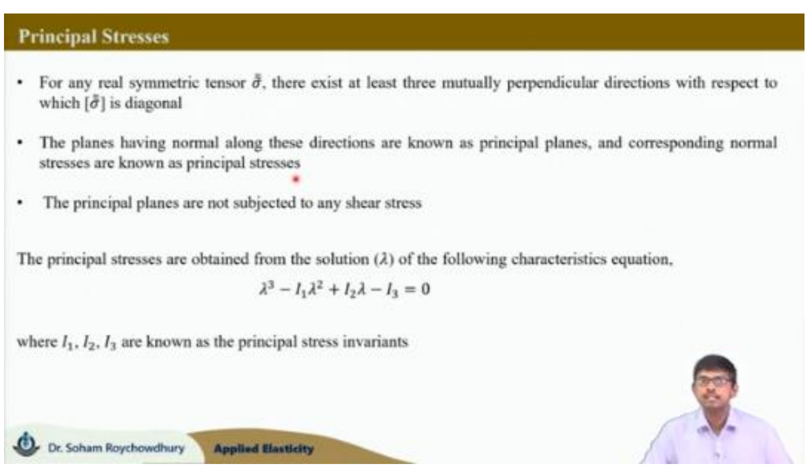
No point couple is acting on the body. In the previous free body diagram, we have only considered the surface tractions and resulting Cauchy stress components. No point couple about the $x, y,$ or z axis was added. So, in the absence of any point couple in the infinitesimal cube, it is possible to show that the Cauchy stress tensor $\tilde{\tilde{\sigma}}$ is a symmetric tensor.



Now, we are moving forward to the concept of stress invariance.

This is something similar to the concept of strain invariants, which we had already discussed. So, instead of strain, here we are just considering the Cauchy stress tensor sigma and invariants are defined to be the quantities which do not change with the choice of the reference frame or our x_1, x_2, x_3 axes. Now, in terms of Cauchy stress components, we can define three stress invariants I_1, I_2, and I_3, which are nothing but the trace of σ equals I_1, I_2 equals half of the trace of $\tilde{\tilde{\sigma}}$ squared minus the trace of the square of $\tilde{\tilde{\sigma}}$, and I_3 is the determinant of $\tilde{\tilde{\sigma}}$.

So, in terms of the Cauchy stress tensor $\tilde{\tilde{\sigma}}$, it is possible to define these three stress invariants I_1, I_2, I_3, which are known as principal stress invariants and are independent of the choice of the reference frame.



Now, with the help of these stress invariants, it is possible to define the principal stress components. So, for any real symmetric tensor $\tilde{\tilde{\sigma}}$, $\tilde{\tilde{\sigma}}$ is always symmetric, as we had already proved, and we are interested only in the real values of the Cauchy stresses. So, thus for our purpose, $\tilde{\tilde{\sigma}}$ is a real symmetric tensor.

Now, for $\tilde{\tilde{\sigma}}$ being a real symmetric tensor, it is possible to find out one mutually orthogonal set of directions. With respect to which, the $\tilde{\tilde{\sigma}}$ matrix is a diagonal matrix or the $\tilde{\tilde{\sigma}}$ tensor becomes a diagonal tensor. The corresponding directions are known as the principal directions, and the normal stresses associated with those directions are known as the principal stresses. So, the planes which are normal—which have normals along these principal directions—are called principal planes. And the normal stresses acting on those planes are called principal stresses.

However, these principal planes are free of any kind of shear stress. So, we are only having the normal stresses present on the principal plane. The values of the principal stresses— σ_1, σ_2, and σ_3 —the diagonal components. Of $\tilde{\tilde{\sigma}}$, when it is expressed in terms of a diagonal matrix or diagonal tensor, can be obtained by getting the solution of λ by solving this characteristic equation, which is a cubic equation of λ. $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3$ equals to 0, where I_1, I_2, I_3 are the principal stress invariants as discussed in the previous slide. So, from this, you would be getting three solutions of lambda, which are nothing but the.

Three principal stress values, and corresponding directions can be obtained as the principal directions. Now, moving forward to the stress transformation, We would be using the concept of orthogonal transformation tensor $\tilde{\tilde{Q}}$, which was discussed in the first week of this course, for transforming two sets of base vectors. One is $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, and the transformed base vectors $\tilde{e}_1', \tilde{e}_2'$, and $\tilde{e}_3'$.

These two can be related by any orthogonal coordinate transformation tensor. So, let us define Q_{ij} as the components of this orthogonal transformation tensor. $\tilde{e}_i'\ \tilde{e}_j$, the dot product of two unit vectors in two different frames, $\tilde{e}_i'\ \tilde{e}_j$, is nothing but the cosine of the angle between these two unit vectors $\tilde{e}_i'\ \tilde{e}_j$. So, like that, we can define Q_j. The opposite one is also possible to define; we can define Q_{ij} as $\tilde{e}_j'.\tilde{e}_i$—that is our choice. So, we can define the orthogonal tensor as the dot product of two unit vectors defined in two different frames, and here it is defined in this particular fashion. Now, for any vector a, we can write it as the summation of $a_j\tilde{e}_j$ in the first frame of reference with respect to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ unit vectors. We can also write the same vector a as the summation over i of $a_i'\tilde{e}_i'$ prime, where $\tilde{e}_i'$ are the second set of transformed unit vectors. So, with respect to both these unit vectors, it is possible to define any vector a. Thus, by using the definition of $a_i'\tilde{e}_i'$ prime, the components of the vector in the transformed frame can be written as Q times a in the initial notation: a_i' prime equals $Q_{ij}a_j$.

Now, coming to the stress transformation—the previous one was vector transformation with the help of Q— now we are going to implement it for the stress transformation, where stress is a second-order tensor. So, considering σ_{ij} to be the stress components of sigma, the Cauchy stress, in the $\tilde{e}_i$ base vector frame, and σ_{ij}' to be the stress components for σ, the Cauchy stress tensor, in the $\tilde{e}_i'$ base vector frame. The components of the traction vectors in both these frames can be related with the corresponding σ components as t equals $[\sigma]^T\{n\}$. And $\{t'\}$ equals $[\sigma']^T\{n'\}$, or $\{t'\}$ prime equals $[\sigma']^T\{n'\}$.

So, this is the relation between the traction vector and stress in the $\tilde{e}_i$ frame. This is the relation between t', the traction vector, and stress components in the $\tilde{e}_i'$ frame. Now, t and n are two vector quantities. So, they follow the transformation law defined in the previous slide. We can write the components of $\{t'\}$ as $[Q]\{t\}$

and $\{n'\}$ as $[Q]\{n\}$.

Now, we will substitute t' as $[Q]\{t\}$ and n' as $[Q]\{n\}$ in this particular equation. So, t' equals σ' transpose n'. On the left-hand side, t' is written as Q times t, and on the right-hand side, n' is written as Q times n. Thus, this equation becomes $[Q]\{t\}$ equals $[\sigma']^T$ Q times n. Now, using the definition of t, if you look at this, t was defined to be $[\sigma]^T$ n. So, comparing these two and taking Q from the left-hand side to the right-hand side, adding a $[Q]^{-1}$, pre-multiplying both sides of this equation with $[Q]^{-1}$, we can get t to be $[Q]^{-1}$ $[\sigma']^T[Q]\{n\}$, and then comparing this expression of t with the initial expression of t.

We can simply write that Q $[\sigma]^T$ equals $[Q]^T[\sigma']^T[Q]$. So, comparing both expressions of t, we can write $[\sigma]^T$ to be $[Q]^T[\sigma']^T[Q]$. And σ and $[\sigma']$ both being the symmetric stress tensor components, we can remove the transpose over σ, but we cannot remove transpose from Q, as Q is not symmetric in general. So, removing transpose from σ and $[\sigma']$ terms, we get $[\sigma]$ equals $[Q]^T[\sigma'][Q]$. This is the transformation law between two stress tensor components for the same Cauchy stress tensor in $\tilde{e}_i$ frame and $\tilde{e}_i'$ frame.

So, $[\sigma']$ is equals to $[Q][\sigma][Q]^T$ through this transformation we should be able to convert the stress components in $\tilde{e}_i$ frame to stress components in $\tilde{e}_i'$ frame. Now, in the initial notation if I write this σ_{ij}' becomes $Q_{ik}\sigma_{kl}Q_{lj}^T$ which is product of 3 matrices $[Q][\sigma][Q]^T$ and removing the transpose sign and flipping the indices for this term $[Q]^T\ lj$ it is written as $Q_{ik}Q_{jl}\sigma_{kl}$. Thus σ_{ij}' becomes $Q_{ik}Q_{jl}\sigma_{kl}$. The components of σ in $\tilde{e}_i$ frame and $\tilde{e}_i'$ frame are related through 2 Q components Q_{ik} and Q_{jl} where Q is the orthogonal transformation tensor.

Moving forward the coordinate independence of the eigenvalues which we will try to prove. So, let us say in $\tilde{e}_i$ frame λ is the eigenvalue or λ are the set of eigenvalues with $\{V\}$ being the corresponding eigenvectors. And thus by definition of of the eigenvalue or eigenvectors for the stress σ stress tensor σ σ acting over $\{V\}$ is equals to λ times $\{V\}$. Similarly, in the $\tilde{e}_i'$ frame lambda i λ' being the eigenvalues and V' being the eigenvectors, we can also write $[\sigma']\{V'\}$ equals to λ $\{V'\}$. This is the eigenvector relation in the sig $\tilde{e}_i'$ frame.

Now, our objective is to show that this λ and V eigenvalue and eigenvectors are the frame invariant quantities. Now, let us start with the $\tilde{e}_i'$ prime frame. So, eigenvalue equation in $\tilde{e}_i'$ frame is this sigma prime $\{V'\}$ is equals to $\lambda'\{V'\}$. Let us assume lambda itself is the eigenvalue of the system in the $\tilde{e}_i'$ frame and starting with this assumption we will try to show that this equation will boil down to this equation. If it is possible to show that then we can claim that this λ and V quantities are independent to the frame.

So, choosing λ to be the eigenvalue of the σ stress tensor in the $\tilde{e}_i'$ frame, the eigenvalue equation can be written as $[\sigma']\{V'\}$ equals λ times V'. Now, moving forward, using the stress transformation rule, we write $[\sigma']$ as $[Q][\sigma][Q]^T$ and the eigenvector V' component as $[Q]\{V\}$. So, following the transformation law of a vector for the eigenvector and the transformation law of a tensor for the Cauchy stress, substituting both σ' and V' on the left-hand side and right-hand side, this is the equation we obtain.

Now, comparing both sides, we can write that $[Q]$ times σ acting over V minus λ times V, so σ V minus λ V, the total thing is multiplied with Q, equals 0. And from this, for any orthogonal transformation Q, which is quite arbitrary, we must have σ V equals λ V. So, we started with σ' V' equals λ V', and with that, we end up with σ V equals λ V, which is the definition of the eigenvalue problem in the $\tilde{e}_i$ frame. So, this proves the frame invariance.

of the eigenvalue λ and eigenvector V. So, if λ equals λ', along with that, V and V' are the components of the same vector, we are getting back the eigenvalue equation in the $\tilde{e}_i$ frame, the initial frame, and thus the eigenvalues λ and eigenvector V are frame-invariant quantities. Now, λ being a frame-invariant quantity, we should also have λ^2, λ^3 coefficients. in the characteristic equation to be frame-invariant quantities, and those were nothing but I1, I2, I3, the principal stress invariants. Thus, λ being frame-invariant.

The coefficients of λ, λ^2, λ^3 in the characteristic equation, which are nothing but I_1, I_2, I_3, are also frame invariant quantities. Now, after this, we will be taking a few specific cases which are commonly available states of stress. The first one is hydrostatic pressure. So, for the case of hydrostatic pressure, no shear stresses are observed or felt by the body. It only applies compressive normal stress, which is the same along all three directions.

This occurs due to the fluid loading or hydrostatic loading acting on the body. So, if you have a submerged body within a fluid, then this kind of hydrostatic state of stress can be achieved. So, for such cases, σ_{ij} can be written as minus p times δ_{ij}, where delta is the δ; only the diagonal terms are non-zero, and all have the value of minus p, where p is the hydrostatic pressure. Normally, this depends on the position of the point.

So, p depends on the point or location of the point at which we are trying to evaluate σ. Now, after hydrostatic pressure, let us consider the case of uniaxial tension or compression, which is applied along the x_1 direction. So, this is the body taken in the x_1-x_3 plane; then, σ is the stress applied along the x_1 direction with this The stress tensor looks like this: only the σ_{11} (σ_{x1}) component along the x_1 direction on the x_1 plane is non-zero; all the rest of the stress components are zero. Thus, the Cauchy stress tensor is σ, 0, 0, then all zeros in both the second and third rows.

where σ is a constant normal stress along the x_1 direction. So, for uniaxial tension, only one non-zero component in the Cauchy stress tensor is possible. Now, moving forward to the next example of uniform shear on the x_2 plane, the x_2 plane means planes which have a normal along the x_2 direction, or you can also call that the x_1x_3 plane. So, considering this x_1x_3 plane, for which x_2 is a constant quantity, and subjecting this to a pure shear or uniform shear τ, as shown in the figure, for this x_2 equals constant plane, we should get the stress matrix.

stress tensor, the Cauchy stress tensor as 0 0 τ 0 0 0 tau 0 0. Now, what are the non-zero components? One is τ in the 1-3 location, another is τ in the 3-1 location. Now, why only these two are non-zero? Because τ is the constant shear stress acting along the x_1 direction on the x_3 plane.

So, if you consider the top face, this top face has an outward normal in the upward direction, which is along x_3. So, this face must be the positive x_3 plane. Now, the direction of tau here is towards the right, which is along the x_1 direction. So, thus this particular tau is nothing but the 1-3 location acting on—sorry, the 3-1 location acting on the third plane in the one direction, whereas, on the right-hand side, this tau is acting on

Plane 1 x_1 plane in the upward direction, that is along the x_3 direction. So, these 2 results, these 2 non-zero terms at 1 3 and 3 1 locations. All the rest of the shear stresses and normal stresses are 0 for this uniform shear problem on x_2 equals to constant plane. Now, taking the next example, the next case of a plane stress problem. So, for a plane stress problem, we can consider the plane stress assumption in any one particular plane.

So, x_1 x_2 plane or x_3 equals to constant plane means the thickness of the material in the x_3 direction is negligible, and thus no out-of-plane stresses would be generated. So, for such cases, all the third row and third column entries of sigma would be 0, and thus the plane stress σ Cauchy stress tensor looks like this: σ_{11}, σ_{12}, 0; σ_{12}, σ_{22}, 0; and 0, 0, 0. Now, considering the case of pure bending about the x_2 axis. So, let us consider this body, a bar or a beam aligned along the x_1 direction, and this is subjected to pure bending moment M about the x_2 axis.

So, as shown here, this M will try to bend the bar and will give rise to the normal stresses along all these fibers aligned along the length of the bar. So, the normal stress generated due to this applied bending moment M would be along the x_1 direction, and that can be obtained as M times x_3 divided by I, where I is the second moment of area, x_3 is the distance of any fiber from the neutral fiber or the center fiber, which refers to x_3 equals to 0. So, this σ_{11}, the bending stress generated along the x_1 direction, is a constant times x_3. where this constant is nothing but M divided by I, C equals to M divided by I.

So, sigma, the Cauchy stress tensor for this problem, becomes Cx_3 0 0 0 0 0 and 0 0 0. The only non-zero component is σ_{11}, which is proportional to x_3. Here, we have considered x_3 equals to 0 to be the neutral plane or mid-plane, which is shown here. The mid-plane of the beam, where the stress is 0, has no elongation or compression of the mid-plane fiber. For all the fibers above the mid-plane, x_3 greater than 0, we are supposed to have tension, and for all the fibers below the mid-plane, x_3 less than 0, we are supposed to have compression. If C is positive, which is the case?

For the given direction of m. So, the value of c, whether it would be positive or negative, depends on the direction of M. If you change the direction of the bending moment, the top set of fibers will be undergoing compression, and the bottom set of fibers will be having tension. Thus, this will be a negative quantity in the 1 1 location. Now, moving forward to the pure torsion about the x_1 axis. So, we are considering a circular shaft or circular cylinder with x_1 being its longitudinal axis, about which

T is the torque supplied, which is trying to give torsion to this shaft with a circular cross-section. Now, the equation of the circle, the cross-section, is given by x_2 square plus x_3 square equals to r^2 square, where r is the radius of the shaft. Now, we can find out tau, the shear stresses in-plane, as τ_{12} and τ_{13}. So, if you consider any point here, if you consider any point here on the circle, on the positive x_2 and positive x_3 values. Due to the applied torque T, the generated shear stress direction will be like this.

Now, this axis is x_2, this axis is x_3. Due to the applied torque about the positive x_1 axis, the component of tau will be along the negative x_2 direction and along the positive x_3 direction. Thus, τ_{12}, the component of shear stress along x_2, in the x_2 direction is minus Cx_3, and τ_{13}, the component of shear stress along x3 directed along x_3, is a positive quantity Cx_2. So, with this, if you write the Cauchy stress tensor, we are having four non-zero terms at τ_{12}, τ_{13}, τ_{21}, and τ_{31} locations.

So, the Cauchy stress tensor looks like this: 0, minus Cx_3, Cx_2, minus Cx_3, 0, 0, Cx_2, 0, 0, where C is nothing, but T divided by J. T is the applied torque, and J is the polar moment of area for this circular cross-sectional shaft. So, in this lecture, we first proved or showed the equality of the cross-shear components for the Cauchy stress tensor, then discussed the concepts of stress invariants, principal stresses, eigenvalues, and eigenvectors for the stresses, and finally, took a few examples of specific commonly available cases for the state of stress.

Thank you.